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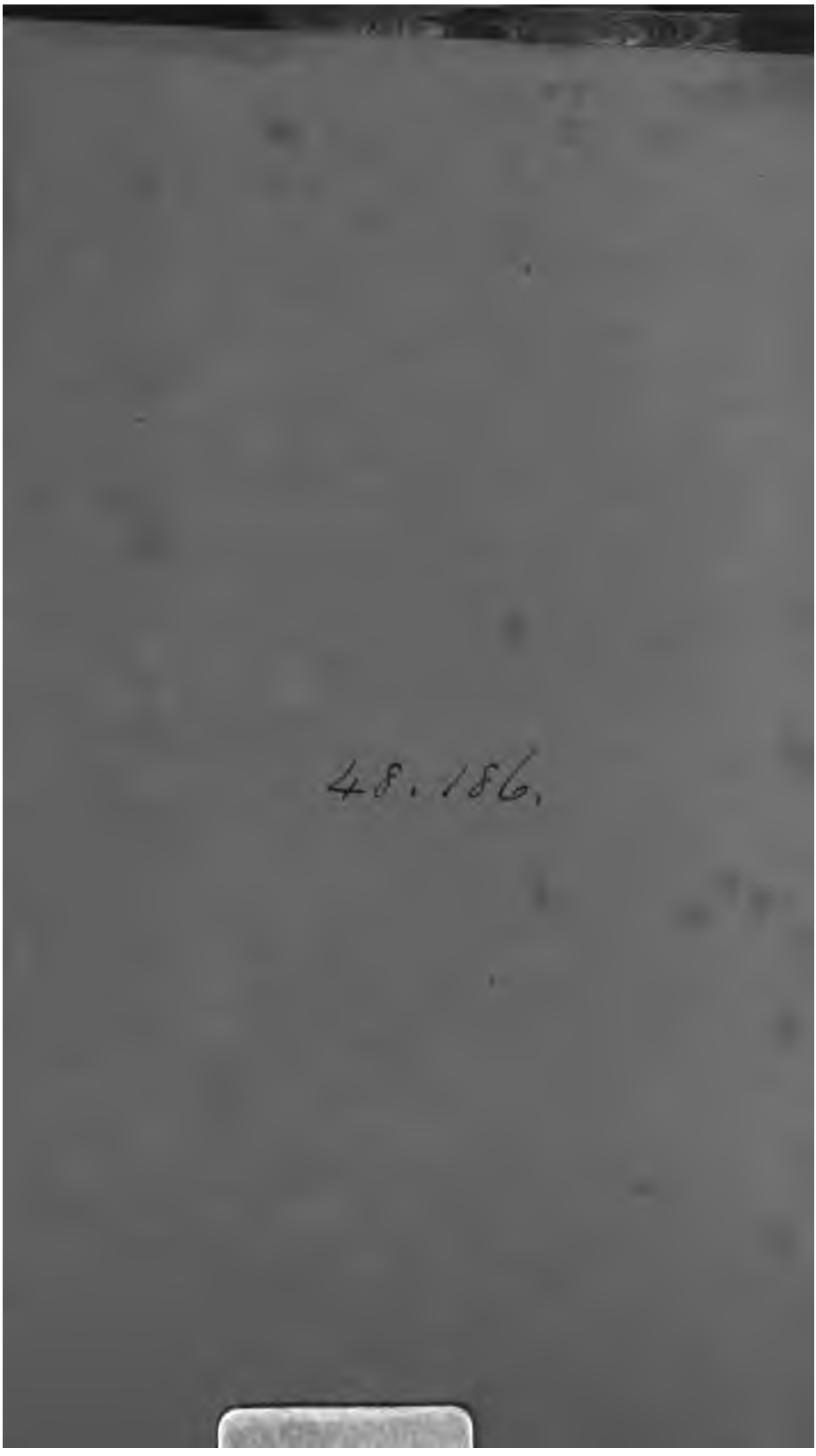
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MATHEMATICS

FOR

PRACTICAL MEN:

BEING

A COMMON-PLACE BOOK

OF

PURE AND MIXED MATHEMATICS,

DESIGNED CHIEFLY FOR THE USE OF

CIVIL ENGINEERS, ARCHITECTS, AND SURVEYORS.

BY OLINTHUS GREGORY, LL.D., F.R.A.S.

THIRD EDITION, REVISED AND ENLARGED

BY HENRY LAW,

CIVIL ENGINEER.

LONDON:
JOHN WEALE, 59, HIGH HOLBORN.

1848.

LONDON:

GEORGE WOODFALL AND SON,

ANGEL COURT, SKINNER STREET.

TO

SIR ISAMBART BRUNEL, F.R.S.,

ETC. ETC. ETC.

THIS WORK,

DESIGNED FOR THE USE OF MEMBERS OF THAT PROFESSION WHICH HE HAS SO GREATLY ADVANCED,

IS INSCRIBED,

IN GRATEFUL ACKNOWLEDGMENT OF MANY KINDNESSES,

BY HIS OBLIGED FRIEND AND PUPIL,

HENRY LAW.

AUTHOR'S PREFACE.

THE work now presented to the public had its origin in a desire which I felt to draw up an Essay on the principles and applications of the mechanical sciences for the use of the younger members of the Institution of Civil Engineers. The eminent individuals who are deservedly regarded as the main pillars of that useful Institution, stand in need of no such instructions as are in my power to impart: but it seemed expedient to prepare an Essay, comprised within moderate limits, which might furnish scientific instruction for the many young men of ardour and enterprise who have of late years devoted themselves to the interesting and important profession, of whose members that Institution is principally constituted. My first design was to compose a paper which might be read at one or two of the meetings of that Society; but, as often happens in such cases, the embryo thought has grown, during meditation, from an essay to a book; and what was first meant to be a very compendious selection of principles and rules, has, in its execution, assumed the appearance of a systematic analysis of principles, theorems, rules, and tables.

Indeed, the circumstances in which the inhabitants of this country are now placed, with regard to the love and acquisition of knowledge impelled me, almost unconsciously, to such an extension of my original plan, as sprung from a desire to contribute to the instruction of that numerous class, the practical mechanics of this country. Besides the early disadvantages under which many of them have laboured, there is another which results from the activity of their pursuits. Unable, therefore, to go through the details of an extensive systematic course, they must, for the most part, be satisfied with imperfect views of theories and principles, and take much upon trust: an evil, however, which the establishment of Societies, and the composition of treatises, with an express view to their benefit, will probably soon diminish.

LORD BROUGHAM, in his "*Practical Observations upon the Education of the People*," remarks that "a most essential service will be

rendered to the cause of knowledge, by him who shall devote his time to the composition of elementary treatises on the Mathematics, sufficiently clear, and yet sufficiently compendious, to exemplify the method of reasoning employed in that science, and to impart an accurate knowledge of the most useful fundamental propositions, with their application to practical purposes; and treatises upon Natural Philosophy, which may teach the great principles of physics, and their practical application, to readers who have but a general knowledge of Mathematics, or who are even wholly ignorant of the science beyond the common rules of arithmetic." And again, "He who shall prepare a treatise simply and concisely unfolding the doctrines of Algebra, Geometry, and Mechanics, and adding examples calculated to strike the imagination, of their connection with other branches of knowledge, and with the arts of common life, may fairly claim a large share in that rich harvest of discovery and invention which must be reaped by the thousands of ingenious and active men, thus enabled to bend their faculties towards objects at once useful and sublime."

I do not attempt to persuade myself that the present volume will be thought adequately to supply the desiderata to which these passages advert: yet I could not but be gratified, after full two-thirds of it were written, to find that the views which guided me in its execution accorded so far with the judgment of an individual, distinguished, as LORD BROUGHAM was, in early life, for the elegance and profundity of his mathematical researches.

With a view to the elementary instruction of those who have not previously studied Mathematics, I have commenced with brief, but, I hope, perspicuous, treatises on *Arithmetic* and *Algebra;* a competent acquaintance with both of these being necessary to ensure that accuracy in computation which every practical man ought to attain, and that ready comprehension of scientific theorems and formulæ which becomes the key to the stores of higher knowledge. As no man sharpens his tool or his weapon, merely that it may be sharp, but that it may be the fitter for use; so no thoughtful man learns Arithmetic and Algebra for the mere sake of knowing those branches of science, but that he may employ them; and these being possessed as valuable pre-requisites, the course of an author is thereby facilitated: for then, while he endeavours to express even common matters so that the learned shall not be disgusted, he may so express the more abstract and difficult that the comparatively ignorant (and the

mere knowledge of *Arithmetic* and *Algebra* is, in our times, comparative ignorance,) may practically understand and apply them.

After the first 97 pages, the remaining matter is synoptical. The general topics of Geometry, Trigonometry, Conic Sections, Curves, Perspective, Mensuration, Statics, Dynamics, Hydrostatics, Hydrodynamics, and Pneumatics are thus treated. The definitions and principles are exhibited in an orderly series; but investigations and demonstrations are only sparingly introduced. This portion of the work is akin in its nature to a syllabus of a Course of Lectures on the departments of science which it treats; with this difference, however, occasioned by the leading object of the publication, that popular illustrations are more frequently introduced, practical applications incessantly borne in mind, and such tables as seemed best calculated to save the labour of Architects, Mechanics, and Civil Engineers, inserted under their appropriate heads. Of these latter, several have been collected from former treatises, &c., but not a few have been either computed or contributed expressly for this Commonplace Book.

In a work like this, it would be absurd to pretend to originality. The plan, arrangement, and execution, are my own; but the materials have long been regarded, and rightly, as common property. It has been my aim to reduce them into the smallest possible space, consistently with my general object; but wherever I have found the work in this respect prepared to my hands, I have transcribed it into the following pages, with the usual references to the sources from whence it was taken. They who are conversant with the best writers on subjects of mixed mathematics and natural philosophy, will know that *Smeaton*, *Robison*, *Playfair*, *Young*, *Du Buat*, *Leslie*, *Hachette*, *Bland*, *Tredgold*, &c., are authors who *ought* to be consulted, in the preparation of a volume like this. I hope it will appear that I have duly, yet, at the same time, honourably, availed myself of the advantages which they supply. I have also made such selections from my own earlier publications as were obviously suitable to my present purpose; but not so copiously, I trust, as to diminish the utility of those volumes, or to make me an unfair borrower even from myself.

Besides our junior Civil Engineers, and the numerous Practical Mechanics who are anxious to store their minds with scientific facts and principles; there are others to whom, I flatter myself, the following pages will be found useful. Teachers of Mathematics, and those departments of natural philosophy which are introduced into

our more respectable seminaries, may probably find this volume to occupy a convenient intermediate station between the merely popular exhibitions of the truths of Mechanics, Hydrostatics, &c., and the larger treatises in which the whole chain of inquiry and demonstration is carefully presented link by link, and the successive portions firmly connected upon irrefragable principles. While students who have recently terminated a scientific course, whether in our universities, or other institutions public or private, may, I would fain believe, find in this Common-place Book an abridged repository of the most valuable principles and theorems, and of hints for their applications to practical purposes.

The only performances with which I am acquainted, that bear any direct analogy to this, are Martin's *Young Student's Memorial Book*, Jones's *Synopsis Palmariorum Matheseos*, and Brunton's *Compendium of Mechanics;* the latter of which I had not seen until the present volume was nearly completed. The first and last mentioned of these are neat and meritorious productions; but restricted in their utility by the narrow space into which they are compressed. The other, written by the father of the late Sir William Jones, is a truly elegant introduction to the principles of Mathematics, considering the time in which it was written (1706); but as it is altogether theoretical, and is, moreover, now becoming exceedingly scarce, it by no means supersedes the *necessity*, for such I have been induced to regard it, of a Compendium like that which I now offer to the public.

In its execution I have aimed at no higher reputation than that of being perspicuous, correct, and useful; and if I shall be so fortunate as to have succeeded in those points, I shall be perfectly satisfied.

OLINTHUS GREGORY.

Royal Military Academy,
Woolwich, October 1st, 1825.

In this new edition I have corrected a few errors which had escaped my notice in the former impression. I have also made a few such additions and improvements as the lapse of time and the progress of discovery rendered desirable; and such as will, I hope, give the work new claims on public approbation.

July 1st, 1833.

THE EDITOR'S PREFACE.

In presenting to the public a new edition of Dr. Gregory's "Mathematics for Practical Men," the Editor feels that a few introductory remarks are due from him to his readers. Not, indeed, to apologise for presenting to them again a work which they have already marked with their approbation, and which has from time to time been favourably noticed by other writers, but to offer some explanation of the numerous alterations which he has taken upon himself to make, and to state his reasons for having done so.

In looking through the former edition, it was evident, that since the date of its publication, many of the subjects which it treated upon had been greatly improved and extended; railways had been brought into successful operation; the steam engine and machinery generally had undergone vast improvements; and almost every branch of mechanical science had progressed in some degree; it was therefore found necessary, in order to render the present edition as extensively useful as its predecessors had been, that the work should undergo an entire revision. In doing this, it was found that so large a quantity of fresh matter would have to be given, as would swell the volume to an inconvenient size, and by a necessary increase of price, render the work less accessible to that important class—the working mechanics—for whose use it was so peculiarly adapted. To obviate, therefore, these disadvantages, a smaller type has been adopted, so that the work, although not much increased in its external dimensions, contains more than a third of its present bulk of new matter.

In order to add to the usefulness of the work, and render it more generally convenient as a book of reference, a more orderly arrangement of its contents has been adopted, and several articles, which, although of much value, did not properly belong to the subject of the work, have been thrown into an Appendix; in addition to which a Table of Contents has been given, showing at once the general ar-

rangement and division of the work, as also a very copious Index at the end.

The Editor has endeavoured, in the following short review of the work in its present form, to point out the principal alterations or additions which have been made:—

The work has been divided into two distinct parts; viz. 1st, PURE MATHEMATICS, comprising chapters upon Arithmetic, Algebra, Geometry, Mensuration, Trigonometry, Conic Sections, and the Properties of Curves; and, 2ndly, MIXED or APPLIED MATHEMATICS, being the application of Mathematics to the general pursuits of the Engineer, the Mechanician, the Surveyor, &c., comprised in chapters upon Mechanics generally, Statics, Dynamics, Hydrostatics, Hydrodynamics, Pneumatics, Mechanical Agents, or Moving Powers, and the Strength of Materials.

In the part upon PURE MATHEMATICS:—

In the First Chapter, upon Arithmetic, the article upon fractions has been extended, and rules given for the reduction of *circulating* decimals; and an entirely new section (IX.) is added upon Logarithmic Arithmetic, containing a description of the Logarithmic Tables given in the Appendix, with the method of using them, and rules for performing the ordinary processes of calculation by their aid.

The Second Chapter, on Algebra, has undergone a very careful revision. In Section V., on Involution, a table of roots and powers of monomials has been added; and in the following section on Evolution, a new rule for finding the roots of powers in general. In Section X. examples have been added of the method for determining the roots of equations. Section XII., on Fractional and Negative Exponents, is entirely new; as is also nearly the whole of the next section on Logarithms; to which have been added tables for converting *Common* Logarithms into *Hyperbolic*, and *vice versâ*.

The Third Chapter, on Geometry, may almost be said to have been rewritten. The definitions and propositions have been arranged in a more orderly manner, and many additional ones have been added. No demonstration of the several propositions will be found, as the size of the work would not allow, nor did its practical character require, that such should be given; but for the convenience of those who may wish to see them demonstrated, a reference has been made to

Euclid, giving the number of the theorem and book in which the demonstration of the same proposition will be there found.

The Fourth Chapter, on Mensuration, has also been almost entirely rewritten: the tables of weights and measures have been put under a more convenient form, and rules and formulæ given for the mensuration of all kinds of superficies and solids.

In the Fifth Chapter, on Trigonometry, very extensive *Trigonometrical Formulæ* have been added, which cannot fail to be found of service in the pursuits of Engineers and Surveyors. The second section is entirely new, being a description of the Tables of Logarithmic sines, cosines, tangents, cotangents, secants, and cosecants given in the Appendix, with the method of using them.

The Sixth Chapter, on Conic Sections, has been almost entirely rewritten, and several additional formulæ and problems relating to them have been added.

The Seventh Chapter, on the Properties of Curves, has also been much extended; the first section of definitions, and the latter part of the fourth section, on the Epicycloid, being entirely new.

In the second division of the work, upon MIXED MATHEMATICS:—

In the First Chapter, on Mechanics in General, many fresh definitions have been added.

In the Second Chapter, the whole of the definitions and formulæ in the first section, on Statical Equilibrium, are new. In the second section, the principle of *parallel pressures* is applied to determine the position of the center of gravity, and an example is given of the centro-baryc method of determining the superficies or solidity of surfaces or solids. In the third section, on the application of Statics to the equilibrium of structures, the formulæ have been more conveniently arranged, and tables have been added of the natural slope assumed by various kinds of materials, as also of the limiting angle and coefficient of friction of the various materials used in the construction of arches, &c., the latter extracted from Professor Moseley's work on the Mechanical Principles of Engineering. An article is also added upon Suspension Bridges, a subject of much importance to Engineers, from the general use into which they are being brought. Formulæ are given for determining all their elements and proportions, the use of which is illustrated by an example. In the Third Chapter, on Dynamics, the definitions in the first section have been much extended; and at the commencement of the second section, on

the General Laws of Uniform and Variable Motion, it has been endeavoured to put in a clear light the long disputed question relative to the *momentum* and *vis viva* of moving bodies. In this section, also, formulæ have been added expressing the relation between the times, velocities, spaces, &c. of bodies in motion. In the third section some practical formulæ are added for determining the proportions for the *conical governor*, so generally used for regulating the velocity of steam engines and water wheels. In the fifth section, on the Mechanical Powers, their true nature is explained, in order to prevent any misconception which might be occasioned by their designation; and a table is added exhibiting the ratio in each, between the power exerted and the effect produced.

In the Fourth Chapter, on Hydrostatics, the fifth section, on Capillary Attraction, is entirely new.

In the Fifth Chapter, on Hydrodynamics, several additional formulæ are given, in the second section, for determining the velocity and discharge through conduit pipes; and the whole of the formulæ have been arranged in a more orderly manner, and numbered for the convenience of reference.

In the Sixth Chapter, on Pneumatics, at the conclusion of the first section, a very exact formulæ of Mr. Galbraith's is given, for determining the velocity of sound, and for measuring distances by the same.

In the Seventh Chapter, on the Mechanical Agents, the third section, on the Steam Engine, is entirely new. The nature and properties of steam are explained, and formulæ and rules are given, and illustrated by examples, for determining its pressure or temperature. The subject of the specific heat of steam is also explained, and formulæ given for determining it from the pressure; and a table is added of the pressure, temperature, and specific volume of steam, from 5 lbs. to 150 lbs. upon the square inch. A general description of the mode of action of the steam engine is given, which is followed by an investigation of its general theory upon the principles first laid down by the Comte de Pambour. The various kinds of engines employed are then classified, and described separately; the principles previously educed are applied to each; and formulæ are deduced for the velocity of the engine, the quantity of steam used, and the useful effect of the engine, for each particular case, which are given both in a general and a more practical form. In this portion of the work, in treating of engines acting expansively, it was

found impossible to dispense with the use of the Differential Calculus, the introduction of which had been studiously avoided, in order that the work might be accessible to those who had not entered upon the higher branches of Mathematics. The use of the Differential Calculus, however, in the present case, while it was necessary for the proper investigation of the subject, will not, it is hoped, in any way abridge the usefulness of the work to those who are not conversant with its principles, since the results obtained are expressed in the ordinary form, while for those who do understand them, it was thought that it would prove much more satisfactory to them to be enabled to examine and satisfy themselves of the truthfulness of the several investigations.

The Eighth Chapter, on the Strength of Materials, has been almost entirely rewritten. This portion of the work has been rendered much more valuable by the introduction of the results of Professor Hodgkinson and Mr. Cubitt's researches upon this subject, so important to the Civil Engineer and Architect.

The woodcuts have been replaced by copperplate engravings, and several additional plates have been given, for three of which the Editor is indebted to Mr. R. A. Rumble.

An Appendix has been added, containing a Table of Logarithmic Differences, in a new, and, it is hoped, more convenient form than that usually adopted, and Tables of the Logarithms of Numbers, and of Logarithmic sines, tangents, &c.: these tables will be found of great utility to Engineers and others, much engaged in calculations. It also contains a new Table (VI.) of various useful numbers, with their logarithms, and a Table (XI.) of the weight of materials frequently employed in construction. The remaining portion of the Appendix consists of matter which stood in the body of the former edition of the work, but which has now been put into the Appendix, because, although too valuable to be omitted, it did not properly belong to, but was only incidentally connected with, the subjects therein treated.

The Editor has thought it only doing justice to the late Dr. Gregory to point out thus in detail the extent of the alterations which he has made, in order that the public may know how far each is responsible for the work in its present form, and to prevent either blame being attached to Dr. Gregory, or credit given to the Editor, where both might have been misplaced.

H. L.

London,
21st October, 1847.

CONTENTS.

PART I.—PURE MATHEMATICS.

CHAPTER I.

ARITHMETIC.

		PAGE
SECT. 1.	Definitions and Notation.	1
2.	Addition of Whole Numbers	5
3.	Subtraction of Whole Numbers	6
4.	Multiplication of Whole Numbers	7
5.	Division of Whole Numbers	10
	Proof of the first Four Rules of Arithmetic	13
6.	Vulgar Fractions	13
	Reduction of Vulgar Fractions	14
	Addition and Subtraction of Vulgar Fractions	17
	Multiplication and Division of Vulgar Fractions	17
7.	Decimal Fractions	18
	Reduction of Decimals	19
	Addition and Subtraction of Decimals	22
	Multiplication and Division of Decimals	22
8.	Complex Fractions used in the Arts and Commerce	23
	Reduction	23
	Addition	24
	Subtraction and Multiplication	25
	Division	26
	Duodecimals	27
9.	Powers and Roots	28
	Evolution	29
10.	Proportion	32
	Rule of Three	34
	Determination of Ratios	37
11.	Logarithmic Arithmetic	40
	Use of the Tables	42
	Multiplication and Division by Logarithms	45
	Proportion, or the Rule of Three, by Logarithms	46
	Evolution and Involution by Logarithms	47
12.	Properties of Numbers	49

CHAPTER II.

ALGEBRA.

SECT. 1.	Definitions and Notation	52
2.	Addition and Subtraction	54

PAGE

SECT. 3. Multiplication 56
4. Division 58
5. Involution 60
6. Evolution 63
7. Surds 65
Reduction 65
Addition, Subtraction, and Multiplication 68
Division, Involution, and Evolution 69
8. Simple Equations 70
Extermination 73
Solution of General Problems 75
9. Quadratic Equations 77
10. Equations in General 81
11. Progression 84
Arithmetical Progression 85
Geometrical Progression 86
12. Fractional and Negative Exponents 88
13. Logarithms 90
14. Computation of Formulæ 95

CHAPTER III.

GEOMETRY.

SECT. 1. Definitions 98
2. Of Angles, and Right Lines, and their Rectangles . . . 99
3. Of Triangles 100
4. Of Quadrilaterals and Polygons 102
5 Of the Circle, and Inscribed and Circumscribed Figures . . 104
6. Of Planes and Solids 109
7. Practical Geometry 112

CHAPTER IV.

MENSURATION.

SECT. 1. Weights and Measures 119
1. Measures of Length 120
2. Measures of Surface 121
3. Measures of Solidity and Capacity 121
4. Measures of Weight 123
5. Angular Measure 124
6. Measure of Time 124
Comparison of English and French Weights and Measures . 125
2. Mensuration of Superficies 127
3. Mensuration of Solids 130

CHAPTER V.

TRIGONOMETRY.

SECT. 1. Definitions and Trigonometrical Formulæ 134
2. Trigonometrical Tables 139

PAGE

SECT. 3. General Propositions 146
4. Solution of the Cases of Plane Triangles 148
Right-angled Plane Triangles 151
5. On the application of Trigonometry to Measuring Heights and Distances 152
Determination of Heights and Distances by Approximate Mechanical Methods 158

CHAPTER VI.

CONIC SECTIONS.

SECT. 1. Definitions 162
2. Properties of the Ellipse 165
Problems relating to the Ellipse 167
3. Properties of the Hyperbola 168
Problems relating to Hyperbolas 170
4. Properties of the Parabola 171
Problems relating to the Parabola 173

CHAPTER VII.

PROPERTIES OF CURVES.

SECT. 1. Definitions 175
2. The Conchoid 176
3. The Cissoid 177
4. The Cycloid, and Epicycloid 178
5. The Quadratrix 179
6. The Catenary 180
Tables of Relations of Catenarian Curves 185

PART II.—MIXED MATHEMATICS.

CHAPTER I.

MECHANICS IN GENERAL 187

CHAPTER II.

STATICS.

SECT. 1. Statical Equilibrium 189
2. Center of Gravity 192
3. General application of the Principles of Statics to the Equilibrium of Structures 196
Equilibrium of Piers or Abutments 196
Pressure of earth against Walls 198
Thickness of Walls 201
Equilibrium of Polygons 202
Stability of Arches 205
Equilibrium of Suspension Bridges 207

CHAPTER III.

DYNAMICS.

		PAGE
SECT. 1.	General Definitions	211
2.	On the General Laws of Uniform and Variable Motion	212
	Motion uniformly Accelerated	214
	Motion of Bodies under the Action of Gravity	214
	Motion over a fixed Pulley	216
	Motion on Inclined Planes	217
3.	Motions about a fixed Center, or Axis	220
	Centers of Oscillation and Percussion	220
	Simple and Compound Pendulums	221
	Center of Gyration, and the Principles of Rotation	229
	Central Forces	232
	Inquiries connected with Rotation and Central Forces	234
4.	Percussion or Collision of Bodies in Motion	236
5.	On the Mechanical Powers	239
	Levers	240
	Wheel and Axle	242
	Pulley	243
	Inclined Plane	244
	Wedge and Screw	245

CHAPTER IV.

HYDROSTATICS.

SECT. 1.	General Definitions	248
2.	Pressure and Equilibrium of Non-elastic Fluids	249
3.	Floating Bodies	254
4.	Specific Gravities	256
5.	On Capillary Attraction	258

CHAPTER V.

HYDRODYNAMICS.

SECT. 1.	Motion and Effluence of Liquids	260
2.	Motion of Water in Conduit Pipes and Open Canals, over Weirs, &c.	262
	Velocities of Rivers	265
3.	Contrivances to Measure the Velocity of Running Waters	273

CHAPTER VI.

PNEUMATICS.

SECT. 1.	Weight and Equilibrium of Air and Elastic Fluids	278
2.	Machines for Raising Water by the Pressure of the Atmosphere	281
3.	Force of the Wind	292

CHAPTER VII.

Mechanical Agents.

		PAGE
Sect. 1.	Water as a Mechanical Agent	294
2.	Air as a Mechanical Agent	297
	Coulomb's Experiments	301
3.	Mechanical Agents depending upon Heat. The Steam Engine	302
	Table of Pressure and Temperature of Steam	311
	General Description of the Mode of Action of the Steam Engine	312
	Theory of the Steam Engine	316
	Description of the various kinds of Engines, and the Formulæ for calculating their Power	322
	Practical application of the foregoing Formulæ	343
4.	Animal Strength as a Mechanical Agent	348

CHAPTER VIII.

Strength of Materials.

Sect. 1.	Results of Experiments, and Principles upon which they should be practically applied	370
2.	Strength of Materials to Resist Tensile and Crushing Strains	374
	Strength of Columns	379
3.	Elasticity and Elongation of Bodies subjected to a Crushing or Tensile Strain	381
4.	On the Strength of Materials subjected to a Transverse Strain	385
	Longitudinal form of Beam of uniform Strength	386
	Transverse Strength of other Materials than Cast Iron	387
	The Strength of Beams according to the manner in which the Load is distributed	388
5.	Elasticity of Bodies subjected to a Transverse Strain	388
6.	Strength of Materials to resist Torsion	392

APPENDIX.

I.	Table of Logarithmic differences	1
II.	Table of Logarithms of Numbers, from 1 to 100	1
III.	Table of Logarithms of Numbers, from 100 to 10,000	1
IV.	Table of Logarithmic Sines, Tangents, Secants, &c.	19
V.	Table of Useful Factors, extending to several places of Decimals	64
VI.	Table of various Useful Numbers, with their Logarithms	66
VII.	A Table of the Diameters, Areas, and Circumferences of Circles and also the Sides of Equal Squares	66
VIII.	Table of the Relations of the Arc, Abscissa, Ordinate and Subnormal, in the Catenary	73
IX.	Tables of the Lengths and Vibrations of Pendulums	76
X.	Table of Specific Gravities	80
XI.	Table of Weight of Materials frequently employed in Construction	82
XII.	Principles of Chronometers	82
XIII.	Select Mechanical Expedients	89
XIV.	Observations on the Effect of Old London Bridge on the Tides, &c.	92
XV.	Professor Farish on Isometrical Perspective	96

LIST OF PLATES.

PLATE I. } General Diagrams.
II. }
III. }
IV. }
V. }

VI. Details of a Breast Water-wheel.
VII. Fenton, Murray, and Wood's Steam Engine.
VIII. High-pressure Expansive Engine, by Middleton.
IX. Details of Steam Engines.
X. Longitudinal Section of a Locomotive Engine.
XI. Two Transverse Sections of the Same.
XII. Details of various Engines.
XIII. Isometrical Perspective.

A

COMMON-PLACE BOOK,

ETC., ETC.

PART I.

PURE MATHEMATICS.

CHAP. I.

ARITHMETIC.

SECT. I. *Definitions and Notation.*

ARITHMETIC is the science of numbers.

We give the name of *number* to any assemblage of *units*, or of any parts of an assumed unit; a *unit* being the quantity which, among all those of the same kind, forms a *whole* which may be regarded as the base or element. Thus, when we speak of *one house*, *one guinea*, we speak of *units*, of which the first is the thing called a house, the second that called a guinea. But when we say *four houses*, *ten guineas*, *three quarters of a guinea*, we speak of *numbers*, of which the first is the unit *house* repeated *four times*; the second is the unit *guinea* repeated *ten times*; the third is the *fourth* part of the unit *guinea* repeated *three times*.

In every particular classification of numbers, the unit is a measure taken arbitrarily, or established by usage and convention.

Numbers formed by the repetition of an unbroken unit are called *whole numbers*, or *integers*, as *seven* miles, *thirty* shillings: those which are formed by the assemblage of *any parts* of a unit are called *fractional numbers*, or simply *fractions*; as *two thirds* of a yard, *three eighths* of a mile.

When the unit is restricted to a certain thing in particular, as *one man*, *one horse*, *one pound*, the collection of many of those units is called a *concrete* number, as *ten men*, *twenty horses*, *fifty pounds*. But if the unit does not denote any particular thing, and is expressed simply by *one*, numbers which are constituted of such units are denominated *discrete* or *abstract*, as *five*, *ten*, *thirty*. Hence, it is evident that abstract numbers can only be compared with their unit, as con-

crete numbers are compared with, or measured by, theirs; but that it is not possible to compare an abstract with a concrete number, or a concrete number of one kind with a concrete number of another; for there can exist no measurable relations but between quantities of the *same* kind.

The series of numbers is indefinite; but only the first nine of them are expressed by different characters, called *figures;* which are as follows:—

Names. one, two, three, four, five, six, seven, eight, nine.
Figures. 1, 2, 3, 4, 5, 6, 7, 8, 9.

These are called *significant* figures, in contradistinction to another character employed, namely 0, called the *cypher* or *zero;* which has no particular value of itself, but by its *position* is made to change the value of any *significant* figures with which it is connected.

In the system of numeration now generally adopted, and which is borrowed from the Indians*, an infinitude of words and characters is

* As the Roman notation is not unfrequently met with, especially in dates, we subjoin the following brief account of it:—The Romans employed only seven numerals, being the following capital letters of their alphabet;—viz. I, for *one;* V, for *five;* X, for *ten;* L, for *fifty;* C, for an *hundred;* D, for *five hundred;* M, for a *thousand;* and for expressing any intermediate or greater numbers they employed various repetitions and combinations, the principles of which are shown in the following examples:—

1 = I	As often as any character is repeated, so many times is its value repeated.
2 = II	
3 = III	
4 = IIII or IV	A less character *before* a greater *diminishes* its value.
5 = V	
6 = VI	A less character *after* a greater *increases* its value.
7 = VII	
8 = VIII	
9 = IX	
10 = X	
40 = XL	
50 = L	
60 = LX	
100 = C	
500 = D or Iↀ	For every ↀ annexed, this becomes ten times as many.
1000 = M or CIↀ	For every C and ↀ, placed one at each end, it becomes ten times as much.
2000 = MM	
5000 = $\overline{\text{V}}$ or Iↀↀ	A bar over any number increases its value 1000-fold.
6000 = $\overline{\text{VI}}$	
10,000 = $\overline{\text{X}}$ or CCIↀↀ	
50,000 = $\overline{\text{L}}$ or Iↀↀↀ	
60,000 = $\overline{\text{LX}}$	
100,000 = $\overline{\text{C}}$ or CCCIↀↀↀ	
1,000,000 = $\overline{\text{M}}$ or CCCCIↀↀↀↀ	
2,000,000 = $\overline{\text{MM}}$	
&c. &c.	

avoided, by a simple yet most ingenious expedient, which is this:—*every figure placed to the left of another assumes ten times the value that it would have if it occupied the place of the latter.*

Thus, to express the number that is the sum of 9 and 1, or ten units, (called *ten*,) we place a 1 to the *left* of a 0, thus 10. So again the sum of 10 and 1, or *eleven*, is represented by 11; the sum of 11 and 1, or of 10 and 2, (called *twelve*,) is represented by 12; and so on for *thirteen*, *fourteen*, *fifteen*, &c., denoted respectively by 13, 14, 15, &c., the figure 1 being all along equivalent to *ten*, because it occupies the second rank.

In like manner, *twenty*, *twenty-one*, *twenty-two*, &c., are represented by 20, 21, 22, because the 2 in the second rank is equivalent to twice ten, or *twenty*. And thus we may proceed with respect to the numbers that fall between twenty and three tens or *thirty* (30), four tens or *forty* (40), five tens or *fifty* (50), six tens or *sixty* (60), seven tens or *seventy* (70), eight tens or *eighty* (80), nine tens or *ninety* (90). After 9 are added to the 90 (ninety), numbers can no longer be expressed by *two* figures, but require a third rank on the left hand of the second.

The figure that occupies the third rank, or of *hundredths*, is expressed by the word *hundred*. Thus 369, is read three *hundred* and sixty-nine; 428, is read four *hundred* and twenty-eight; 837, eight *hundred* and thirty-seven: and so on for all numbers that can be represented by three figures.

But if the number be so large that more than three figures are required to express it, then it is customary to divide it into periods of *three* figures each, reckoning from the right hand towards the left, and to distinguish each by a peculiar name. The second period is called that of *thousands*, the third that of *millions*, the fourth that of *milliards* or *billions**, the fifth that of *trillions*, and so on; the terms units, tens, and hundreds, being successively applied to the first, second, and third ranks of figures from the right towards the left, in each of these periods.

Thus, 1,111, is read one thousand one hundred and eleven.

23,456, twenty-three thousands, four hundred and fifty-six.

421,835, four hundred and twenty-one thousands, eight hundred and thirty-five.

732,846,915, seven hundred and thirty-two millions, eight hundred and forty-six thousands, nine hundred and fifteen.

The manner of estimating and expressing numbers which we have here described is conformable to what is denominated the *decimal notation*. But, besides this, there are other kinds invented by philosophers, and others indeed in common use: as the *duodecimal*, in

* It has been customary in England to give the name of *billions* to millions of millions, of *trillions* to millions of millions of millions, and so on: but the method here given of dividing numbers into periods of three figures instead of six, is universal on the Continent; and, as it seems more simple and uniform than the other, I have adopted it.

which every superior name contains *twelve* units of its next inferior name; and the *sexagesimal*, in which *sixty* of an inferior name are equivalent to one of its next superior. The former of these is employed in the measurement and computation of artificers' work; the latter in the division of the circle, and of an hour in time.

To the head of notation we may also refer the explanation of the principal symbols or characters employed to express operations or results in computation. Thus,

The sign + (*plus*) belongs to addition, and indicates that the numbers between which it is placed are to be added together. Thus, 5 + 7 expresses the sum of 5 and 7, or that 5 and 7 are to be added together.

The sign — (*minus*) indicates that the number which is placed after it is to be subtracted from that which precedes it. So, 9 — 3 denotes that 3 is to be taken from 9.

The sign ~ denotes *difference*, and is placed between two quantities when it is not immediately evident which of them is the greater.

The sign × (*into*), for multiplication, indicates the product of two numbers between which it is placed. Thus 8 × 5 denotes 8 times 5, or 40.

The sign ÷ (*by*), for division, indicates that the number which precedes it is to be divided by that which follows it; and the quotient that results from this operation is often represented by placing the first number over the second with a small bar between them. Thus, 15 ÷ 8 denotes that 15 is to be divided by 8, and the quotient is expressed thus $\frac{15}{8}$.

The sign =, two equal and parallel lines placed horizontally, is that of equality. Thus, 2 + 3 + 4 = 9, means that the sum of 2, 3, and 4, is equal to 9.

Inequality is represented by two lines so drawn as to form an angle, and placed between two numbers, so that the angular point turns towards the least. Thus, 7 > 4, and A > B, indicate that 7 is *greater than* 4, and the quantity represented by A *greater than* the quantity represented by B; and, on the other hand, 3 < 5 and C < D indicate that 3 is *less than* 5, and C *less than* D.

Colons and double colons are placed between quantities to denote their proportionality. So, 3 : 5 : : 9 : 15, signifies that 3 are to 5 as 9 are to 15, or $\frac{3}{5} = \frac{9}{15}$.

The extraction of roots is indicated by the sign √, with a figure occasionally placed over it to express the *degree* of the root; or by a *fraction* (having unity for its numerator, and the figure expressing the *degree* of the root for its denominator) placed above and to the right of the quantity to have its root extracted; thus, √ 4 or $4^{\frac{1}{2}}$, signifies the square root of 4; $\sqrt[3]{27}$, or $27^{\frac{1}{3}}$, the cube root of 27; $\sqrt[4]{16}$ or $16^{\frac{1}{4}}$, the fourth or biquadrate root of 16.

The raising of powers is expressed by a *whole* number similarly placed, the figure denoting the power to which the quantity is to be raised; thus 6^2, signifies the square of 6; 8^3, the cube of 8; and 3^4,

the fourth power of 3. The figures thus used to indicate the power, whether *whole* or *fractional* numbers, are termed *indices* or *exponents*.

When both operations are to be successively performed upon a quantity, that is, when some root is to be extracted, and then that root to be raised to some different power, the operation is very simply expressed by a fraction placed as before, the *denominator* of which indicates the *root* to be extracted, and the *numerator* the *power* to which that root is to be afterwards raised. Thus $27^{\frac{2}{3}}$, denotes that the cube root of 27 is to be extracted, and that the root so obtained is to be squared, or raised to the second power. It is immaterial which operation is first performed, for the result would be the same whether we first extracted the cube root, as above, and then squared it, or whether we squared the number first, and then extracted the cube root of the power so obtained.

Although the above signs * are principally employed in Algebra and the higher branches of Mathematics, they are given here, as their use in Arithmetic frequently affords brevity without a sacrifice of perspicuity.

SECT. II. *Addition of Whole Numbers.*

ADDITION is the rule by which two or more numbers are collected into one aggregate or *sum*.

Suppose it were required to find the sum of the numbers 3731, 349, 12487, and 54. It is evident that if we computed separately the sums of the units, of the tens, of the hundreds, of the thousands, &c., their combined results would still amount to the same. We should thus have 15 thousands + 14 hundreds + 20 tens + 21 units, or 15000 + 1400 + 200 + 21; operating again upon these, in like manner, rank by rank, we should have 10 thousands + 6 thousands + 6 hundreds + 2 tens + 1, or 16621, which is the sum required.

But the calculation is more commodiously effected in the following manner:—

Rule.—Place the given numbers under each other, so that units stand under units, tens under tens, hundreds under hundreds, &c. Add up all the figures in the column of units, and observe for every *ten* in its amount to carry *one* to the place of tens in the second column, putting the overplus figure in the first column. Proceed in the same manner with the second column, then with the third, and so on till all the columns be added up: the figures thus obtained in the several amounts indicate, according to the rules of notation, the sum required.

Note.—Whether the addition be conducted upwards or downwards, the result will be the same; but the operation is most frequently conducted by adding upwards.

* There are other signs employed in the processes of Algebra, an explanation of which will be found at page 52, *et seq.*

Example.—Taking the same numbers as before, and disposing them as the rule directs, we have 4 + 7 + 9 + 1 = 21, of which we put down the 1 in the place of *units*, and carry the 2 to the tens: then 2 + 5 + 8 + 4 + 3 = 22, of which we put down the left hand 2 in the place of *tens*, and carry the other to the hundreds: then 2 + 4 + 3 + 7 = 16, of which the 6 is put in the *hundreds*, and the 1 carried to the thousands. This progress continued will give the same sum as before.

3731	
349	
12487	
54	
16621	

Other Examples.

57	6475	77756	10376786
762	9830	3388	789632
5389	2764	9763	1589
97615	5937	90257	73
103823	25006	181164	11168080

SECT. III. *Subtraction of Whole Numbers.*

SUBTRACTION is the rule by which one number is taken from another, so as to show the difference, or excess.

The number to be subtracted or taken away is called the *subtrahend;* the number from which it is to be taken, the *minuend;* and the quantity resulting, the *remainder.*

Rule.—Write down the minuend, and beneath it the subtrahend, units under units, tens under tens, and so on. Then beginning at the place of units, take each figure in the subtrahend from its corresponding figure in the minuend, and write the difference under those figures in the same rank or place.

But if the figure in the subtrahend be greater than its corresponding figure in the minuend, add ten to the latter, and then take the figure in the subtrahend from the sum, putting down the remainder, as before; and in this case add 1 to the next figure to the *left* in the subtrahend, to compensate for the ten borrowed in the preceding place. Thus proceed till all the figures are subtracted.

Example: . .

Minuend	26565874
Subtrahend	9853642
Remainder	16712232

Here the five figures on the right of the subtrahend are each less than the corresponding figures in the minuend, and may therefore be taken from them, one by one. But the sixth figure, viz. 8, cannot be taken from the 5 above it. Yet, as a unit in the *seventh* place is equivalent to 10 in the *sixth*, this unit *borrowed* (for such is the

technical word here employed) makes the 5 become 15. Then 8 taken from 15 leaves 7, which is put down; and 1 is added to the 9 in the seventh place of the subtrahend, to compensate or balance the 1 which was borrowed from the seventh place in the minuend. Recourse must be had to a like process whenever a figure in the subtrahend exceeds the corresponding one in the minuend.

Other Examples.

From 8217	From 44444	Take 21498	Take 45624
Take 3456	Take 3456	From 76262	From 80200
Remains 4761	Remains 40988	Remains 54764	Remains 34576

Note.—Although it is customary to place the minuend above the subtrahend, this is not absolutely necessary. Indeed, it is often convenient in computation to find the difference between a number and a greater that naturally stands beneath it: it is, therefore, expedient to practise the operation in both ways, so that it may, however it occurs, be performed without hesitation.

SECT. IV. *Multiplication of Whole Numbers.*

MULTIPLICATION of whole numbers is a rule by which we find what a given number will amount to when it is repeated as many times as are represented by another number *.

The number to be multiplied, or repeated, is called the *multiplicand*, and may be either an abstract or a concrete number.

The number to be multiplied by is called the *multiplier*, and *must* be an abstract number, because it simply denotes the number of times the multiplicand is to be repeated.

Both multiplicand and multiplier are called *factors*.

The number that results from the multiplication is called the *product*.

Before any operation can be performed in multiplication, the learner must commit to memory the following table of products, from 2 times 2, to 12 times 12.

* This definition, though not the most scientific that might be given, is placed here, because others depend, implicitly if not explicitly, on proportion, and therefore cannot logically be introduced thus early in the course.

times	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

It is very advantageous, in practice, to have this table carried on, at least intellectually, to 20 times 20. All the products to this extent are easily remembered.

The learner will perceive that in this table 7 times 5 is equal to 5 times 7, or $7 \times 5 = 35 = 5 \times 7$. In like manner that $8 \times 3 = 24 = 3 \times 8$, $4 \times 11 = 44 = 11 \times 4$, and so of other products. This is often made a subject of formal proof, as well as that $3 \times 5 \times 8 = 3 \times 8 \times 5 = 5 \times 3 \times 8 = 5 \times 8 \times 3$, &c. But to attempt the demonstration of things so nearly axiomatical as these is quite unnecessary.

Previously to exhibiting the rules for performing multiplication, let us take a simple example, and multiply 4827 by 8. Here placing the numbers as in the margin, and multiplying in their order 7 units by 8, 2 tens by 8, 8 hundreds by 8, 4 thousands by 8, the several products are 56 units, 16 tens, 64 hundreds, 32 thousands: these placed in their several ranks, according to the rules of notation, and then added up, give for the sum of the whole, or for the product of 4827 multiplied by 8, the number 38616.

4827
8
56
16
64
32
38616

Or the same example may be worked thus:—

8	×	7	=	56
8	×	20	=	160
8	×	800	=	6400
8	×	4000	=	32000
				38616

Which is evidently the same in effect as before.

CASE I.—*To multiply a number, consisting of several figures, by a number not exceeding* 12.

Rule.—Multiply each figure of the multiplicand by the multiplier, beginning at the units; write under each figure the units of the product, and carry on the tens to be added as units to the product following.

Examples.

Multiply	4827	218043	440052	8765400
By	8	9	11	12
Products	38616	1962387	4840572	105184800

CASE II.—*To perform multiplication when each factor exceeds* 12.

Rule.—Place the factors under each other (usually the smallest at bottom), and so that units stand under units, tens under tens, and so on. Multiply the multiplicand by the figure which stands in the unit's place of the multiplier, and dispose the product so that its unit's place shall stand under the unit of the multiplicand; then multiply successively by the figure in the place of tens, hundreds, &c., of the multiplier, and place the first figure of each product under that figure of the multiplier which gave the said product. The sum of these products will be the product required.

Example.

Multiply 8214356 by 132.

Multiplicand	8214356
Multiplier	132
8214356 × 2 = ...	16428712
8214356 × 3 tens = ...	24643068
8214356 × 1 hundred = ...	8214356
8214356 × 132 = ...	1084294992

Other Examples.

Multiply 821436 by 672576		Multiply 8210075 by 420306	
4928616		49260450	
5750052		24630225	
4107180		16420150	
1642872		32840300	
5750052			
4928616		Product	3450743782950
Product	552478139136		

Note.—Multiplication may frequently be shortened by separating the multiplier into its component parts or factors, and multiplying by them in succession. Thus, since 132 times any number are equal to 12 times 11 times that number, the first example may be performed in this manner:

Multiply	8214356	
by	11	
And this product	90357916	Here one line of multiplication, and one of addition, are saved.
by	12	
Product as before	1084294992	

So, again, the multiplier of the second example, viz. 672576, divides into three numbers, 600000, 72000, and 576; where, omitting the cypher, we have $72 = 12 \times 6$, and $576 = 8 \times 72$. Hence the operation may be performed thus:—

Multiplicand	821436	
Multiply by	6	in the 6th place.
	4928616	
Previous product × 12 ...	59143392	for 72 thousands.
Second product × 8	473147136	for 576 units.
Same product as before	552478139136	: three lines saved.

Other modes of contraction will appear as we proceed.

SECT. V. *Division of Whole Numbers.*

DIVISION is a rule by which we determine how often one number is contained in another. Or, it is a rule by which, when we know a product and one of the factors which produced it, we can find the other.

The number to be divided is called the *dividend*; that by which it

is divided, the *divisor;* and that which results from the division, the *quotient.* When division and multiplication are regarded as reciprocal operations, the *dividend* is equivalent to the *product*, the *divisor* is equivalent to the *multiplier*, and the *quotient* is equivalent to the *multiplicand.*

Rule.—Draw a curved line both on the right and left of the dividend, and place the divisor on the left; then find the number of times the divisor is contained in as many of the left-hand figures of the dividend as are just necessary, and place that number on the right. Multiply the divisor by that number, and place the product under the above-mentioned figures of the dividend. Subtract the said product from that part of the dividend under which it stands, and bring down the next figure of the dividend to the right of the remainder. Divide the remainder thus increased, as before; and if at any time it be found less than the divisor, put a cypher in the quotient, bring down the next figure of the dividend, and continue the process till the whole is finished: the figures thus arranged will be the quotient required.

Examples.

Divide 743256 by 324.

Dividend.

Divisor 324)743256(2294 Quotient.

648	
952	
648	
3045	
2916	
1296	
1296	
Remain	

Divisor	324
Quotient	2294
	1296
	2916
	648
	648
Proof	743256

71)29754(419$\frac{5}{71}$

284	
135	
71	
644	
639	
5 Remain.	

131)135076(1031$\frac{15}{131}$

131	
407	
393	
146	
131	
15 Remain.	

In these two examples the numbers which remain are placed *over* their respective divisors, and attached to the quotients; the meaning of which will be explained when we treat of *fractions.*

Note.—When the divisor does not exceed 12, the operation may readily be performed in a single line; as will appear very evident if

the following example be compared with the two methods of working the first example in multiplication.

Divide 38616 by 8.

8)38616(4827	Dividend	38616
32	Divisor	8
66	Quotient	4827
64		
21		
16		
56		
56		
...		

Here 38 contains 8 four times, leaving a remainder of 6; these carried as 6 tens to the next 6, make 66, which contains 8 eight times, leaving 2, which carried as 2 tens to the next figure 1, make 21: and so of the rest.

In division, also, upon the same principle as in multiplication, the labour may often be abridged by taking component parts of the divisor. Thus, in the first example, the divisor is equal to 4 times 81, or 4 times 9 times 9. Hence the dividend may be divided by 4, 9, and 9, successively, as in the margin, and the result will be the same as before.

Divide	743256
by	4
this quotient	185814
by	9
and this	20646
by	9
Quotient	2294

Since 25 is a fourth part of 100, and 125 the 8th part of 1000, it will be easy to multiply or to divide by either of these numbers in a single line—thus,

To multiply 4827 by 25, put two cyphers on the right, which is equivalent to multiplying by 100; and divide by 4.

4)482700

120675 Answer.

To multiply 6218 by 125, put 3 cyphers, which is equivalent to multiplying by 1000; then divide by 8.

8)6218000

777250 Answer.

To divide 582100 by 25, strike off two figures on the right hand, which is equivalent to dividing by 100; then multiply by 4.

5821|00

4

23284 Answer.

To divide 4567000 by 125, strike off three figures on the right hand, which is equivalent to dividing by 1000; then multiply by 8.

4567|000

8

36536 Answer.

PROOF OF THE FIRST FOUR RULES OF ARITHMETIC.

Simple as these four rules are, it is not unusual to commit errors in working them: it is, therefore, useful to possess modes of proof.

1. Now, *addition* may be proved by adding downwards, as well as upwards, and observing whether the two sums agree; or, by dividing the numbers, to be added into two portions, finding the sum of each, and then the sum of those two separate amounts. Thus, in the margin, the sum of the four numbers is 7355; the sum of the two upper ones 5857, of the two lower ones 1498, and *their* sum is 7355, the same as before.

2758	2758
3099	3099
469	——
1029	469
——	1029
7355	——
	5857
	——
	1498
	——
	7355

2. The proof of *subtraction* is effected by adding the remainder to the subtrahend; if their sum agrees with the minuend the work is right, otherwise not.

3. *Multiplication* and *division* reciprocally prove each other.

There is also another proof for multiplication, known technically by the phrase *casting out the nines*. Add together the numbers from left to right in the multiplicand, dropping 9 whenever the sum exceeds 9, and carry on the remainder, dropping the nines as often as the amount is beyond them; and note the last remainder. Do the same with the multiplier and with the product; then multiply the first two remainders and cast the nines out of their product; if the remainder is equal to the last remainder, this is regarded as a test that the work is right. Thus, taking the second example in multiplication, the figures in the multiplicand amount to 6 above two nines, those in the multiplier to 6 above three nines, those in the product to 0 above six nines; the product 6×6 of the two first excesses is 36, or 0 above four nines: the coincidence of the two 0's is the proof. It is plain, however, that the proof will be precisely the same so long as the figures in the product be the same, whatever be their *order:* the proof, therefore, though ingenious, is defective *.

A similar proof applies to *division*.

SECT. VI. *Vulgar Fractions.*

THE fractions of which we have already spoken in Sect. I., are usually denominated *Common* or *Vulgar Fractions*, to distinguish

* The correctness of this proof, with the exception above specified, may be shown algebraically, thus:—put M and N = the number of nines in the multiplicand and multiplier respectively, m and n their excesses; then, $9M + m$ = the multiplicand, and $9N + n$ = the multiplier, and the product of those factors will be = $81MN + 9Mn + 9Nm = mn$; but the three first terms are each a precise number of nines; because one of the factors in each is so; these, therefore, being neglected, there remains mn to be divided by nine; but mn is the product of the two former excesses: therefore the truth of the method is evident. Q. E. D.

them from another kind, hereafter to be mentioned, called *Decimal Fractions.*

A fraction is an expression for the value of any part of an integer, or *whole* number, such number being considered as *unity.* Thus, if a *pound* sterling be the *unit*, then a shilling will be the *twentieth* part of that unit, and *four pence* will be four twelfths of that twentieth part. These represented according to the usual notation of vulgar fractions, will be $\frac{1}{20}$ and $\frac{4}{12}$ of $\frac{1}{20}$, respectively.

The lower number of a fraction thus represented (denoting the number of parts into which the integer is supposed to be divided) is called the *denominator;* and the upper figure (which indicates the number of those parts expressed by the fraction) the *numerator.* Thus, in the fractions $\frac{5}{7}$, $\frac{8}{15}$, 7 and 15 are *denominators*, 5 and 8 *numerators.*

Vulgar fractions are divided into proper, improper, mixed, simple, compound, and complex.

Proper fractions have their numerators less than their denominators, as $\frac{3}{4}$, $\frac{5}{8}$, &c.

Improper fractions have their numerators equal to, or greater than, their denominators, as $\frac{4}{4}$, $\frac{9}{8}$, &c.

Mixed fractions, or numbers, are those compounded of whole numbers and fractions, as $7\frac{5}{8}$, $12\frac{3}{4}$, &c.

Simple fractions are expressions for parts of whole numbers, as $\frac{4}{9}$, $\frac{5}{8}$, &c.

Compound fractions are expressions for the parts of given fractions, as $\frac{3}{4}$ of $\frac{5}{8}$, $\frac{2}{3}$ of $\frac{7}{11}$, &c.

Complex fractions have either one or both terms fractional numbers, as $\frac{5\frac{2}{3}}{24}$, $\frac{12}{14\frac{3}{4}}$, $\frac{6\frac{5}{8}}{12\frac{2}{3}}$, &c.

The value of a fraction is not altered by multiplying or dividing both its numerator and denominator by the same number; thus $\frac{1}{2}$, $\frac{2}{4}$, $\frac{12}{24}$, $\frac{144}{288}$, are all equal, although successively multiplied by 2, 6, and 12.

Any number which will divide two or more numbers without remainder, is called their *common measure.*

REDUCTION OF VULGAR FRACTIONS.

This consists principally in changing them into a more commodious form for the operations of addition, subtraction, &c.

CASE I.—*To reduce fractions to their lowest terms.*

Rule.—Divide the numerator and denominator of a fraction by any number that will divide them both, without a remainder; the quotient again, if possible, by any other number: and so on, till 1 is the greatest divisor.

Thus, $\frac{1470}{2205} = \frac{294}{441} = \frac{98}{147} = \frac{14}{21} = \frac{2}{3}$, where 5, 3, 7, 7, respectively, are the divisors.

Or, $\frac{1470}{2205} = \frac{2}{3}$, by dividing at once by 735.

Note.—This number 735 is called the *greatest* common measure of the terms of the fraction: it is found thus—Divide the greater of the two numbers by the less; the last divisor by the last remainder, and so on till nothing remains: the last divisor is the greatest common measure required*.

CASE II.—*To reduce an improper fraction to its equivalent whole or mixed number.*

Rule.—Divide the numerator by the denominator, and the quotient will be the answer: as is evident from the nature of division.

Ex.—Let $\frac{957}{43}$ and $\frac{5480}{274}$ be reduced to their equivalent whole or mixed numbers.

43)957(22$\frac{11}{43}$ Answer.
86
97
86
11

274)5480(20 Answer.
548
0

CASE III.—*To reduce a mixed number to its equivalent improper fraction; or a whole number to an equivalent fraction having any assigned denominator.*

Rule.—This is, evidently, the reverse of Case II.; therefore multiply the whole number by the denominator of the fraction, and add the numerator to obtain the numerator of the fraction required.

Ex.—Reduce 22$\frac{11}{43}$ to an improper fraction, and 20 to a fraction whose denominator shall be 274.

$(22 \times 43) + 11 = 957$ new numerator, and $\frac{957}{43}$ the 1st fraction.

$20 \times 274 = 5480$ new numerator, and $\frac{5480}{274}$ the 2nd fraction.

* The following theorems are useful for abbreviating Vulgar Fractions:—

THEOREMS.

1. If the last digit of any number be divisible by 2, the whole number is divisible by 2. If the two last digits be divisible by 4, the whole number is divisible by 4. If the three last digits be divisible by 8, the whole number is divisible by 8. And, generally, if the last n digits of any number be divisible by 2^n, the whole number is divisible by 2^n.

2. If a number terminate with 5, it is divisible by 5; and if it terminate in 0, it is divisible by either 10 or 5.

3. If the sum of the digits constituting any number be divisible by 3 or 9, the whole is divisible by 3 or 9; and if also the last digit be even, the whole number is divisible by 18.

4. If the sum of the digits constituting any number be divisible by 6, and the right-hand digit by 2, the whole is divisible by 6: for by the data it is divisible both by 2 and 3.

5. If the sum of the 1st, 3rd, 5th, &c., digits constituting any number be equal to that of the 2nd, 4th, 6th, &c., that number is divisible by 11: for if a, b, c, d, e, m, n, be the digits, constituting any number, its digits, when multiplied by 11, will become

(8)	(7)	(6)	(5)	(4)	(3)	(2)	(1)
a,	$a+b$,	$b+c$,	$c+d$,	$d+e$,	$e+m$,	$m+n$,	n;

where the odd terms are = to the even.

CASE IV.—*To reduce a compound fraction to an equivalent simple one.*

Rule.—Multiply all the numerators together for the numerator, and all the denominators together for the denominator, of the simple fraction required.

If part of the compound fraction be a mixed or a whole number, reduce the former to an improper fraction, and make the latter a fraction by placing 1 under the numerator.

When like factors are found in the numerators and denominators, cancel them both.

Ex.—Reduce $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{5}{7}$ of $\frac{7}{9}$ of $\frac{8}{11}$ to a simple fraction.

$$\frac{2\times3\times5\times7\times8}{3\times4\times7\times9\times11}=\frac{2\times5\times8}{4\times9\times11}=\frac{1\times5\times8}{2\times9\times11}=\frac{1\times5\times4}{1\times9\times11}=\frac{20}{99}$$

Here the 3 and 7 common to numerator and denominator are first cancelled; then the fraction is divided by 2; and then by 2 again.

Ex.—Reduce three farthings to the fraction of a pound sterling.

A farthing is the fourth of a penny, a penny the twelfth of a shilling, and a shilling the twentieth of a pound.

Therefore $\frac{3}{4}$ of $\frac{1}{12}$ of $\frac{1}{20}=\frac{3}{960}=\frac{1}{320}$ the answer.

Ex.—Simplify the complex fraction $\frac{2\frac{2}{3}}{4\frac{1}{5}}$.

Here, reducing the mixed numbers to improper fractions, we have $\frac{\frac{8}{3}}{\frac{21}{5}}$: multiplying by 3, to get quit of the denominator of the upper fraction, we have $\frac{8}{\frac{21}{5}}$: multiplying by 5, to get quit of the denominator of the lower fraction, we have $\frac{40}{21}$: dividing both terms of this fraction by 8, there results $\frac{5}{9}$ for the simple fraction required.

CASE V.—*To reduce fractions of different denominators to equivalent fractions having a common denominator.*

Rule.—Multiply each numerator into all the denominators except its own, for new numerators; and all the denominators together for a common denominator.

Ex.—Reduce $\frac{2}{3}$, $\frac{6}{7}$, and $\frac{5}{9}$, to equivalent fractions having a common denominator.

$$\left.\begin{array}{l}2\times7\times9=126\\6\times3\times9=162\\5\times3\times7=105\end{array}\right\}\text{the numerators.}$$

$3\times7\times9=189$, the common denominator.

Hence the fractions are $\frac{126}{189}$, $\frac{162}{189}$, $\frac{105}{189}$, or $\frac{42}{63}$, $\frac{54}{63}$, $\frac{35}{63}$, when divided by 3.

Hence, also, it appears that $\frac{2}{3}$ exceed $\frac{5}{9}$, and that $\frac{6}{7}$ exceed $\frac{2}{3}$.

Ex.—Reduce $\frac{4}{5}$ of a penny, and $\frac{2}{3}$ of a shilling, each to the fraction of a pound; and then reduce the two to fractions having a common denominator.

$\frac{4}{5}$ of a penny $=\frac{4}{5}$ of $\frac{1}{12}$ of $\frac{1}{20}=\frac{4}{1200}=\frac{1}{300}$ of a pound.

$\frac{2}{3}$ of a shilling $=\frac{2}{3}$ of $\frac{1}{20}=\frac{2}{60}=\frac{1}{30}=\frac{10}{300}$ of a pound.

Hence $\frac{2}{3}$ of a shilling are 10 times as much as $\frac{4}{5}$ of a penny.

Note.—Other methods of reduction will occur to the student after tolerable practice, and still more after the principles of algebra are acquired.

ADDITION AND SUBTRACTION OF FRACTIONS.

Rule.—If the fractions have a common denominator, add or subtract the numerators, and place the sum or difference as a new numerator over the common denominator.

If the fractions have not a common denominator, they must be reduced to that state before the operation is performed.

In addition of mixed numbers, it is usually best to take the sum of the integers, and that of the fractions, separately; and then *their* sum, for the result required.

Examples.

1. Find the sum of $\frac{2}{3}$, $\frac{5}{7}$, and $\frac{3}{4}$.
   $\frac{2}{3} + \frac{5}{7} + \frac{3}{4} = \frac{56}{84} + \frac{60}{84} + \frac{63}{84} = \frac{179}{84} = 2\frac{11}{84}$.
2. Take $\frac{3}{4}$ of a shilling from $\frac{1}{12}$ of a pound sterling.
   $\frac{3}{4}$ of a shilling $= \frac{3}{4}$ of $\frac{1}{20} = \frac{3}{80}$ of a pound $= \frac{180}{4800}$.
   Also $\frac{1}{12}$ of a pound $= \frac{400}{4800}$. Hence $\frac{400}{4800} - \frac{180}{4800} = \frac{220}{4800}$
   $= \frac{11}{240} = 11$ pence.
3. Find the difference between $12\frac{2}{6}$ and $8\frac{3}{5}$.
   $12\frac{2}{6} - 8\frac{3}{5} = \frac{77}{6} - \frac{43}{5} = \frac{185}{30} - \frac{258}{30} = \frac{127}{30} = 4\frac{7}{30}$.

MULTIPLICATION AND DIVISION OF FRACTIONS.

Rule 1. To multiply a fraction by a whole number, *multiply* the *numerator* by that number, and retain the denominator:—Or, *divide* the *denominator* by the same number, (if a multiple of it,) and retain the numerator.

2. To divide a fraction by a whole number, *multiply* the *denominator* by that number, and retain the numerator:—Or, *divide* the *numerator* by the same number, (if a multiple of it,) and retain the denominator.

3. To multiply two or more fractions is the same as to take a fraction of a fraction; and is, therefore, effected by taking the product of the numerators for a new numerator, and of the denominators for a new denominator. (The product is evidently smaller than either factor when each is less than unity.)

4. To divide one fraction by another, invert the divisor, and proceed as in multiplication. (The quotient is always greater than the dividend when the divisor is less than unity.)

Examples.

1. Multiply $\frac{3}{4}$ by 2, and divide $\frac{6}{7}$ by 5.
   $\frac{3}{4} \times 2 = \frac{6}{4} = \frac{3}{2} = 1\frac{1}{2}$, and $\frac{6}{7} \div 5 = \frac{6}{7 \times 5} = \frac{6}{35}$, *Ans.*

2. Multiply $2\frac{2}{3}$ by $\frac{3}{4}$, and divide $\frac{8}{9}$ by $\frac{4}{5}$.
$2\frac{2}{3} \times \frac{3}{4} = \frac{8}{3} \times \frac{3}{4} = \frac{8}{4} = 2$, and $\frac{8}{9} \div \frac{4}{5} = \frac{8}{9} \times \frac{5}{4} = \frac{10}{9} = 1\frac{1}{9}$, *Ans.*

3. Multiply £2 13*s.* 4*d.* by $3\frac{1}{3}$, and divide £4 15*s.* by $3\frac{1}{3}$.
£2 13*s.* 4*d.* $= 2 + \frac{13}{20} + \frac{4}{1.2}$ of $\frac{1}{20} = 2\frac{2}{3} = \frac{8}{3}$, and $\frac{8}{3} \times 3\frac{1}{3}$
$= \frac{8}{3} \times \frac{7}{3} = \frac{56}{9} = 2\frac{8}{9} = 9\frac{1}{3} =$ £9 6*s.* 8*d.*
£4 15*s.* $\div 3\frac{1}{3} = 4\frac{3}{4} \div 3\frac{1}{3} = \frac{19}{4} \div \frac{10}{3} = \frac{19}{4} \times \frac{3}{10} = \frac{57}{40} =$
$1\frac{17}{40} =$ £1 8*s.* 6*d.*

Note.—In the multiplication of mixed numbers, it is often less laborious to perform the multiplication of each part separately, and collect their sum, as in the margin, than to reduce the mixed numbers to improper fractions, and reduce their product back again to a mixed number.

Multiply	$45\frac{3}{4}$
By	$17\frac{2}{3}$
45×7	$= 315$
45×1 ten	$= 45.$
$\frac{2}{3} \times \frac{3}{4} =$	$\ldots\frac{1}{2}$
$45 \times \frac{2}{3} =$	$.30$
$17 \times \frac{3}{4} =$	$.12\frac{3}{4}$
Product	$808\frac{1}{4}$

SECT. VII. *Decimal Fractions.*

THE embarrassment and loss of time occasioned by the computation of quantities expressed in vulgar or ordinary fractions, have inspired the idea of *fixing* the denominator so as to know what it is without actually expressing it. Hence originate two dispositions of numbers, *decimal fractions* and *complex numbers*. Of the latter, such, for example, as when we express lineal measures in yards, in feet (or thirds of a yard), and inches (or twelfths of a foot), we shall treat in the following section, and shall here confine ourselves to the former.

Decimal fractions, or substantively, *decimals*, are fractions always having some power of *ten* for their denominator; but for the sake of brevity only the numerator is expressed, being written as a *whole* number with a dot placed on its left hand, which dot determines the value of the denominator, the number of cyphers in the denominator being always equal to the number of figures to the *right* of the dot, or as it is termed decimal point; if the number of significant figures in the numerator is not sufficient, cyphers are added to the left hand. It is evident that the values of decimals decrease in the same tenfold proportion from the point towards the *right* hand, as those of integers increase towards the left:—thus

The vulgar	fraction,	$\frac{1}{10}$	is written	·1
,,	,,	$\frac{1}{100}$	,,	·01
,,	,,	$\frac{1}{1000}$	,,	·001
,,	,,	$\frac{1}{10000}$	,,	·0001
,,	,,	$\frac{7}{10}$	,,	·7
,,	,,	$\frac{43}{100}$	,,	·43
,,	,,	$\frac{125}{10000}$	,,	·0125

The vulgar fraction,		$7\frac{3}{10}$	is written	7·3
„	„	$42\frac{85}{100}$	„	42·85
„	„	$57\frac{217}{1000}$	„	57·217
		&c.		&c.

The value of a decimal fraction is not altered by cyphers on the right hand: for ·500, or $\frac{500}{1000}$, is in value the same as $\frac{5}{10}$, or ·5, that is $\frac{1}{2}$.

When decimals terminate after a certain number of figures, they are called *finite*, as ·125 = $\frac{125}{1000}$ = $\frac{1}{8}$, ·958 = $\frac{958}{1000}$ = $\frac{479}{500}$.

When one or more figures in the decimal become repeated, it is called a *repeating* or *circulating* decimal; and a dot is placed over the figure to be repeated, if only one, or if more than one, over the first and last figures; thus:—

·333333, &c.	= $\frac{1}{3}$ is written	$\dot{3}$
·666666, &c.	= $\frac{2}{3}$ „	$\dot{6}$
·428571428571, &c.	= $\frac{3}{7}$ „	$\dot{4}28571\dot{4}$
·29504504, &c.	= $\frac{131}{444}$ „	$\cdot29\dot{5}0\dot{4}$

When the circulating portion of the decimal is preceded by other figures which do not circulate, (as in the last example,) it is called a *mixed circulate*.

Rules for the management of this latter kind of decimals are given by several authors; but, in general, it is more simple and commodious to perform the requisite operations by means of the equivalent vulgar fractions, the method of obtaining which is given in Case III.

REDUCTION OF DECIMALS.

Reduction of Decimals is a rule by which the known parts of given integers are converted into equivalent decimals, and *vice versâ*.

CASE I. — *To reduce a given vulgar fraction to an equivalent decimal.*

Rule.—Annex as many cyphers to the numerator as may be necessary, then divide by the denominator, and point off in the quotient as many places of decimals as the number of cyphers added to the numerator; if the quotient does not contain so many figures, the deficiency must be made up by cyphers placed on the *left* hand.

Examples.

1. Reduce $\frac{1}{2}$, $\frac{3}{4}$, $\frac{7}{16}$, $\frac{6}{64}$, to equivalent decimals.

2) 1·0

·5 decimal = $\frac{1}{2}$;

4) 3·00

·75 decimal = $\frac{3}{4}$:

16 { 4)7·0000 / 4)1·7500 }

·4375 decimal = $\frac{7}{16}$;

64 { 8)6·000 / 8) ·75000 }

·09375 decimal = $\frac{6}{64}$

2. Reduce $\frac{4}{27}$ and $\frac{11}{63}$ to equivalent decimals.

$$27 \left\{ \begin{array}{l} 3)4\cdot000000 \\ \hline 9)1\cdot333333 \\ \hline \cdot148148, \text{ \&c.} = \end{array} \right. \qquad 63 \left\{ \begin{array}{l} 7)11\cdot0000000 \\ \hline 9)\ 1\cdot5714285714285 \\ \hline \cdot1746031746031, \text{ \&c.} = \end{array} \right.$$

$\dot{\cdot}1\dot{4}8$ decimal $= \frac{4}{27}$. $\qquad$ $\dot{\cdot}\dot{1}7460\dot{3}$ decimal $= \frac{11}{63}$.

These two are evidently circulating decimals, in the former of which the figures 148 become indefinitely repeated, in the latter the figures 174603.

3. Reduce 14*s.* 6*d.* to the decimal of a pound.

First, 14*s.* 6*d.* $= \frac{14}{20} + \frac{1}{2}$ of $\frac{1}{20} = \frac{28}{40} + \frac{1}{40} = \frac{29}{40}$.

Then $\frac{29}{40} = \frac{7\cdot25}{10} = \cdot725$, the decimal required.

4. Reduce $\frac{4}{5}\frac{4}{7}$ to its equivalent decimal.

57) 44·000000 (·77192, &c. decimal $= \frac{44}{57}$.

399

410
399

110
57

530
513

170
114

56

Note.—The above fraction is $= \frac{4}{5} \times \frac{11}{19}$, of which the two denominators are both *prime numbers*, (that is, divisible by no other number than unity,) the entire equivalent decimal is a circulate of 18 places, *i. e.* one less than the last prime .... ·771929824561403508, 7719, &c. over again *ad infinitum* *.

* There are many curious properties of fractions whose denominators are prime numbers, one of which may be here shown in reference to fractions having the denominator 7. The circulating figures of the equivalent decimals are precisely the same, for $\frac{1}{7}$, $\frac{2}{7}$, &c., and in the same order: the circulate merely commences at a different place for each numerator.

$\frac{1}{7} = \dot{\cdot}142857\dot{}$, &c.
$\frac{2}{7} = \dot{\cdot}285714\dot{}$, &c.
$\frac{3}{7} = \dot{\cdot}428571\dot{}$, &c.
$\frac{4}{7} = \dot{\cdot}571428\dot{}$, &c.
$\frac{5}{7} = \dot{\cdot}714285\dot{}$, &c.
$\frac{6}{7} = \dot{\cdot}857142\dot{}$, &c.

CASE II.—*Any decimal being given to find its equivalent vulgar fraction; or to express its value by integers of lower denominations.*

Rule.—When the equivalent vulgar fraction is required, place under the decimal as a denominator a unit with as many cyphers as there are figures in the proposed decimal; and let the fraction so constituted be reduced to its lowest terms.

Or, if the value of the decimal be required in lower denominations, multiply the given decimal by the number of parts in the next less denomination contained in its integer; and point off, from right to left, as many figures of the product as there were places in the given decimal. Multiply the decimal last pointed off by the value of its integer, in the next inferior order, pointing off the same number of decimals as before: and thus continue the process to the lowest integer, or until the decimals cut off become all cyphers; then will the several numbers on the left of the separating points, together with the remaining decimal, if any, express the required value of the given decimal.

Examples.

1. Find the vulgar fractions equivalent to ·25 and ·375.
·25 = $\frac{25}{100}$ = $\frac{1}{4}$; and ·375 = $\frac{375}{1000}$ = $\frac{3}{8}$, *Answers.*

2. Find the value in shillings, &c. of ·528125 of a pound.

·528125	
20	
10·562500	*Ans.* 10*s.* 6$\frac{3}{4}$*d.*
12	
6·7500 = 6$\frac{3}{4}$	

3. Find the value of ·74375 of an acre.

·74375	
4	
2·97500	*Ans.* 2 roods 39 perches.
40	
39·000	

CASE III.—*To reduce a circulating decimal to its equivalent vulgar fraction.*

Rule.—Take the figures in the decimal and place them as a whole number for the numerator, and under them for a denominator as many 9's as there are figures in the circulate, and the fraction thus formed will be equivalent in value to the given decimal.

If the decimal is a *mixed* circulate, subtract the finite part (or the figures which are not repeated) from the whole mixed circulate, (both

considered as whole numbers,) for the numerator; and for the denominator, take as many 9's as there are figures in the circulating portion of the decimal, with as many cyphers to the right as there are figures in the finite portion of the same.

Examples.

The circulate $\dot{3} = \frac{3}{9} = \frac{1}{3}$

„ $\dot{0}\dot{6} = \frac{6}{99} = \frac{2}{33}$

„ $\dot{5}4\dot{9} = \frac{549}{999} = \frac{61}{111}$

„ $7\cdot6\dot{3}0 = 7\frac{630}{999} = 7\frac{70}{111}$

The *mixed* circulate $\cdot64\dot{0}\dot{9} = \frac{6409-64}{9900} = \frac{1}{2}\frac{41}{20}$

„ $5\cdot29\dot{5}0\dot{4} = 5\frac{29504-29}{999000} = 5\frac{13}{44}\frac{1}{4}$

ADDITION AND SUBTRACTION OF DECIMALS.

These operations are performed precisely as in whole numbers, the figures being so arranged that units stand under units, tens under tens, &c., or, (which amounts to the same thing,) so that the decimal points stand under one another. Thus,

Add together	421·75 32·8165 ·0027 11·	From	2486·173 Take 14·56789
		Remains	2471·60511
Sum	465·5692	Proof	2486·17300

MULTIPLICATION AND DIVISION OF DECIMALS.

Here, again, the operations are performed as in integers: Then, in multiplication, let the product contain as many decimal places as there are in both the multiplier and multiplicand, cyphers being prefixed, if necessary, to make that number; and, in division, point off as many decimals in the quotient as the number in the dividend (including the cyphers supplied, if there be any) exceeds that in the divisor.

Examples.

1. Multiply 43·7 by 3·91, and 2·4542 by ·0053.

43·7 39·1 437 3933 1311 170·867	=Here 43·7 × 3·91 $\frac{437}{10} \times \frac{391}{100} = \frac{170867}{1000} = 170\frac{867}{1000}$, as in the decimal operation.	2·4542 ·0053 73626 122710 ·01300726	Here one cypher is prefixed to make the requisite number of decimals in the product.

2. Divide 172·8 by ·144, and 192 by 5·423.

·144) 172·8 (1200· *quotient.*	5·423) 192·000 (35·40475
144	16269
288	29310
288	27115
···00	21950
	21692
	25800
	21692
	41080
	37961
	31190
	27115
	4075

In the first of these examples, the two cyphers brought down, together with the decimal 8, make the number of decimals in the dividend the same as in the divisor, therefore the quotient is composed entirely of integers. In the second example, 3, the number of decimal places in the divisor, taken from 8, the number in the dividend (including those brought down), leave 5 for the decimal places in the quotient.

SECT. VIII. *Complex Fractions used in the Arts and Commerce.*

IN the arts and in commerce, it is customary to assume a series of units having a constant relation to each other, so that the units of one denomination become fractions of another. One farthing, for example, is $\frac{1}{4}$ of a penny, 1 penny $\frac{1}{12}$ of a shilling, 1 shilling $\frac{1}{20}$ of a pound, or $\frac{1}{21}$ of a guinea. One lineal inch, again, is $\frac{1}{12}$ of a foot, 1 foot $\frac{1}{3}$ of a yard; and so on, according to the relations expressed in the tables in Chap. V. on Mensuration, Sect. I. The arithmetical operations on complex numbers of these kinds are usually effected by simpler rules than those which apply to vulgar fractions generally; of which it will, therefore, be proper here to specify a few.

REDUCTION.

Here we have two general cases:

CASE I.— *When the numbers are to be reduced from a higher denomination to a lower:—*

Rule.—Multiply the number in the higher denomination by as many of the next lower as make an integer, or one, in that higher, and set down the product. To this product add the number, if any, which was in this lower denomination before; and multiply the sum by as many of the next lower denomination as make an integer in the present one. Proceed in the same manner through all the denominations to the lowest, and the number last found will be the value

of all the numbers which were in the higher denominations taken together.

CASE II.—*When the numbers are to be reduced from a lower denomination to a higher:—*

Rule.—Divide the given number by as many of that denomination as make one of the next higher, and set down what remains. Divide the quotient by as many of this as make one of the next higher denomination, and set down what remains in like manner as before. Proceed in the same manner through all the denominations to the highest; and the quotient last found, together with the several remainders, if any, will be of the same value as the first number proposed.

The method of proof is to work the question back again.

Examples.

1. Reduce £14 to shillings, pence, and farthings; and 24316 farthings into pounds, &c.

14		4) 24316
20		
280	shillings	12) 6079 pence
12		20) 506 7
3360	pence	£25 6*s.* 7*d.*
4		
13440	farthings	

2. Reduce 22 Ac. 3 R. 24 P. into perches; and 52187 perches into acres.

a.	*r.*	*p.*		
22	3	24		40) 52187
4				
91	roods			4) 1304 27
40				Ac. 326 0 R. 27 P.
3664	perches.			

ADDITION.

Rule.—Place the quantities to be added so that those of the same denomination may be all under each other. Then add up the numbers in the right hand column, and divide their sum by the number of units of that denomination contained in an unit of the next denomination to the left, write the remainder (if any) at the foot of the first column, and carry the quotient on to the addition of the second column, and thus proceed until all have been added up.

Examples.

	£	s.	d.
Add	368	10	3
	257	10	5
	88	11	4½
	33	10	0
	12	13	5
	8	8	8½
Sum	769	4	2

	lb.	oz.	dwt.	gr.
Add	14	6	12	13
	17	5	3	12
	15	0	9	16
	2	7	15	20
	13	2	10	19
	4	1	5	21
Sum	66	11	18	5

	lb.	oz.	dwt.	gr.
Add	10	8	11	17
	42	5	16	12
	12	2	14	18
	51	6	0	22
	24	9	17	17
	29	4	18	22
Sum	171	2	0	12

SUBTRACTION.

Rule.—Write the smaller number beneath the greater, taking care to keep the same denominations under each other. Then begin at the right hand and subtract the lower number of each denomination from the upper, writing the remainder underneath.

When the lower number of any denomination is greater than the upper one of the same, add to the latter as many units as are contained in one unit of the next greater denomination, always taking care, when such has been done, to add one to the next lower number to the left.

Examples.

	£.	s.	d.
From	16	12	8¾
Take	10	11	6¼
Rem.	6	1	2½

	£.	s.	d.
From	21	13	4¾
Take	18	9	8½
Rem.	3	3	8¼

	lb.	oz.	dwt.	gr.
From	18	9	10	8
Take	9	10	15	20
Rem.	8	10	14	12

MULTIPLICATION.

Rule.—Place the multiplier under the lowest denomination of the multiplicand.—Multiply the number in the lowest denomination by the multiplier, and find how many integers of the next higher denomination are contained in the product, and write down what remains.—Carry the integers, thus found, to the product of the next higher denomination, with which proceed as before; and so on, through all the denominations to the highest; and this product, together with the several remainders, taken as one number, will be the whole amount required.

If the multiplier exceed 12, multiply successively by its component parts; as in the following examples:—

Examples.

1. Multiply £ s. d. 4 17 6¼ by 441, and a. r. p. 3 2 14 by 531.

£	s.	d.		a.	r.	p.	
						10	
4	17	6¼		35	3	20	
		9	9 × 7 × 7 = 441			10	
43	17	10¼		358	3	0	for 100
		7				5	
307	4	11¼		1793	3	0	for 500
		7		107	2	20	3 times 10
				3	2	14	1 top line.
Ans. £2150	14	10¼		*Ans.* 1904	3	34	

DIVISION.

Rule.—Place the divisor and dividend as in simple division.—Begin at the left hand, or highest denomination of the dividend, which divide by the divisor, and write down the quotient.—If there be any remainder after this division, find how many integers of the next lower denomination it is equal to, and add them to the number, if any, which stands in that denomination.—Divide this number, so found, by the divisor, and write the quotient under its proper denomination. —Proceed in the same manner through all the denominations to the lowest, and the whole quotient, thus found, will be the answer required.

2. Divide £ s. d. 521 18 6 by 432.

$432 = 12 \times 12 \times 3.$

Therefore, by short division:

12)521 18 6

12)43 9 10½

3)3 12 5¼ + ½ a farthing.

Quotient £1 4 1¾ + ⅝ of a farthing.

By long division:—

£	s.	d.	£	s.	d.	
432)521	18	6	(1	4	$1\frac{3}{4}$	+ $\frac{5}{6}$ of a farthing.
432						
89						
20						
432)1798(4						
1728						
70						
12						
432)846(1						
432						
414						
4						
432)1656(3						$\frac{360}{432} = \frac{10}{12} = \frac{5}{6}$.
1296						
360						

DUODECIMALS.

Fractions whose denominators are multiples of 12, as 144, 1728, &c., are called *duodecimals;* and the division and sub-division of the integer are *understood* without being expressed, as in *decimals*. The method of operating by this class of fractions is principally in use among artificers, in computing the contents of work, of which the dimensions are taken in *feet*, *inches*, and *twelfths* of an inch.

Rule.—Set down the two dimensions to be multiplied together, one under the other, so that feet shall stand under feet, inches under inches, &c. Multiply each term in the multiplicand, beginning at the lowest, by the feet in the multiplier, and set the result of each immediately under its corresponding term, observing to carry 1 for every 12, from the inches to the feet. In like manner, multiply all the multiplicand by the inches of the multiplier, and then by the twelfth parts, setting the result of each term *one* place removed to the right hand when the multiplier is inches, and *two* places when the parts become the multiplier. The sum of these successive products will be the answer required.

Or, instead of multiplying by the inches, &c., take such parts of the multiplicand as these are of a foot.

Examples.

1. Multiply 12 *ft.* $7\frac{1}{3}$ *ins.* by 7 *ft.* $3\frac{3}{4}$ *ins.*

ft.	*ins.*	′			
12	7	4			
7	3	9			
88	3	4			
3	1	10	0		
	9	5	6	0	
92	2	7	6	0	

or,

ins.
3 = $\frac{1}{4}$ of 1 ft.
9′ = $\frac{1}{4}$ of 3 ins.

ft.	*ins.*	′
12	7	4
		7 ft.
88	3	4
3	1	10
	9	$5\frac{1}{2}$
92	2	$7\frac{1}{2}$

2. Multiply 35 *ft.* $4\frac{1}{2}$ *ins.* into 12 *ft.* $3\frac{1}{3}$ *ins.*

ft.	*ins.*	′			
35	4	6			
12	3	4			
424	6	0			
8	10	1	6		
	11	9	6	0	
434	3	11	0	0	

or,

ins.
3 = $\frac{1}{4}$ of 1 ft.
4′ = $\frac{1}{9}$ of 3 ins.

ft.	*ins.*	′
35	4	6
		12 ft.
424	6	0
8	10	$1\frac{1}{3}$
	11	$9\frac{1}{2}$
434	3	11

The feet in the answers are square feet, but the numbers standing in the place of inches are not square inches, but twelfth parts of square feet, each part being equal to 12 square inches; and the numbers in the *third* place being twelfth parts of these are square inches: in like manner, if the operation be carried further, every successive place will be a twelfth part of that preceding it.

SECT. IX. *Powers and Roots.*

A *power* is a quantity produced by multiplying any given number, called the *root* or *radix*, a certain number of times successively by itself. The operation of thus raising powers is called *involution*. Thus, if 3 is the root,

3 = 3 is the 1st power of 3.
3 × 3 = 3^2 = 9, is the 2d power, or *square* of 3.
3 × 3 × 3 = 3^3 = 27, is the 3d power, or *cube* of 3.
3 × 3 × 3 × 3 = 3^4 = 81, do. 4th power, or *biquadrate* of 3,
&c. &c. &c.

Table of the first Nine Powers of the first Nine Numbers.

1st	2d	3d	4th	5th	6th	7th	8th	9th
1	1	1	1	1	1	1	1	1
2	4	8	16	32	64	128	256	512
3	9	27	81	243	729	2187	6561	19683
4	16	64	256	1024	4096	16384	65536	262144
5	25	125	625	3125	15625	78125	390625	1953125
6	36	216	1296	7776	46656	279936	1679616	10077696
7	49	343	2401	16807	117649	823543	5764801	40353607
8	64	512	4096	32768	262144	2097152	16777216	134217728
9	81	729	6561	59049	531441	4782969	43046721	387420489

So again, $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$ = square of $\frac{2}{3}$; $\frac{4}{9} \times \frac{2}{3} = \frac{8}{27}$ = cube of $\frac{2}{3}$; $\frac{8}{27} \times \frac{2}{3} = \frac{16}{81}$, biquadrate of $\frac{2}{3}$; and so of others. Where it is evident, that while the powers of integers become successively larger and larger, the powers of pure or proper fractions become successively smaller and smaller.

EVOLUTION.

Evolution, or the extraction of roots, is the reverse of involution.

Any power of a given number may be found exactly; but we cannot, conversely, find every root of a given number exactly *. Thus, we know the *square root* of 4 exactly, being 2; but we cannot assign exactly the *cube root* of 4. So, again, though we know the *cube root* of 8, viz. 2, we cannot exactly assign the *square root* of 8. But, of 64 we can assign both the square root and the cube root, the former being 8, the latter 4.

By means of decimals we can in all cases *approximate* to the root to any proposed degree of exactness.

Those roots which only approximate are called surd roots, or *surds*, or *irrational* numbers; as $\sqrt{2}$, $\sqrt[3]{5}$, $\sqrt[4]{9}$, &c., while those which can be found exactly are called *rational;* as $\sqrt{9} = 3$, $\sqrt[3]{125} = 5$, $\sqrt[4]{16} = 2$.

1.—*To extract the square root.*

Rule.—Divide the given number into periods of two figures each, by setting a point *over the place of units*, another over the place of

* For the method of extracting roots by logarithms, see page 47.

hundreds, and so on over every second figure, both to the left hand in integers, and to the right hand in decimals. Find the greatest square in the first period on the left hand, and set its root on the right hand of the given number, after the manner of a quotient figure in division. Subtract the square thus found from the said period, and to the remainder annex the two figures of the next following period, for a dividend. Double the root above-mentioned for a divisor; and find how often it is contained in the said dividend, exclusive of its right-hand figure; and set that quotient figure both in the quotient and divisor. Multiply the whole augmented divisor by this last quotient figure, and subtract the product from the said dividend, bringing down to it the next period of the given number, for a new dividend. Repeat the same process, viz., find another new divisor, by doubling all the figures now found in the root; from which, and the last dividend, find the next figure of the root as before; and so on through all the periods, to the last*.

Note.—The best way of doubling the root, to form the new divisors, is by adding the last figure always to the last divisor, as appears in the following Examples.—Also, after the figures belonging to the given number are all exhausted, the operation may be continued into decimals at pleasure, by adding any number of periods of cyphers, two in each period.

Examples.

1. Find the square root of 17·3056.

$$
\begin{array}{rl|l}
 & \dot{1}\dot{7}\cdot\dot{3}0\dot{5}\dot{6} & (4\cdot16 \text{ the root: in which the number of} \\
 & 16 & \text{decimal places is the same as the} \\
\hline
81 & 130 & \text{number of decimal } \textit{periods} \text{ into which} \\
1 & 81 & \text{the given number was divided.} \\
\hline
826 & 4956 & \\
6 & 4956 & \\
\hline
 & \ldots\ldots &
\end{array}
$$

* The reason for separating the figures of the dividend into periods or portions of two places each, is, that the square of any single figure never consists of more than two places; the square of a number of two figures of not more than four places, and so on. So that there will be as many figures in the root as the given number contains periods so divided or parted off.

And the reason of the several steps in the operation appears from the algebraic form of the square of any number of terms, whether two or three, or more. Thus, $35^2 = 30^2 + 2 \cdot 30 \cdot 5 + 5^2$, or generally $(a + b)^2 = a^2 + 2ab + b^2 = a^2 + (2a + b)b$, the square of two terms; where it appears that a is the first term of the root, and b the second term; also a the first divisor, and the new divisor is $2a + b$, or double the first term increased by the second. And hence the manner of extraction is as in the rule.

2. Find the square root of 2, to six decimals.

2(1·414213 root.
1

24 |	100
4 |	96
281 |	400
1 |	281
2824 |	11900
4 |	11296
28282 |	60400
2 |	56564
282841 |	383600
1 |	282841
2828423 |	10075900
	8485269
	1590631

3. Find the square root of $\frac{5}{12}$.

$\frac{5}{12}$ = ·416666666, &c.

$\dot{0}\cdot4\dot{1}6\dot{6}6\dot{6}$(0·64549, &c.
36

124 |	566
4 |	496
1285 |	7066
5 |	6425
12904 |	64166
4 |	51616
129089 |	1255066
	1161801
	93265

Note.—In cases where the square roots of all the integers up to 1000 are tabulated, such an example as the above may be done more easily by a little reduction. Thus $\sqrt{\frac{5}{12}} = \sqrt{(\frac{5}{12} \times \frac{12}{12})} = \sqrt{\frac{60}{144}} = \frac{1}{12}\sqrt{60} = \frac{7\cdot7459667}{12}$ = ·645497, &c.

2.—*To extract cube and higher roots.*

The rules usually given in books of arithmetic for the cube and higher roots, are very tedious in practice: on which account it is advisable to work either by means of approximating rules, or by means of logarithms*. The latter is, generally speaking, the best method. We shall merely present here Dr. Hutton's approximating rule for the *cube root.*

Rule.—By trials take the nearest rational cube to the given number, whether it be greater or less, and call it the assumed cube.

Then say, by the Rule of Three, as the sum of the given number and double the assumed cube, is to the sum of the assumed cube and double the given number, so is the root of the assumed cube, to the root required, nearly. Or, as the first sum is to the difference of the given and assumed cube, so is the assumed root, to the difference of the roots, nearly.

* See page 47.

Again, by using, in like manner, the cube of the root last found as a new assumed cube, another root will be obtained still nearer. And so on as far as we please; using always the cube of the last found root, for the assumed cube.

Example.

To find the cube root of 21035·8.

Here we soon find that the root lies between 20 and 30, and then between 27 and 28. Taking therefore 27, its cube is 19683, which is the assumed cube. Then

19683		21035·8		
2		2		
39366		42071·6		
21035·8		19683		
As 60401·8	:	61754·6	:: 27	: 27·6047
		27		
		4322822		
		1235092		

60401·8)1667374·2(27·6047 the root nearly.

459338

36525

284

42

Again, assuming 27·6 and working as before, the root will be found to be 27·60491.

SECT. X. *Proportion.*

Two magnitudes may be compared under two different points of view, that is to say, either by inquiring what is the *excess* of one above the other, or *how often* one is contained in the other. The result of this comparison is obtained by *subtraction* in the first case, by *division* in the second, the quotient resulting being termed the *ratio* of the two numbers. Thus 3 may be regarded as the ratio of 12 to 4, since $\frac{12}{4}$ or 3 is the quotient of the numbers 12 and 4.

The first of two numbers constituting a ratio is called the *antecedent*, the second the *consequent*.

The *difference* of two numbers is not changed by *adding* one and the same number to each, or by *subtracting* the same number from each.

Thus $12 - 5 = (12 + 2) - (5 + 2) = 14 - 7 = (12 - 2) - (5 - 2) = 10 - 3$.

In like manner, a *ratio* is not changed by either *multiplying* both its terms, or *dividing* both its terms by the same number.

Thus $\frac{14}{4} = (\frac{14}{4} \times \frac{3}{3}) = \frac{42}{12} = (\frac{14}{4} \div \frac{2}{2}) = \frac{7}{2}$.

Equality of differences, or *equidifference*, is a term used to indicate that the difference between two numbers is the same as the difference between two other numbers. Such, for example, as $12 - 9 = 8 - 5 = 7 - 4$.

Equality of ratios, or *proportion*, is similarly employed to denote that the ratio of two numbers is the same as that between two others. Thus 20 and 10, 14 and 7, have 2 for the measure of the ratio: we have therefore a proportion between 20 and 10, 14 and 7, which is thus expressed, 20 : 10 : : 14 : 7, and thus read 20 *are to* 10 *as* 14 *are to* 7. The same proportion may also be represented thus, $\frac{20}{10} = \frac{14}{7}$. Though, by whatever notation it be represented, it is best to read or enumerate it as above. It is true, however, that in all cases when two fractions are equal, the numerator of one of them is to its denominator, as the numerator of the other is to its denominator.

In a proportion, as 20 : 10 : : 14 : 7, the second and third terms are called the *means*, the first and fourth the *extremes*.

When the two means are equal, the proportion is said to be *continued*. Thus 3 : 6 : : 6 : 12 are in continued proportion. This is usually expressed thus $\div\div$ 3 : 6 : 12; and the second term is called the *mean proportional*.

In the case of *equidifference*, as $12 - 9 = 7 - 4$, the *sum* of the extremes $(12 + 4)$ is equal to that of the means $(9 + 7)$. In like manner in a proportion, as 20 : 10 : : 14 : 7, the *product* of the extremes (20×7) is equal to that of the means (10×14). The converse of this likewise obtains, that if $20 \times 7 = 10 \times 14$, then 20 : 10 : : 14 : 7. Hence,

1. If there be four numbers, 5, 3, 15, 9, such that the products 5×9 and 3×15 are found equal, we may infer the equality of their ratios, or the proportion $\frac{5}{3} = \frac{15}{9}$, or 5 : 3 : : 15 : 9. So that a proportion may always be constituted with the factors of two equal products.

2. If the means are equal, their product becomes a square; therefore the mean proportional between two numbers is equal to the square root of their product. Thus, between 4 and 9 the mean proportional is $\sqrt{(4 \times 9)} = 6$.

3. If a proportion contain an unknown term, such, for example, as 5 : 3 : : 15 : the unknown quantity; since 5 times the unknown quantity must be equal to 3×15 or 45, that quantity itself is equal to $45 \div 5$ or 9. Or generally, one of the extremes is equal to the product of the means divided by the other extreme; and one of the means is equal to the product of the extremes divided by the other mean.

4. We may, without affecting the correctness of a proportion, subject the several terms which compose it to all the changes which can be made, while the product of the extremes remains equal to that of the means. Thus, for 5 : 3 : : 15 : 9, which gives $5 \times 9 = 3 \times 15$, we may

I. Change the places of the means without changing those of the extremes, or change the places of the extremes without changing those of the means: this is denoted by the term *alternando*.

Thus, $5 : 3 :: 15 : 9$
become $5 : 15 :: 3 : 9$
or $9 : 3 :: 15 : 5$
or $9 : 15 :: 3 : 5$

II. Put the extremes in the places of the means; this is called *invertendo;* as

$$3 : 5 :: 9 : 15$$

III. Multiply or divide the two antecedents or the two consequents by the same number.

It also appears, with regard to proportions, that the sum or the difference of the antecedents is to that of the consequents, as either antecedent is to its consequent.

And, that the sum of the antecedents is to their difference, as the sum of the consequents is to their difference.

Thus $\frac{5 \pm 15}{3 \pm 9} = \frac{5}{3} = \frac{15}{9}$, and $\frac{5 + 15}{3 + 9} = \frac{5 \sim 15}{3 \sim 9}$.

If there be a series of equal ratios represented by $\frac{6}{3} = \frac{10}{5} = \frac{14}{7} = \frac{30}{15}$, we shall have $\frac{6 + 10 + 14 + 30}{3 + 5 + 7 + 15} = \frac{60}{30} = \frac{6}{3} = \frac{10}{5}$ = &c.

Therefore, in a series of equal ratios, the sum of the antecedents is to the sum of the consequents, as any one antecedent is to its consequent.

If there be two proportions, as $30 : 15 :: 6 : 3$, and $2 : 3 :: 4 : 6$, then multiplying them term by term, we shall have $30 \times 2 : 15 \times 3 :: 6 \times 4 : 3 \times 6$, which is evidently a proportion, because $30 \times 2 \times 3 \times 6 = 15 \times 3 \times 6 \times 4 = 1080$.

Thus, also, any powers of quantities in proportion are in proportion; and conversely of the roots. Thus,

If $2 : 3 :: 6 : 9$ then $2^2 : 3^2 :: 6^2 : 9^2$
$2 : 3 :: 6 : 9$ „ $\sqrt{2} : \sqrt{3} :: \sqrt{6} : \sqrt{9}$
$2 : 3 :: 6 : 9$ „ $2^3 : 3^3 :: 6^3 : 9^3$

RULE OF THREE.

When the elements of a problem may be so disposed that they form a proportion of which the quantity sought is the last term; that is, when the first bears the same proportion to the second as the third does to the fourth or unknown quantity, its value may be easily determined, and the problem is said to belong to the *Golden Rule*, or *Rule of Three.*

Rule.—Of the three given terms set down that which is of the same kind as the number sought, then consider from the nature of the problem whether this number will be greater or less than the term so put down; if *greater*, write on its *right* hand the greater, if *less*, the lesser, of the two remaining terms, and place the other on its *left* hand.

Then multiply the second and third terms together, and divide their product by the first; the quotient will be the number sought. It

must be observed, that the first and third terms must be reduced to the same denomination; and if the second term is a compound number, it should be reduced to the lowest name mentioned; unless the third term is a composite number, in which case it is generally better to multiply the second term (without any previous reduction) by the component parts of the third, as in compound multiplication, after which divide the compound product by the first term, or by its factors. The answer will be of the same denomination as the second term.

Examples.

1. If 3 gallons of brandy cost 19*s.*, what will 126 gallons cost at the same rate?

gal. s. gal.
3 : 19 :: 126 : ?
19
1134
126
3)2394(798 shillings
21
or 39*l.* 18*s.* *Ans.*
29
27
24
24

2. How much brandy may be bought for 39*l.* 18*s.*, at the rate of 3 gallons for 19 shillings?

s. gal. £ s.
19 : 3 :: 39 18 : ?
20
798*s.*
3
gal.
19)2394(126 *Ans.*
19
49
38
114
114

3. If 21 yards of cloth cost 24*l.* 10*s.*, what will 160 yards cost?

yds. £ s. yds.
Here, 21 : 24 10 :: 160 : ?
4
4 × 4 × 10 = 160
98 0
4
392 0
10
(3)3920 0
21
(7)1306 13 4
£186 13*s.* 4*d.* *Ans.*

4. If by selling cloth at 1*l.* 2*s.* per yard, 10 per cent. is gained, what would be gained if it had been sold at 1*l.* 5*s.* per yard?

£ s. £ s.
Here, 1 2 : 110 :: 1 5 : ?
20 20
22 25
110
2)2750
22
11)1375
Amount £125
Deduct 100
Gain per cent. £25

5. What is the simple interest of 560*l.* for 5 years, at 4 per cent. per annum?

£ £ £
Here, 100 : 4 :: 560
4
100) 2240
£22·4
20
8·0

Interest for 1 year, £22 8*s.*

y. £ *s.* *y.*
Then 1 : 22 8 :: 5
5

£112 0 *Answer.*

6. If 100 workmen can finish a piece of work in 12 days, how many men working equally hard would have finished it in 3 days?

d. *w.* *d.*
12 : 100 :: 3 : ?
12
3) 1200
Answer 400 *workmen.*

A distinct rule is usually given for the working of problems in *Compound Proportion;* but they may generally be solved with greater mental facility by means of separate statings. Thus:—

7. If a person travel 300 miles in 10 days of 12 hours each, in how many days of 16 hours each may he travel 600 miles?

First, if the days were of the same length, it would be, by simple proportion,

m. *d.* *m.*
As 300 : 10 :: 600 : 20 days.

But these would be days of 12 hours each, instead of 16, of which fewer will be required. Hence, again, by simple proportion,

h. *d.* *h.* *d.*
As 12 : 20 :: 16 : 15

So that the answer is 15 days.

8. If a family of 9 persons spend 480*l.* in 8 months, how much will serve a family (living upon the same scale) of 24 persons 16 months?

p. £ *p.*
First, as 9 : 480 :: 24 : £1280.

But this would only be the expense for 8 months. Hence, again,

m. £ *m.* £
As 8 : 1280 :: 16 : 2560, the expense of the 24 persons for 16 months.

Note.—The Rule of Three receives its application in questions of *Interest*, *Discount*, *Fellowship*, *Barter*, &c.

DETERMINATION OF RATIOS.

To find the ratio of two numbers, A *and* B, *to each other.*

Rule 1.—Divide *B* by *A*, then, *unity* or 1 : *the quotient*, will be the first ratio; and if the quotient be an integral number, the ratio will be expressed in the least terms possible. If, however, it contain a fraction, proceed as follows:—first write, 1 : *integral portion of the quotient* + *the numerator of the fraction;* then, add 1 to the integral, and subtract the numerator of the fraction from the denominator, and write under the former, 1 : *integral portion* + 1 — *the ~ of the numerator and denominator.*

Then, if the numerator and this difference be nearly equal, add the two ratios together; but if the numerator and difference are not nearly equal, divide the greater by the less; then, multiply that ratio ending with either the numerator or the difference, whichever was the divisor, by the integral portion of this last quotient, and add to it the other ratio. With the three ratios thus obtained proceed in the same manner, from whichever two have the numbers appended by the signs + and —, nearest equal, to obtain a fourth ratio, and thus continue, until this appended number has been eliminated.

Note.—The ratios thus found will be alternately greater and less than the true one, but continually approaching nearer to it. And that is the nearest in small numbers, which is immediately followed by much larger numbers: the excess or defect of any one is equal to a fraction, having the number appended to that ratio by the sign + or — for its numerator, and the denominator belonging to the first quotient, for its denominator.

Example 1.

To find the ratio of 10000 (*A*) to 7854 (*B*) in small numbers.

$$7854 \div 10000 = 0\frac{7854}{10000}$$

Then,

	1st	1 :	0+7854	or ratio of 1 to the integer and num.
I.	2nd	1 :	1—2146	or integer+1—the ~ of num. and den.
		2146) 7854 (3		
	3rd	3 :	3—6438	or 2nd ratio × by 3.
II.	4th	4 :	3+1416	or 1st and 3rd ratios added together.
III.	5th	5 :	4— 730	or 2nd „ 4th „ „
IV.	6th	9 :	7+ 686	or 4th „ 5th „ „
V.	7th	14 :	11— 44	or 5th „ 6th „ „
		44) 686 (15		
	8th	210 :	165— 660	or 7th ratio × by 15.
VI.	9th	219 :	172+ 26	or 6th and 8th ratios added together.
VII.	10th	233 :	183— 18	or 7th „ 9th „ „
VIII.	11th	452 :	355+ 8	or 9th „ 10th „ „
		8) 18 (2		

	12th	904 : 710 +	16 or 11th ratio × by 2.
IX.	13th	1137 : 893 —	2 or 10th and 12th ratios added together.
		2) 8 (4	
	14th	4548 : 3572 —	8 or 13th ratio × by 4.
X.	15th	5000 : 3927 +	0 or 11th and 14th ratios added together.

The ratios are numbered according to their *convergence*, with Roman numerals to the left hand, and are as follows:—

$\frac{1}{1}, \frac{4}{3}, \frac{5}{4}, \frac{9}{7}, \frac{14}{11}, \frac{219}{172}, \frac{233}{183}, \frac{452}{355}, \frac{1137}{893}, \frac{5000}{3927}$.

Of these the nearest in small numbers is $\frac{14}{11}$, differing from the true ratio by only $\frac{44}{10000}$, and is indicated (as alluded to in the preceding note) by being immediately followed by the much larger numbers $\frac{219}{172}$.

Example 2.

To find the ratio of 268·8 to 282 in the least numbers.

2688) 2820 ($1\frac{132}{2688}$

2688

132

Then,

I.	1st	1 : 1 +	132 or ratio of 1 to the integer and num.
	2nd	1 : 2 —	2556 or integer + 1 — the ~ of num. and den.
		132) 2556 (19	
	3rd	19 : 19 +	2508 or 1st ratio × by 19.
II.	4th	20 : 21 —	48 or 2nd and 3rd ratios added together.
		48) 132 (2	
	5th	40 : 42 —	96 or 4th ratio × by 2.
III.	6th	41 : 43 +	36 or 1st and 5th ratios added together.
IV.	7th	61 : 64 —	12 or 4th „ 6th „ „
		12) 36 (3	
	8th	183 : 192 —	36 or 7th ratio × by 3.
V.	9th	224 : 235	or 6th and 8th ratios added together.

Therefore the several ratios are $\frac{1}{1}$, $\frac{20}{21}$, $\frac{41}{43}$, $\frac{61}{64}$, and $\frac{224}{235}$. And the excess or defect of any one is seen by inspection; thus, $\frac{41}{43}$ differs from the true ratio only $\frac{36}{2688}$; and $\frac{20}{21}$, but $\frac{48}{2688}$.

RULE 2.—Divide the greater number by the less, and the divisor by the remainder, and the last divisor by the last remainder, and so on till 0 remain. Then,

1 divided by the first quotient, gives the first ratio:

And the terms of the first ratio multiplied by the second quotient, and 1 added to the denominator, give the second ratio:

And in general the terms of any ratio, multiplied by the next quotient, and the terms of the foregoing ratio added, give the next succeeding ratio.

Example 3.

Let the numbers be 10000 and 31416, or the ratio $\frac{10000}{31416}$.

```
10000 ) 31416 ( 3
          30000
          -----
   1416 ) 10000 ( 7
            9912
            ----
     88 ) 1416 ( 16
            88
            --
           536
           528
           ---
             8 ) 88 ( 11
                 88
                 --
                  0
                 ==
```

Then,

$$\frac{1}{3} = \text{1st or least ratio.}$$

$$\frac{1}{3} \times 7 = \frac{7}{21} \text{ and } \frac{7}{21+1} = \frac{7}{22} = \text{2nd ratio.}$$

$$\frac{7}{22} \times 16 = \frac{112}{352} \text{ and } \frac{112+1}{352+3} = \frac{113}{355} = \text{3rd ratio.}$$

$$\frac{113}{355} \times 11 = \frac{1243}{3905} \text{ and } \frac{1243+7}{3905+22} = \frac{1250}{3927} = \text{4th ratio.}$$

Example 4.

The ratio of 268·8 to 282 is required.

```
2688 ) 2820 ( 1
         2688
         ----
  132 ) 2688 ( 20
          264
          ---
   48 ) 132 ( 2
          96
          --
   36 ) 48 ( 1
          36
          --
   12 ) 36 ( 3
          36
          --
           0
          ==
```

Then,

$$\frac{1}{1} = \text{1st ratio.}$$

$$\frac{1}{1} \times 20 = \frac{20}{20} \text{ and } \frac{20}{20+1} = \frac{20}{21} = \text{2nd ratio.}$$

$$\frac{20}{21} \times 2 = \frac{40}{42} \text{ and } \frac{40+1}{42+1} = \frac{41}{43} = \text{3rd ratio.}$$

$$\frac{41}{43} \times 1 = \frac{41}{43} \text{ and } \frac{41+20}{43+21} = \frac{61}{64} = \text{4th ratio.}$$

$$\frac{61}{64} \times 3 = \frac{183}{192} \text{ and } \frac{183+41}{192+43} = \frac{224}{235} = \text{5th ratio.}$$

SECT. XI. *Logarithmic Arithmetic.*

As the nature and properties of logarithms are described in a subsequent part of this work, being so placed because such description could not have been properly understood without a certain acquaintance with algebra, we shall here only explain the use of the tables given in the Appendix, and the method of employing logarithms to facilitate the common processes of arithmetic.

By an inspection of Table II., which contains the logarithms of all numbers from 1 to 100, it will be seen that each logarithm consists of two distinct parts, separated by a decimal point; thus, the logarithm of 13 is 1·113943; the number to the left of the decimal point (or 1 in the above example), is called the *index* or *characteristic**, and its value depends *only* upon the *number of digits* in the quantity whose logarithm it is, without any regard to the *value* of that quantity, and it is always 1 less than that number of digits; thus, in the example, the characteristic of the logarithm of 13, which contains two digits, is 1, or one less than that number; and it will be seen from the Table, that 1 is the characteristic of all the logarithms from 10 to 99, but that, for numbers below 10, the index is 0, and for 100 is 2, in each case 1 less than the number of digits in the quantity of whose logarithm it is the characteristic. The characteristic, therefore, of the logarithms of all numbers

equal to or greater than	1	and less than	10	is 0·
,, ,,	10	,,	100	,, 1·
,, ,,	100	,,	1000	,, 2·
,, ,,	1000	,,	10000	,, 3·
,, ,,	10000	,,	100000	,, 4·
	&c.		&c.	&c.

* In order to avoid confusion from the use of the word index to signify two things, we shall throughout this work employ the term *characteristic* when speaking of logarithms, and *index* when speaking of roots or powers.

When the quantity is less than unity, the characteristic of its logarithm becomes negative, and its value is determined by the number of cyphers which occur between the decimal point and the first significant figure, (the fraction being decimally expressed), and is always 1 greater than such number of cyphers; or it is equal to the difference in the number of figures in the numerator and decimal denominator; thus, the characteristic of the logarithm of

·1	or	$\frac{1}{10}$	is $\overline{1}$·*
·01	,,	$\frac{1}{100}$	,, $\overline{2}$·
·001	,,	$\frac{1}{1000}$	,, $\overline{3}$·
·0001	,,	$\frac{1}{10000}$	,, $\overline{4}$.
&c.		&c.	&c.

The *decimal* part of the logarithm, or that lying to the right of the decimal point, depends entirely on the *relative* value of the figures composing the quantity whose logarithm it is, and not at all upon the actual *numerical* value of that quantity; thus, in the example already given, the decimal part of the logarithm of 13 is ·113943, which is also the decimal part of the logarithm of 1·3, or 130, or 1300, for in each case the 1 and the 3 have the same *relative* value. So that the decimal portion of a logarithm is always the same for the same figures, and is not altered by the addition of any number of cyphers either to the right or to the left hand of those figures, or what is equivalent, by the multiplication or division of the quantity by 10, or any power of 10; it is only the characteristic of the logarithm which alters its value, 1 being *added* to the characteristic for every 10 by which the quantity is *multiplied*, or *subtracted* from it for every 10 by which the quantity is *divided*. Thus,

the logarithm of	745800	being	5·872622
that of	74580	is	4·872622
,,	7458	,,	3·872622
,,	745·8	,,	2·872622
,,	74·58	,,	1·872622
,,	7·458	,,	0·872622
,,	·7458	,,	$\overline{1}$·872622
,,	·07458	,,	$\overline{2}$·872622
,,	·007458	,,	$\overline{3}$·872622

It must be borne in mind, that in the logarithm of a fractional quantity, it is only the *characteristic* which has a *negative* value, and that the decimal part of a logarithm is always positive. It is, however, sometimes convenient to have the whole logarithm expressed negatively, both characteristic and decimal; for which purpose, sub-

* The negative sign (—) is always placed *above* the characteristic, thus $\overline{2}$, instead of before it, in order to avoid its being misunderstood for the sign of subtraction.

tract the last right hand figure in the decimal portion from 10, and all the others from 9, and the result will be what is termed the *arithmetical complement* of the decimal, to which prefix the former characteristic less 1, and the result will be a negative logarithm, equivalent in value to the original logarithm having only a negative characteristic; for example, the logarithm of ·07458, as above, is $\bar{2}$·872622, which is equivalent to — 1·127378. It is also frequently convenient to take the *arithmetical complement* of the whole logarithm, and this is obtained by subtracting the right hand figure of the decimal from 10, and all the others from 9, including the characteristic when *positive*, but if *negative* it must be *added* to 9. Thus, the arithmetical complement

of 3·146128 is 6·853872
„ $\bar{2}$·076276 „ 11·923714
„ $\bar{5}$·322839 „ 4·677161
„ $\bar{1}$·986772 „ 10·013228

USE OF THE TABLES.

To find the logarithm of any given number.

If the number is less than 100, its logarithm will be found in Table II., with its proper characteristic prefixed; but if the number contains more than two figures, its logarithm may be found from Table III. as follows:—If there are only three figures in the number, look for that number in the first column of the table, and on the same line in the next column to the right, under 0, will be found the decimal portion of the required logarithm, to which the proper characteristic must be prefixed, according to the rules which we have just explained. If the quantity contains four figures, look for the first three figures in the first column as before, and the four last figures of the logarithm of the required number will be found on the same line with those three figures, and in that column which has at its head the fourth figure of the given number; the two first figures of the logarithm will be found in the second column (headed 0), and which figures being common to all the logarithms enclosed by each pair of horizontal lines, it is unnecessary to repeat. Where these first figures change their value in the middle of a line, the same is indicated by a break in the horizontal line, thus, 139879 | 0194, which shows that the two first figures (13) have changed to 14, and the right hand logarithm is therefore 140194. The heading figures are repeated at every tenth line in the body of the tables, in order to facilitate their use.

Examples.

Required the logarithm of 734.

In Table III., on the same line with 734 and under 0, are found 5696, the four last figures of the logarithm, to which the *common*

figures 86 and the proper characteristic 2 being prefixed, we obtain 2·865696, the logarithm required.

Find the logarithm of 3476.

Here, on the same line with 347 and under 6, will be found 1080, which, with the two first figures and the characteristic prefixed, is 3·541080, the logarithm required.

The log. of 5·84 is 0·766413
„ ·0932 „ $\bar{2}$·969416
„ 10·24 „ 1·010300
„ 3708 „ 3·569140.

When the quantity whose logarithm is required contains more than four figures, proceed as follows:—Find the logarithm for the first four figures as above, then look in the first column of Table I. for the first figures, and on the same line in the column having at its head the fifth figure will be found the quantity which must be added to the logarithm already taken out, to give the logarithm of the quantity first required. If the first four figures are not found in the first column of the table, then take the line containing the next less number to it. If the number whose logarithm is required contains more than five figures, proceed as above to obtain the logarithm of the first five figures, then, on the same line of Table I. that the number added to the logarithm for the fifth figure was found, and in the column having at its head the sixth figure, will be found a quantity, which, divided by 10 (or what is the same, having its right hand figure taken away *), and added to the logarithm already found, will give the logarithm of the first six figures; again, on the same line and in the column having at its head the seventh figure, will be found a quantity, which, divided by 100 (or having two figures cut off from the right hand), and added, will give the logarithm for seven figures †.

Examples.

Required the logarithms of 11488, 621547, 768654, 7642·179.

Log. of the first four figures from Tab. III.....	=	4·059942	
From Tab. I. on line with 1148 and under 8 ...	=	302	4
Logarithm of 11488, as required	=	4·060244	
Log. of 6215	=	5·793441	
From Tab. I. on line with 6160, the next less No. in the tab. to 6215 and under 4	=	28	
On same line under 7	=	4	9
Logarithm of 621547	=	5·793474	

* If the figure thus cut off exceeds five, one must be added to the first right hand figure left.

† See remark at page 45, with regard to the number of places to be depended upon.

Log. of 7686	=	5·885700	
From Tab. I. on same line with 7686 under 5	=	28	
On same line under 4	=	2	2
Logarithm of 768654	=	5·885730	
Log. of 7642	=	3·883207	
From Tab. I. on same line with 7552 under 1	=	5	7
On same line under 7	=	3	99
On same line under 9	=		513
Logarithm of 7642·179	=	3·883217	

To find the number answering to any given logarithm.

Look in Table III. for the given logarithm, or the next less in value to it that can be found, then on the same line, in the first column, will be found the first three figures, and at the head of the column in which the logarithm was found, the fourth figure of the number sought. If the given logarithm is found exactly in the table, the figures thus obtained will be the required number, care being taken to point off one more figure to the left hand than there are units in the characteristic of the given logarithm, cyphers being attached to the right hand of the number, if requisite. If, however, the given logarithm is not found exactly in the table, subtract from it the next less logarithm found, calling the remainder the first difference; then look in Table I. on the same line with the four figures already obtained from Table III. (or the next less figures which can be found) for this difference, and at the head of the column in which it is found will be the fifth figure of the number sought. If the first difference is not found exactly in the table, look for the next less number to it, which subtract from the first difference for the second difference; then add a cypher to this second difference, and look for it on the same line of Table I. as before, and the figure at the head of the column containing the nearest number to it, either greater or less, will be the sixth figure of the number required.

Examples.

Required the number answering to the logarithm 3·241756.

Given log. = 3·241756

Next less log. in Tab. III. = 3·241546 = the log. of 1744·

	210 first dif.	
In Tab. I. on same line with the next less No. to 1744	199 is found in col. 8	·8
	110 second dif.	
On the same line	99 is found in col. 4	·04
	The No. required =	1744·84

In this example the next less logarithm which can be found in Table III. is 3·241546, the number answering to which, 1744, is the first four figures of the number sought; then subtracting this logarithm from the given logarithm, we obtain for the first difference 210, and looking in Table I. on a line with 1740 (the next less number to 1744), for the next less number to 210, we find 199, at the head of the column containing which is 8, the fifth figure required; then subtracting 199 from 210, we obtain the second difference, 11, and adding a cypher, the nearest number which we find on the same line is 99, at the head of the column containing which is 4, the sixth figure required.

Required the numbers answering to the following logarithms:— 3·510009, 2·475771, 5·871624.

The number answering to the logarithm 3·510009 is found at once to be 3236.

Given log.	= 2·475771		
Next less log.	= 2·475671	= the log. of	299·0
	100	= 1st dif.	
From Tab. I....	87	...	·06
	130	= 2nd dif.	
	130	...	·009
	No. required =		299·069

Given log.	= 5·871624		
Next less log.	= 5·871573	= the log. of	744000·
	51	= 1st dif.	
From Tab. I....	46	...	80·
	50		
	52	...	9·
	The No. required =		744089·

It should be observed here, that the number of figures which may be depended upon in any result obtained by logarithms, will be equal to the number of decimal places in the logarithms employed; thus, in using the tables appended to this work, the results obtained will be accurate to six figures, except towards the end of the tables, in which only five figures should be trusted.

MULTIPLICATION AND DIVISION BY LOGARITHMS.

To *multiply* two numbers together, *add* together their logarithms,

and the sum will be the logarithm of their product; or, to *divide* one number by another, *subtract* the logarithm of the divisor from the logarithm of the dividend, and the remainder will be the logarithm of the quotient of the two numbers.

Ex.—Multiply 30 × 43 × 72·54; and divide the product of (7143 × 6278) by 314·5.

Add	Log. 30 = 1·477121	Log. 7143 = 3·853881	
	log. 43 = 1·633468	add log. 6278 = 3·797821	
	log. 72·54 = 1·860578		7·651702
		sub. log. 314·5 = 2·497621	
	Log. 93576·4 = 4·971167	Log. 142589 = 5·154081	

PROPORTION OR THE RULE OF THREE BY LOGARITHMS.

The Rule of Three is very readily performed with the aid of logarithms, by simply adding together the logarithms of the 2nd and 3rd terms, and subtracting the logarithm of the 1st, the remainder being the logarithm of the 4th term, or number required to be found. Or, instead of *subtracting* the logarithm of the 1st term, we may *add* its complement, (the method of obtaining which has been already explained at page 42,) and subtract 10 from the characteristic of the result, which will, as before, be the logarithm of the 4th term.

Examples.

The following are the same as the 1st, 3rd, 4th, and 5th Examples given in the Rule of Three, at pages 35 and 36.

1st *Ex.*		3rd *Ex.*	
Log. 19	= 1·278754	Log. 245	= 1·389166
+ log. 126	= 2·100370	+ log. 160	= 2·204120
	3·379124		3·593286
− log. 3	= 0·477121	− log. 21	= 1·322219
Log. 798	= 2·902003	Log. 186·667	= 2·271067

4th *Ex.*		5th *Ex.*	
Comp. of log. 22	= 8·657577	Comp. of log. 100	= 8·000000
+ log. 110	= 2·041393	+ log. 4	= 0·602060
+ log. 25	= 1·397940	+ log. 560	= 2·748188
Log. 125	= 2·096910	Log. 22·4	= 1·350248

EVOLUTION AND INVOLUTION BY LOGARITHMS.

To perform the operation of *involution*, or the raising of powers, it is only necessary to *multiply* the number, any power of which is required, by the index of that power, and the product will be the logarithm of the required power; and, inversely, the operation of *evolution*, or the extraction of roots, is performed by simply *dividing* the logarithm of the number by the index of the root required, the quotient will be the logarithm of the root.

Examples.

Square 84, cube 13, and raise 7 to the sixth power.

Log. $84 = 1{\cdot}924279 \times 2 = 3{\cdot}848558 = 7056 = 84^2$
log. $13 = 1{\cdot}113943 \times 3 = 3{\cdot}341829 = 2197 = 13^3$
log. $7 = 0{\cdot}845098 \times 6 = 5{\cdot}070588 = 117649 = 7^6$.

Extract the square root of 576, the cube root of 4913, and the sixth root of 46656.

Log. $576 = 2{\cdot}760422 \div 2 = 1{\cdot}380211 = 24 = \sqrt{576}$
log. $4913 = 3{\cdot}691347 \div 3 = 1{\cdot}230449 = 17 = \sqrt[3]{4913}$
log. $46656 = 4{\cdot}668908 \div 6 = 0{\cdot}779151 = 6 = \sqrt[6]{46656}$.

It is necessary here to make a few remarks on performing the operations of evolution and involution on logarithms with *negative* characteristics. In doing this, it must be borne in mind that it is only the *characteristic* which has a *negative* value, the *decimal* part of the logarithm being always *positive;* therefore, if it is required to multiply a logarithm with a negative characteristic by any number, first multiply the decimal part of the logarithm, pointing off as many decimal figures in the product as there were in the logarithm, then multiply the characteristic, and subtract from the product the number (if any) pointed off to the *left* in the first product, the result will be the negative characteristic, and the decimals pointed off in the first product will be the decimal part of the required logarithm.

Examples.

Required the square of ·25, the cube of ·375, and the sixth power of ·7.

The logarithm of $\cdot 25 = \bar{1}{\cdot}397904$
First multiply ·397940 by 2
2
·795880
Then $\bar{1} \times 2 = \bar{2}$
$\bar{2}{\cdot}795880 = \cdot 0625 = \cdot 25^2$.

The logarithm of ·375 is $\bar{1}$·574031.

$$\begin{array}{r} \cdot 574031 \\ 3 \\ \hline 1\cdot 722093 \\ \bar{1} \times 3 = \bar{3}\cdot \\ \hline \bar{2}\cdot 722093 = \cdot 05273437 = \cdot 375^3. \end{array}$$

The logarithm of ·7 is $\bar{1}$·845098

$$\begin{array}{r} \cdot 845098 \\ 6 \\ \hline 5\cdot 070588 \\ \bar{1} \times 6 = \bar{6}\cdot \\ \hline \bar{1}\cdot 070588 = \cdot 117649 = \cdot 7^6. \end{array}$$

To divide a logarithm with a negative characteristic by any number:—If the characteristic is a *multiple* of that number, that is, if it is *divisible* by it without remainder, proceed as in ordinary division; if not, separate the characteristic from the decimal, and add to the characteristic a number which will make it *divisible*, and prefix to the decimal the same number, then divide both by the given divisor, and the quotients will be the characteristic and decimal of the logarithm required.

Examples.

Required the square root of ·0625, the cube root of ·74, and the fifth root of ·543.

The logarithm of ·0625 is $\bar{2}$·795880

Then $2)\bar{2}\cdot 795880$

$$\bar{1}\cdot 397940 = \cdot 25 = \sqrt{\cdot 0625}$$

The logarithm of ·74 $= \bar{1}$·869232.

Then $\bar{1} + \bar{2} = \bar{3} \div 3 = \bar{1}\cdot$

and $2\cdot 869232 \div 3 = \cdot 956411$

$$\sqrt[3]{\cdot 74} = \cdot 9045 = \bar{1}\cdot 956411$$

The logarithm of ·543 is $\bar{1}$·734800

Then $\bar{1} + \bar{4} = \bar{5} \div 5 = \bar{1}\cdot$

and $4\cdot 734800 \div 5 = \cdot 946960$

$$\sqrt[5]{\cdot 543} = \cdot 885034 = \bar{1}\cdot 946960$$

SECT. XII. *Properties of Numbers.*

To render these intelligible to the student, we shall here collect a few definitions.

Def. 1. A *unit*, or *unity*, is the representation of any thing considered individually, without regard to the parts of which it is composed.

2. An *integer* is either a unit or an assemblage of units; and a *fraction* is any part or parts of a unit.

3. A *multiple* of any number is that which contains it some exact number of times.

4. One number is said to *measure* another, when it divides it without leaving any remainder.

5. And if a number exactly divides two, or more numbers, it is then called their *common measure*.

6. An *even number*, is that which can be halved, or divided into two equal parts.

7. An *odd number*, is that which cannot be halved, or which differs from an even number by unity.

8. A *prime number*, is that which can only be measured by 1, or unity.

9. One number is said to be *prime* to another when unity is the only number by which they can both be measured.

10. A *composite number*, is that which can be measured by some number greater than unity.

11. A *perfect number*, is that which is equal to the sum of all its divisors, or aliquot parts:—thus $6 = \frac{6}{2} + \frac{6}{3} + \frac{6}{6}$.

Prop. 1. The sum or difference of any two *even* numbers is an *even* number.

2. The sum or difference of any two *odd* numbers is *even*; but the sum of three *odd* numbers is *odd*.

3. The sum of any *even* number of *odd* numbers is *even*; but the sum of any *odd* number of *odd* numbers is *odd*.

4. The sum or difference of an *even* and an *odd* number is *odd*.

5. The product of any number of *even* numbers is *even*; and any power of an *even* number is *even*.

6. The product of any number of *odd* numbers is *odd*; and every power of an *odd* number is *odd*.

7. The product of any number of *even* numbers, by any number of *odd* numbers, is *even*.

8. An *odd* number cannot be divided by an *even* number, without a remainder.

9. If an *odd* number divides an *even* number, it will also divide the half of it.

10. If a number consist of many parts, and each of those parts have a common divisor *d*, then will the whole number, taken collectively, be divisible by *d*.

11. Neither the sum nor the difference of two fractions, which are in their lowest terms, and of which the denominator of the one con-

tains a factor not common to the other, can be equal to an integral number.

12. If a square number be either multiplied or divided by a square, the product or quotient is a square; and conversely, if a square number be either multiplied or divided by a number that is not a square, the product or quotient is not a square.

13. The product arising from two different prime numbers cannot be a square number.

14. The product of no two different numbers prime to each other can make a square, unless each of those numbers be a square.

15. The square root of an integral number, that is not a complete square, can neither be expressed by an integer nor by any rational fraction.

16. The cube root of an integer that is not a complete cube cannot be expressed by either an integer or a rational fraction.

17. Every prime number greater than 2, is of one of the forms $4n + 1$, or $4n - 1$.

18. Every prime number greater than 3, is of one of the forms $6n + 1$, or $6n - 1$.

19. No algebraical formula can contain prime numbers only.

20. The number of prime numbers is infinite.

21. The first twenty prime numbers are 1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, and 67.

22. A square number cannot terminate with an *odd* number of cyphers.

23. If a square number terminate with a 4, the last figure but one (towards the right hand) will be an *even* number.

24. If a square number terminate with 5, it will terminate with 25.

25. If a square number terminate with an *odd* digit, the last figure but one will be *even;* and if it terminate with any *even* digit, except 4, the last figure but one will be *odd.*

26. No square number can terminate with two equal digits, except two *cyphers* or two *fours.*

27. No number whose last, or right-hand digit is 2, 3, 7, or 8, is a square number.

28. If a cube number be divisible by 7, it is also divisible by the cube of 7.

29. The difference between any integral cube and its root is always divisible by 6.

30. Neither the sum nor the difference of two cubes can be a cube.

31. A cube number may end with any of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 or 0.

32. If any series of numbers, beginning from 1, be in continued geometrical proportion, the 3rd, 5th, 7th, &c. will be squares; the 4th, 7th, 10th, &c. cubes; and the 7th, of course, both a square and a cube.

33. All the powers of any number that end with either 5 or 6, will end with 5 or 6, respectively.

34. Any power, n, of the natural numbers, 1, 2, 3, 4, 5, 6, &c. has as many orders of differences as there are units in the common exponent of all the numbers; and the last of those differences is a constant quantity, and equal to the continual product $1 \times 2 \times 3 \times 4 \times \ldots\ldots \times n$, continued till the last factor, or the number of factors be n, the exponent of the powers. Thus,

The 1st powers 1, 2, 3, 4, 5, &c., have but one order of differences 1 1 1 1 &c., and that difference is 1.

The 2nd powers 1, 4, 9, 16, 25, &c., have two orders of differences 3 5 7 9
2 2 2
of which the last is constantly $2 = 1 \times 2$.

The 3rd powers 1, 8, 27, 64, 125, &c., have three orders of differences 7 19 37 61
12 18 24
6 6
of which the last is $6 = 1 \times 2 \times 3$.

In like manner, the 4th, or last, differences of the 4th powers, are each $= 24 = 1 \times 2 \times 3 \times 4$; and the 5th, or last differences of the 5th powers, are each $125 = 1 \times 2 \times 3 \times 4 \times 5$.

35. If unity be divided into any two unequal parts, the sum of the square of either of those parts added to the other is the same. Thus, of the two parts $\frac{1}{5}$ and $\frac{4}{5}$, $\frac{1}{5} + (\frac{4}{5})^2 = \frac{4}{5} + (\frac{1}{5})^2 = \frac{21}{25}$; so, again, of the parts $\frac{2}{5}$ and $\frac{3}{5}$, $\frac{2}{5} + (\frac{3}{5})^2 = \frac{3}{5} + (\frac{2}{5})^2 = \frac{19}{25}$.

For the demonstrations of these and a variety of other properties of numbers, those who wish to pursue this curious line of inquiry may consult Legendre "Sur la Theorie des Nombres," the "Disquisitiones Arithmeticæ" of Gauss, or Barlow's "Elementary Investigation of the Theory of Numbers."

Also, for the highly interesting properties of *Circulating Decimals*, and their connexion with *prime numbers*, consult the curious works of the late Mr. H. Goodwyn, entitled "A First Centenary," and "A Table of the Circles arising from the Division of a Unit by all the Integers from 1 to 1024."

CHAP. II.

ALGEBRA.

SECT. I. *Definitions and Notation.*

ALGEBRA is the science of the computation of magnitudes in general, as arithmetic is the particular science of the computation of numbers.

Every figure or arithmetical character has a determinate and individual value; the figure 5, for example, represents always one and the same number, namely, the collection of 5 units, of an order depending upon the position and use of the figure itself. Algebraical characters, on the contrary, must be, in general, independent of all particular signification, and proper to represent all sorts of numbers or quantities, according to the nature of the questions to which we apply them. They should, moreover, be simple and easy to trace, so as to fatigue neither the attention nor the memory. These advantages are obtained by employing the letters of the alphabet, a, b, c, &c. to represent any kinds of magnitudes which become the subjects of mathematical research. The consequence is, that when we have resolved by a single algebraical computation all the problems of the same kind proposed, in the utmost generality of which they are susceptible; the application of the investigation to all particular cases requires no more than arithmetical operations.

It is usual, though by no means absolutely necessary, to represent quantities that are *known* by the commencing letters of the alphabet, as a, b, c, d, &c., and those that are *unknown* by the concluding letters w, x, y, z. But it is often convenient, especially as it assists the memory, to represent any quantity which enters an investigation, whether known or unknown, by its initial letter; as *sum* by s, *product* by p, *density* by d, *velocity* by v, *time* by t; and so of others.

In addition to the signs already explained as being used in arithmetic, the following symbols and modes of expressing certain operations, are employed in algebra.

The product of two or more quantities is expressed either by interposing the sign of multiplication, as $a \times b \times c \times d$; or by interposing *dots*, which have the same signification, as $a \,.\, b \,.\, c \,.\, d$; or, more simply, by placing the letters merely in juxtaposition, as $a\ b\ c\ d$. And as it is immaterial in what order the multiplication is performed, it is usual to write the letters after each other in the order in which they would stand in the alphabet, placing any figures which may occur in the product,

at the commencement. Thus, the continual product of $2b$, x, $7e$, and $3c$, may be written either $2bx7e3c$, or $x2b7e3c$, &c., but it is preferable and usual to write them $42bcex$. The figures 2, 3, 7 and 42, by which the letters are multiplied, are termed their *coefficients;* when a letter is not preceded by any figure (as x in the example), its coefficient is 1, or unity.

We give the name *term* to any quantity separated from another by the sign + or —. When an algebraical expression consists of only

1 term, as	$4a$	it is called a	*monomial.*
2 „	$ac - 4ab$ . . .	„	*binomial*.*
3 „	$a + b - ac$ . . .	„	*trinomial.*
4 „	$c + d - 2f + x$.	„	*quadrinomial.*
Many „	$3b - d + 2g - 7k + l + n$	„	*multinomial,* or *polynomial.*

The signs + and —, which in arithmetic simply indicate the operations of addition and subtraction, are employed more extensively in algebra, to denote, besides addition and subtraction, any two operations or any two states which are as opposed in their nature as addition and subtraction are. And if, in an algebraical process, the sign + is prefixed to a quantity to mark that it exists in a certain state, position, direction, &c., then, whenever the sign — occurs in connexion with such quantity, it *must* indicate precisely the contrary state, position, &c. and no intermediate one. This is a matter of pure convention, and not of metaphysical reasoning. Other characters *might* have been contrived to denote this opposition; but they would be superfluous, because the characters + and —, though originally restricted to denote addition and subtraction, may safely be extended to other purposes.

Thus if + a signifies any thing. { added / above / to the right, / forwards, } — a signifies { subtracted / below, / to the left, / backwards. }

If + a signifies any assigned { Increase, / Gravity, / Money due, / Motion upwards, } — a signifies a corresponding { Decrease, / Levity, / Money owing, / Motion downward. }

And so on in every species of contrariety. And two such equal quantities connected together in any case destroy each other's effect, or are equal to nothing, as $+a - a = 0$. Thus, if a man has but 10*l.* and at the same time owes 10*l.* he is worth *nothing*. And, if a vessel which would, otherwise, sail six miles an hour, be carried back six miles an hour by a current, it makes no advance.

* When the second term of a binomial has the negative sign (—) it is called a *residual*.

Like signs are either all positive (+), or all negative (−). And *unlike* are when some are positive and others negative. If there be no sign before a quantity, the sign + is understood.

Like quantities are such as contain the same letters, and differ only in their *coefficients*, as $a - 7a$, or $9abx + 2abx - 6abx$. *Unlike quantities* are such as contain different letters, as $ab - cf + bkx$.

When any number of terms are collected under a *vinculum* or *bar*, thus, $\overline{a + b - cf}^2$, or inclosed in parentheses thus, $(a + b - cf)^2$, it denotes that the whole quantity so enclosed is to be taken collectively, and subjected to whatever operation or process is indicated by the symbol without the bar or parenthesis; thus, in the example, that the compound quantity $a + b - cf$, is to be squared.

An *equation* is when two sets of quantities which make an equal aggregate are placed with the sign of equality (=) between them:

As $12 + 5 = 20 - 3$, or $x + y = a + b - cd$.

The quantities placed on both sides the sign of equality are called respectively the *members* of the equation.

The sign ∞ indicates that the quantity before which it is placed is *infinite*, or unlimited in its value.

The symbol $\propto$, placed between two quantities, signifies that the second *varies as* the first: thus, $a \propto b$ is read a varies as b.

The word *therefore* being of frequent occurrence in deducing the successive steps of algebraic processes, is denoted by $\therefore$.

SECT. II. *Addition and Subtraction.*

IN algebra, the operations answering to those of addition and subtraction in arithmetic cannot with propriety be called by those names, as *either* of them in algebra frequently involves the actual use of both those processes. They would be better expressed by some general term denoting the incorporation or striking a balance between the several quantities employed.

When two simple quantities, or *monomials*, are to be added together, as $2b$ and c, or $4a$ and $5a$, it is done by connecting them together with the sign of addition (+); as $2b + c$, $4a + 5a$; but when the quantities are similar, as in the second example, the expression may be simplified by adding together the two coefficients, and subjoining the common letter, which would then become $9a$.

In like manner, the subtraction of simple quantities, or *monomials*, may be expressed by the sign of subtraction (−), or in the case of simple like quantities by actually subtracting the coefficients, and subjoining the common letter; thus, the subtraction of b from a, f from c, $4bc$ from $7bc$, is performed thus, $a - b$, $c - f$, $3bc$. It not unfrequently occurs that the quantity to be subtracted is greater than the quantity from which it is to be taken, the consideration of which being totally different from all that the learner has been accustomed to in arithmetical operations, involves a difficulty in its appre-

hension, but this will soon be removed by an attentive perusal of the paragraph at page 53, explaining the sense in which the signs + and − are used in algebra. Where such occurs, the less quantity must be taken from the greater, and the negative sign prefixed to the remainder.

To add together compound quantities, or polynomials, as $(a + 2b + c)$ and $(4a + 5c - d)$, write one after the other, with their proper signs attached, omitting the parentheses, as $a + 2b + c + 4a + 5c - d$, which may be simplified by the further addition or subtraction of its separate terms, and then becomes $5a + 2b + 6c - d$.

When quantities to be added are presented promiscuously, it is best to classify them previous to their addition.

Thus, $3a^2, -3bc, +2c^2, +4d, +7a^2, +5bc, +a^2, -2c^2, -4bc$, when arranged become as in the margin, and their sum is readily obtained, as in the fourth line.

$$\begin{array}{r} 3a^2 + 5bc - 2c^2 \\ 7a^2 - 3bc + 4d \\ a^2 - 4bc + 2c^2 \\ \hline 11a^2 - 2bc + 4d \end{array}$$

To subtract one compound quantity from another, change the sign of each term of the *subtrahend*, and then proceed as in addition.

Thus,	$4ab - 3bc$	And	$4ab - 3c^2 + bc$
	$-(2ab - 6bc)$		$-(ab - c^2 - 2bc)$
become	$4ab - 3bc$	become	$4ab - 3c^2 + bc$
	$-2ab + 6bc$		$-ab + c^2 + 2bc$
Result	$2ab + 3bc$	Result	$3ab - 2c^2 + 3bc$

The reason of changing the signs of the subtrahend may be explained as follows:—Let it be required to subtract $(c-d)$ from $(a+b)$; let us first subtract c, and the result will be $a + b - c$; but it is evident that in subtracting c from $(a + b)$, we have taken away more than we ought, since it is only $c - d$ which should have been subtracted, we have therefore taken away too much by d, and to obtain the correct result, must add it again to $a + b - c$, which then becomes $a + b - c + d$, in which it is obvious that the signs of the subtrahend $(c - d)$ are changed.

In addition and subtraction of *algebraic fractions*, the quantities must be reduced to a common denominator, and occasionally undergo other reductions similar to those in vulgar fractions in arithmetic; and then the sum or the difference of the numerators may be placed over the common denominator, as required.

Thus, $\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad + bc}{bd}$.

And, $\frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab} + \frac{ab}{c} + \frac{ac}{b} + \frac{bc}{a} =$

$\frac{a^2}{abc} + \frac{b^2}{abc} + \frac{c^2}{abc} + \frac{a^2b^2}{abc} + \frac{a^2c^2}{abc} + \frac{b^2c^2}{abc} =$

$\frac{a^2 + b^2 + c^2 + a^2b^2 + a^2c^2 + b^2c^2}{abc}$

Also, $\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$.

And, $\frac{a + x}{d} - \frac{a - x}{c} = \frac{ac + cx}{cd} - \frac{ad - dx}{cd} =$

$\frac{ac + cx - ad + dx}{cd} = \frac{a(c - d) + x(c + d)}{cd}$

And, $\frac{b + x}{b - x} - \frac{b - x}{b + x} =$

$\frac{(b^2 + 2bx + x^2) - (b^2 - 2bx + x^2)}{b^2 - x^2} = \frac{4bx}{b^2 - x^2}$.

SECT. III. *Multiplication.*

1. To multiply one *monomial* by another, multiply their coefficients for the coefficient of the product, and subjoin to it the letters composing the two quantities in their natural order, and if the same letter occurs in both, add together the two exponents for its exponent in the product, 1 being understood where no other exponent is expressed. In attaching the proper sign to the product, observe that the product of two factors having *contrary* signs is *negative*, and of two factors having the *same* sign *positive*. Or briefly, that *like* signs give +, and *unlike* signs −.

Note.—The general rule for the signs may be rendered evident from the following definition; multiplication is the finding a magnitude which has to the multiplicand the proportion of the multiplier to unity. Hence, the multiplier must be an abstract number, and, if a simple term, can have neither + nor − prefixed to its notation. Now first, $+a \times +m = +ma$, for the quality of a cannot be altered by increasing or diminishing its value in any proportion; therefore the product is of the quality *plus*, and ma by the definition is the product of a and m. Secondly, $-a \times +m = -ma$, for the same reasons as before, *mutatis mutandis*. Thirdly, $+a \times -m$ has no meaning; for m must be an abstract number, therefore here

we can have no proof. But $+ a \times (m - n) = m a - n a$, n being less than m; for a taken as often as there are units in m is $= m a$ by the first case; but a was to have been taken only as often as there are units in $m - n$; therefore a has been taken too often by the units in n; consequently a taken n times, or $n a$, must be subtracted; and of course $m a - n a$ is the true product. Fourthly, $- a \times (m - n) = - m a + n a$. For $- a \times m = - m a$ (by case 2); but this, as above, is too great by $- n a$; therefore $- m a$ with $n a$ subtracted from it is the true product; but this, by the rule of subtraction, is $= - m a + n a$.

2. To find the product of two *polynomials*, multiply each term of the one into all those of the other, following the rule given for monomials.

3. To multiply algebraic *fractions*, take the product of the numerators for the new numerator, and that of the denominators for the new denominator.

Examples.

1. $4 a b \times 5 c d = 4 . 5 . a b . c d = 20 a b c d.$

2. $8 a^2 b^3 \times 4 a^5 b = 8 . 4 . a^2 . a^5 . b^3 . b = 32 a^{2+5} b^{3+1} = 32 a^7 b^4.$

3. Multiply $2 a + b c - 2 b^2$
By $2 a - b c + 2 b^2$

$$4 a^2 + 2 a b c - 4 a b^2$$
$$- 2 a b c - b^2 c^2 + 2 b^3 c$$
$$+ 4 a b^2 + 2 b^3 c - 4 b^4$$

Product, $4 a^2 - b^2 c^2 + 4 b^3 c - 4 b^4$

4. $\begin{cases} a + b \\ a + b \end{cases}$

$$a^2 + a b$$
$$a b + b^2$$
$$a^2 + 2 a b + b^2$$

5. $\begin{cases} a^2 + 2 a b + b^2 \\ a + b \end{cases}$

$$a^3 + 2 a^2 b + a b^2$$
$$+ a^2 b + 2 a b^2 + b^3$$
$$a^3 + 3 a^2 b + 3 a b^2 + b^3$$

6. $\begin{cases} a + b \\ a - b \end{cases}$

$$a^2 + a b$$
$$- a b - b^2$$
$$a^2 - b^2$$

7. $\frac{a+b}{c} \times \frac{a-b}{d} = \frac{(a+b)(a-b)}{c \times d} = \frac{a^2-b^2}{c\,d}$.

8. $\frac{2x}{a} \times \frac{3ab}{c} \times \frac{3ac}{2b} = \frac{18a^2bcx}{2abc} = \frac{9ax}{1} \times \frac{2abc}{2abc} = 9ax$.

Note.—From the above examples (4, 5, and 6) we may learn—

1. That the square of the sum of two quantities is equal to the sum of the squares of the two quantities together with twice their product.

2. That the cube of the binomial $a + b$, is $a^3 + 3a^2b + 3ab^2 + b^3$.

3. That the product of the sum and difference of two quantities is equal to the difference of their squares.

SECT. IV. *Division.*

1. To divide one *monomial* by another, divide the coefficient of the dividend by the coefficient of the divisor for the coefficient of the quotient, and subjoin to it a fraction having for its numerator the letters composing the dividend, and for its denominator those of the divisor, and if the same letter occurs in both, subtract the exponents for the exponent of the same letter in the quotient; if the exponents as well as the letters are alike, both the quantities may be struck out. The same rule applies to the signs in division as in multiplication.

2. To divide a *polynomial* by a *monomial*, divide each term of the polynomial by the monomial according to Rule 1, and connect the results by their proper signs.

3. To divide two *polynomials* one by the other, arrange them with respect to the powers of the same letter, then divide the first terms one by the other, and thence will result one term of the quotient; multiply the divisor by this, and subtract the product from the dividend: proceed with the remainder in the same manner.

4. To divide algebraic *fractions*, invert the terms of the divisor, and proceed as in multiplication.

Examples.

1. $12a^3b^2c \div 3ab = {}^{12}_{3}\,a^{3-1}b^{2-1}c = 4a^2bc$.

2. $15a^3b^3 \div 5a^2b^2 = {}^{15}_{5}\,a^{3-2}b^{3-2} = 3ab^1$.

3. $12a^4 - 2a^2c + c \div 2a^2 = 6a^2 - c + \frac{c}{2a^2}$.

4. $6x^2 + 12xy - 9xyz \div 3x = \frac{6x^2}{3x} + \frac{12xy}{3x} - \frac{9xyz}{3x} = 2x + 4y - 3yz$.

5. $16\,a^2\,b^2 - 12\,a^2 + 8\,a \div 4\,a\,b = 4\,a\,b - 3\,\frac{a}{b} + \frac{2}{b}$.

6. Divide $x^3 - 3\,x^2\,z + 3\,x\,z^2 - z^3$ by $x - z$.

$x - z)\ x^3 - 3\,x^2\,z + 3\,x\,z^2 - z^3\ (x^2 - 2\,x\,z + z^2$ *quotient.*

$x^3 - x^2\,z$

$-2\,x^2\,z + 3\,x\,z^2$

$-2\,x^2\,z + 2\,x\,z^2$

$x\,z^2 - z^3$

$x\,z^2 - z^3$

* *

7. Divide $a^5 - b^5$ by $a - b$.

$a - b)\ a^5 - b^5\ (a^4 + a^3\,b + a^2\,b^2 + a\,b^3 + b^4$ *quotient.*

$a^5 - a^4\,b$

$a^4\,b$

$a^4\,b - a^3\,b^2$

$a^3\,b^2$

$a^3\,b^2 - a^2\,b^3$

$a^2\,b^3$

$a^2\,b^3 - a\,b^4$

$a\,b^4 - b^5$

$a\,b^4 - b^5$

Here the second term of the dividend is brought down to stand over the corresponding term in the last product.

8. Divide 1 by $1 - x$

$1 - x)\ 1\ (1 + x + x^2 + x^3 + x^4 + \frac{x^5}{1 - x}$

$1 - x$

x

$x - x^2$

x^2

$x^2 - x^3$

x^3

$x^3 - x^4$

x^4

$x^4 - x^5$

x^5

9. $\dfrac{2x^2}{a^3+x^3} \div \dfrac{x}{a+x} = \dfrac{2x^2}{a^3+x^3} \times \dfrac{a+x}{x} = \dfrac{2x^2(a+x)}{(a^3+x^3)x}$

$= \dfrac{2x}{x^2-ax+a^2}$.

10. $\dfrac{x^4-b^4}{x^2-2bx+b^2} \div \dfrac{x^2+bx}{x-b} = \dfrac{x^4-b^4}{(x-b)^2} \times \dfrac{x-b}{x(x+b)} =$

$\dfrac{x^4-b^4}{x(x+b)(x-b)} = \dfrac{x^4-b^4}{x(x^2-b^2)} = \dfrac{x^2+b^2}{x} = x + \dfrac{b^2}{x}$.

11. Divide $96 - 6a^4$ by $6 - 3a$. *Quot.* $16 + 8a + 4a^2 + 2a^3$.

12. Divide $10a^3 + 11a^2b - 19abc - 15a^2c + 3ab^2 + 15bc^2 - 5b^2c$ by $3ab + 5a^2 - 5bc$. *Quot.* $2a + b - 3c$.

13. Divide $x^2 + y^2 + \dfrac{y^4}{x^2}$ by $x + y + \dfrac{y^2}{x}$. *Quot.* $x - y + \dfrac{y^2}{x}$.

SECT. V. *Involution.*

1. *To involve or raise monomials to any proposed power.*

Rule.—Involve the coefficient to the power required, for a new coefficient. Multiply the index of each letter by the index of the required power. Place each product over its respective letter, and prefix the coefficient found as above: the result will be the power required.

All the powers of an *affirmative* quantity will be +: of a *negative* quantity, the *even* powers, as the 2nd, 4th, 6th, &c., will be +; the *odd* powers, as the 3rd, 5th, 7th, &c., will be −.

To involve fractions, apply these rules to both numerator and denominator.

The application of these rules is fully exemplified in the following table of

ROOTS AND POWERS OF MONOMIALS.

Root	a	$-b$	$\sqrt[3]{x}$	$\dfrac{c}{a}$	$-\dfrac{a^3}{b^4x}$	$\dfrac{2y}{1}$	$-a^7x^3$
Square	a^2	b^2	$\sqrt[3]{x^2}$	$\dfrac{c^2}{a^2}$	$\dfrac{a^6}{b^8x^2}$	$\dfrac{4y^2}{1}$	$a^{14}x^6$
Cube	a^3	$-b^3$	x	$\dfrac{c^3}{a^3}$	$-\dfrac{a^9}{b^{12}x^3}$	$\dfrac{8y^3}{1}$	$-a^{21}x^9$
4th Power	a^4	b^4	$\sqrt[3]{x^4}$	$\dfrac{c^4}{a^4}$	$\dfrac{a^{12}}{b^{16}x^4}$	$\dfrac{16y^4}{1}$	$a^{28}x^{12}$
5th Power	a^5	$-b^5$	$\sqrt[3]{x^5}$	$\dfrac{c^5}{a^5}$	$-\dfrac{a^{15}}{b^{20}x^5}$	$\dfrac{32y^5}{1}$	$-a^{35}x^{15}$

2. *To involve polynomials.*

Rule.—Multiply the given quantity into itself as many times, wanting one, as there are units in the index of the required power, and the last product will be the power required.

Example.

Cube $x \pm z$ and $2x - 3z$.

	$x \pm z$	$2x - 3z$
	$x \pm z$	$2x - 3z$
	$x^2 \pm xz$	$4x^2 - 6xz$
	$\pm xz + z^2$	$- 6xz + 9z^2$
Squares	$x^2 \pm 2xz + z^2$	$4x^2 - 12xz + 9z^2$
	$x \pm z$	$2x - 3z$
	$x^3 \pm 2x^2z + xz^2$	$8x^3 - 24x^2z + 18xz^2$
	$\pm x^2z + 2xz^2 + z^3$	$- 12x^2z + 36xz^2 - 27z^3$
Cubes	$x^3 \pm 3x^2z + 3xz^2 + z^3$...	$8x^3 - 36x^2z + 54xz^2 - 27z^3$

The operation required by the preceding rules, however simple in its nature, becomes tedious when even a binomial is raised to a high power. In such cases it is usual to employ

Sir Isaac Newton's Rule for involving a Binomial.

1. *To find the terms without the coefficients.*—The index of the first, or leading quantity, begins with that of the given power, and decreases continually by 1, in every term to the last; and in the following quantity the indices of the terms are 0, 1, 2, 3, 4, &c.

2. *To find the unciæ or coefficients.*—The first is always 1, the second is the index of the power: and, for the others, if the coefficient of any term be multiplied by the index of its leading quantity, and the product be divided by the number of terms to that place, it will give the coefficient of the term next following*.

Note.—The whole number of terms will be one more than the index of the given power; and when both terms of the root are +, all the terms of the power will be +; but if the second term of the root be −, all the odd terms will be +, and the even terms −.

* This rule, expressed in general terms, is as follows:—

$$(a+b)^n = a^n + n \cdot a^{n-1}b + n \cdot \frac{n-1}{2} a^{n-2}b^2 + n \cdot \frac{n+1}{2} \cdot \frac{n-2}{3} a^{n-3}b^3, \text{ \&c.}$$

$$(a-b)^n = a^n - n \cdot a^{n-1}b + n \cdot \frac{n-1}{2} a^{n-2}b^2 - n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} a^{n-3}b^3, \text{ \&c.}$$

The same theorem applied to fractional exponents, and with a slight modification, serves for the extraction of roots in infinite series; as will be shown a little farther on.

Examples.

1. Let $a + x$ be involved to the fifth power.

The terms, without the coefficients, will be

$$a^5,\ a^4x,\ a^3x^2,\ a^2x^3,\ ax^4,\ x^5,$$

and the coefficients will be

$$1,\ 5,\ \frac{5\times 4}{2},\ \frac{10\times 3}{3},\ \frac{10\times 2}{4},\ \frac{5\times 1}{5};$$

and therefore the fifth power is

$$a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5.$$

Here we have, for the sake of perspicuity, exhibited separately the manner of obtaining the several terms and their respective coefficients. But in *practice* the separation of the two operations is inconvenient. The best way to obtain the *coefficients* is to perform the *division* first, upon either the requisite coefficient or exponent (one or other of which may always be divided without a remainder), and to multiply the quotient into the other. Thus, the result may be obtained at once in a single line, nearly as rapidly as it can be written down.

2. $(x + y)^7 = x^7 + 7x^6z + 21x^5z^2 + 35x^4z^3 + 35x^3z^4 + 21x^2z^5 + 7xz^6 + z^7.$

3. $(x - z)^8 = x^8 - 8x^7z + 28x^6z^2 - 56x^5z^3 + 70x^4z^4 - 56x^3z^5 + 28x^2z^6 - 8xz^7 + z^8.$

For Trinomials and Quadrinomials.—Let two of the terms be regarded as *one*, and the remaining term or terms as the other; and proceed as above.

Example.

Involve $x + y - z$ to the fourth power.

Let x be regarded as one term of the binomial, and $y - z$ as the other: then will $(x + y - z)^4 = \{x + (y - z)\}^4 = x^4 + 4x^3(y - z) + 6x^2(y - z)^2 + 4x(y - z)^3 + (y - z)^4$, where the powers of $y - z$ being expanded by the same rule, and multiplied into their respective factors, we shall at length have $x^4 + 4x^3y - 4x^3z + 6x^2y^2 - 12x^2yz + 6x^2z^2 + 4xy^3 - 12xy^2z + 12xyz^2 - 4xz^3 + y^4 - 4y^3z + 6y^2z^2 - 4yz^3 + z^4$, the fourth power required.

Had $(x + y)$ and $- z$ been taken for the two terms of the binomial, the result would have been the same.

Note.—The rule for the involution of multinomials is too complex to be given in this place.

SECT. VI. *Evolution.*

1. *To find the roots of monomials.*

Rule.—Extract the corresponding root of the coefficient for the new coefficient: then multiply the index of the letter or letters by the index of the root, (fractionally expressed, as explained at page 4,) the result will be the exponents of the letter or letters to be placed after the coefficient for the root required.

Examples.

1. Find the fourth root of $81\,a^4 z^8$.
   First $\sqrt[4]{81} = \sqrt{9} = 3$, new coefficient.
   Then $4 \times \frac{1}{4} = 1$, exponent of a; and $8 \times \frac{1}{4} = 2$, exponent of z.
   Hence $3\,a\,z^2$ is the root required.

2. $\sqrt[5]{(32\,a^5 x^{10})} = \sqrt[5]{32} \times a^{5\times\frac{1}{5}} \times x^{10\times\frac{1}{5}} = 2\,a\,x^2$.

3. $\sqrt[3]{\frac{8\,x^3}{27\,z^9}} = \frac{\sqrt[3]{8} \times x^{3\times\frac{1}{3}}}{\sqrt[3]{27} \times z^{9\times\frac{1}{3}}} = \frac{2\,x}{3\,z^3}$.

2. *To find the square root of a polynomial.*

Proceed as in the extraction of the square root, in arithmetic, as explained at page 29.

Examples.

1. Extract the square root of $a^4 + 4\,a^3 x + 6\,a^2 x^2 + 4\,a\,x^3 + x^4$.

$$a^4 + 4\,a^3 x + 6\,a^2 x^2 + 4\,a\,x^3 + x^4 \,(a^2 + 2\,a\,x + x^2 \quad \textit{[root required.}$$
$$a^4$$

$$\begin{array}{l|l} 2a^2 + 2ax & 4\,a^3 x + 6\,a^2 x^2 \\ \quad 2ax & 4\,a^3 x + 4\,a^2 x^2 \end{array}$$

$$\begin{array}{l|l} 2a^2 + 4ax + x^2 & 2\,a^2 x^2 + 4\,a\,x^3 + x^4 \\ \quad x^2 & 2\,a^2 x^2 + 4\,a\,x^3 + x^4 \end{array}$$

* * *

2. Extract the square root of $x^4 - 2x^3 + \frac{3}{2}x^2 - \frac{1}{2}x + \frac{1}{16}$.

$$x^4 - 2x^3 + \tfrac{3}{2}x^2 - \frac{x}{2} + \frac{1}{16} \,(x^2 - x + \tfrac{1}{4} \quad \textit{root.}$$
$$x^4$$

$$\begin{array}{l|l} 2x^2 - x & -2x^3 + \frac{3}{2}x^2 \\ \quad x & -2x^3 + x^2 \end{array}$$

$$\begin{array}{l|l} 2x^2 - 2x + \frac{1}{4} & \frac{1}{2}x^2 - \frac{x}{2} + \frac{1}{16} \\ \quad \frac{1}{4} & \frac{1}{2}x^2 - \frac{x}{2} + \frac{1}{16} \end{array}$$

* * *

3. *To find the roots of powers in general.*

When the power (n) of the root to be extracted is not very high, the following method may be employed for extracting the nth root*.

Rule.—Range the several terms in the order of the powers of one of the unknown quantities; having done which, extract the nth root of the first or leading term, which place in the quotient, and cancel such term; then bring down the second term for a dividend, which divide by the term of the root just found, raised to the power of $(n-1)$, and multiplied by n, the quotient will be the second term of the root. Then to find the third term of the root, involve the two already obtained to the nth power as far as the *third* term, which subtract from the *third* term of the given power for a dividend, which divided by the same divisor, gives the *third* term of the required root. And, in like manner, to find the *fourth* term of the root, involve those already obtained to the nth power as far as the *fourth* term, which subtracted from the *fourth* term of the given power, gives a fresh dividend; and thus proceed until the root is extracted.

Examples.

1. Extract the cube root of $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$.

$x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$ ($x^2 - 2x + 1$ [*root required.*

x^6

$(x^2)^2 \times 3 = 3x^4$ | $* - 6x^5$

$x^6 - 6x^5 + 12x^4 = (x^2 - 2x)^3$ to the third term.

$3x^4$ | $*$ $*$ $3x^1$

$x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1 = (x^2 - 2x + 1)^3$

$*$ $*$ $*$ $*$ $*$

2. Find the 4th root of $16a^4 - 96a^3x + 216a^2x^2 - 216ax^3 + 81x^4$.

$16a^4 - 96a^3x + 216a^2x^2 - 216ax^3 + 81x^4$ ($2a - 3x$

$16a^4$

$8a^3 \times 4 = 32a^3$)$* - 96a^3x$

$16a^4 - 96a^3x + 216a^2x^2 - 216ax^3 + 81x^4 = (2a - 3x)^4$

$*$ $*$ $*$ $*$ $*$

* By this mode of expression is meant *any root whatever;* the rule may be made to apply to any particular case by substituting for n, the power of the root required, which in the first Example above is 3, and in the second is 4.

Note.—In the higher roots proceed thus:—

For the *biquadrate*, extract the *square root* of the *square root*.
„ *sixth root*, „ *cube root* of the *square root*.
„ *eighth root*, „ *sq. rt.* of the *sq. rt.* of the *sq. rt.*
„ *ninth root*, „ *cube root* of the *cube root*.

Examples, however, of such high roots seldom occur in any practical inquiries.

SECT. VII. *Surds.*

A *Surd*, or irrational quantity, is a quantity under a radical sign or fractional index, the root of which cannot be exactly obtained. (See ARITH. Sect. 9. *Evolution.*)

Surds, as well as other quantities, may be considered as either simple or compound, the first being *monomials*, as $\sqrt{3}$, $a^{\frac{1}{2}}$, $\sqrt[5]{a\,b^2}$, the others *polynomials*, as $\sqrt{3} + \sqrt{5}$, $\sqrt[3]{a} + \sqrt{b} - \sqrt{c^3}$, $\sqrt[3]{(d - \sqrt{3})}$, $\sqrt{a + 3}$, &c.

Rational quantities may be expressed in the *form* of surds, and the operation, when effected, often diminishes subsequent labour.

REDUCTION.

1. *To reduce surds into their simplest expressions.*

CASE I. *If the surd be not fractional, but consist of integers or integral factors under the radical sign:*

Rule.—Divide the given power by the greatest power, having the same index, contained therein, that measures it without remainder; let the quotient be affected by the radical sign, and have the root of the divisor prefixed as a coefficient, or connected by the sign $\times$.

Examples.

1. $\sqrt{75} = \sqrt{(25 \times 3)} = \sqrt{25} \times \sqrt{3} = 5\sqrt{3}$.
2. $\sqrt[3]{448} = \sqrt[3]{(64 \times 7)} = \sqrt[3]{64} \times \sqrt[3]{7} = 4\sqrt[3]{7}$.
3. $\sqrt[4]{176} = \sqrt[4]{(16 \times 11)} = \sqrt[4]{16} \times \sqrt[4]{11} = 2\sqrt[4]{11}$.
4. $\sqrt{(8x^3 - 12x^2y)} = \sqrt{4x^2(2x - 3y)} = \sqrt{4x^2} \times \sqrt{(2x - 3y)} = 2x\sqrt{(2x - 3y)}$.
5. $\sqrt[3]{108x^3y^4} = \sqrt[3]{(27x^3y^3 \times 4y)} = \sqrt[3]{27x^3y^3} \times \sqrt[3]{4y} = 3xy\sqrt[3]{4y}$.
6. $\sqrt[3]{(56x^3y + 8x^3)} = \sqrt[3]{8x^3(7y + 1)} = \sqrt[3]{8x^3} \times \sqrt[3]{(7y + 1)} = 2x\sqrt[3]{(7y + 1)}$.

CASE II. *If the surd be fractional, it may be reduced to an equivalent integral one, thus:—*

Rule.—Multiply the numerator of the fraction under the radical sign by that power of its denominator whose exponent is one less

than the exponent of the surd, and place it as a whole number under the radical sign, and prefix to it for a coefficient, the coefficient of the original surd (whether unity or any quantity) divided by the denominator of the given fraction taken from under the radical sign.

Note.—This reduction saves the labour of actually dividing by an approximated root; and will often enable the student to value any surd expressions by means of a table of roots of integers.

Examples.

1. $\sqrt{\frac{1}{3}}$; then $1 \times 3 = 3$, and $\frac{1}{3}\sqrt{3} = \sqrt{\frac{1}{3}}$.
2. $\sqrt{\frac{1}{5}}$; then $1 \times 5 = 5$, and $\frac{1}{5}\sqrt{5} = \sqrt{\frac{1}{5}}$.
3. $\sqrt{\frac{1}{n}}$; then $1 \times n = n$, and $\frac{1}{n}\sqrt{n} = \sqrt{\frac{1}{n}}$.
4. $\sqrt[3]{\frac{4}{5}}$; then $4 \times 5^2 = 100$, and $\frac{1}{5}\sqrt[3]{100} = \sqrt[3]{\frac{4}{5}}$.
5. $\sqrt[3]{\frac{2a}{5x}}$; then $2a \times \overline{5x}\,|^2 = 50\,a\,x^2$; and $\frac{1}{5x}\sqrt[3]{50\,a\,x^2} = \sqrt[3]{\frac{2a}{5x}}$.
6. $\sqrt[3]{\frac{16}{81}} = 2\sqrt[3]{\frac{2}{81}}$; then $2^3 \times 81^2 = 13122$, and $\frac{2}{81}\sqrt[3]{13122} = \frac{2}{9}\sqrt[3]{18} = \sqrt[3]{\frac{16}{81}}$.

CASE III. *If the denominator of the fraction be a binomial or residual, of which one or both terms are irrational and roots of squares:*

Rule.— Multiply this fraction by another which shall have its numerator and denominator alike, and each to contain the same two quantities as the denominator of the given expression, but connected with a different sign.

Note 1.—By means of this rule, since any fraction whose numerator and denominator are the same, is equal to *unity*, the quantity to be reduced assumes a new appearance without changing its value; while the expression becomes freed from the surds in the denominator, because the product of the sum and difference of two quantities is equal to the difference of their squares.

Examples.

1. $\frac{8}{\sqrt{5}-\sqrt{3}} = \frac{8}{\sqrt{5}-\sqrt{3}} \cdot \frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}} = \frac{8(\sqrt{5}+\sqrt{3})}{2}$
$= 4(\sqrt{5}+\sqrt{3})$.

2. $\frac{3}{\sqrt{5}+\sqrt{2}} = \frac{3}{\sqrt{5}+\sqrt{2}} \cdot \frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}} = \frac{3(\sqrt{5}-\sqrt{2})}{3}$
$= \sqrt{5}-\sqrt{2}$.

3. $\dfrac{\sqrt{20}-\sqrt{12}}{\sqrt{5}+\sqrt{3}}=\dfrac{\sqrt{20}-\sqrt{12}}{\sqrt{5}+\sqrt{3}}\cdot\dfrac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-\sqrt{3}}$

$=\dfrac{\sqrt{100}-2\sqrt{60}+\sqrt{36}}{5-3}=\dfrac{16-2\sqrt{60}}{2}=8-\sqrt{60}=8-2\sqrt{15}.$

4. $\dfrac{\sqrt{ab}}{\sqrt[4]{5}+\sqrt[4]{3}}=\dfrac{\sqrt{ab}}{\sqrt[4]{5}+\sqrt[4]{3}}\cdot\dfrac{\sqrt[4]{5}-\sqrt[4]{3}}{\sqrt[4]{5}-\sqrt[4]{3}}=\dfrac{\sqrt[4]{5}-\sqrt[4]{3}}{\sqrt{5}-\sqrt{3}}\sqrt{ab}$

$=\dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}\cdot\dfrac{\sqrt[4]{5}-\sqrt[4]{3}}{\sqrt{5}-\sqrt{3}}\sqrt{ab}=(5^{\frac{1}{2}}-5^{\frac{1}{4}}3^{\frac{1}{4}}+5^{\frac{1}{4}}3^{\frac{1}{4}}-3^{\frac{1}{2}}).(\tfrac{1}{2}\sqrt{ab}).$

Note 2.—Upon the same general principle *any* binomial or residual surd, as $\sqrt[n]{A}\pm\sqrt[n]{B}$ may be rendered rational by taking $\sqrt[n]{A^{n-1}}\mp\sqrt[n]{(A^{n-2}B)}+\sqrt[n]{(A^{n-3}B^2)}\mp\sqrt[n]{(A^{n-4}B^3)}+$ &c. for a multiplier: where the upper signs must be taken with the upper, the lower with the lower, and the series continued to n terms.

Thus, the expression $\sqrt[4]{a^3}-\sqrt[4]{b^3}$, multiplied by $\sqrt[4]{a^2}+\sqrt[4]{a\,b}+\sqrt[4]{a^2b^2}+\sqrt[4]{b^2}$, gives the rational product a^3-b^3.

2. *To reduce surds having different exponents to equivalent ones that have a common exponent.*

Involve the powers reciprocally, according to each other's exponent, for new powers: and let the product of the exponents be the common exponent.

Note.—Hence, rational quantities may be reduced to the *form* of any assigned root; and roots with rational coefficients may be so reduced as to be brought entirely under the radical sign.

Examples.

1. $a^{\frac{1}{n}}$ and $b^{\frac{1}{m}}$, become $a^{\frac{1}{n}\cdot\frac{m}{m}}$ or $a^{\frac{m}{mn}}$ and $b^{\frac{1}{m}\cdot\frac{n}{n}}$ or $b^{\frac{n}{mn}}$.

2. $a^{\frac{1}{2}}$ and $b^{\frac{1}{3}}$, become $a^{\frac{1}{2}\cdot\frac{3}{3}}$ or $a^{\frac{3}{6}}$ and $b^{\frac{1}{2}\cdot\frac{2}{2}}$ or $b^{\frac{2}{6}}$.

3. $3^{\frac{1}{2}}$ and $2^{\frac{1}{3}}$, become $3^{\frac{3}{6}}$ and $2^{\frac{2}{6}}$, or $\sqrt[6]{3^3}$ and $\sqrt[6]{2^2}$, or $\sqrt[6]{27}$ and $\sqrt[6]{4}$.

4. $(a+b)^{\frac{1}{3}}$, and $(a-b)^{\frac{1}{4}}$, become $\sqrt[12]{(a+b)^4}$ and $\sqrt[12]{(a-b)^3}$.

5. The rational quantity a^2, becomes $\sqrt{a^4}$, $\sqrt[3]{a^6}$, $\sqrt[4]{a^8}$, or $\sqrt[n]{a^{2n}}$.

6. $4a\sqrt[3]{5\,b}$, becomes $\sqrt[3]{(4\,a)^3}\times\sqrt[3]{5\,b}$, $\sqrt[3]{64\,a^3}\times\sqrt[3]{5\,b}$, or $\sqrt[3]{320\,a^3\,b}$.

These and other obvious reductions which will at once suggest themselves, being effected, the operations of addition, subtraction, &c. are so easily performed upon such surd quantities as usually occur, that it will suffice merely to present a few examples without detailing rules.

ADDITION.

Ex. 1. $\sqrt{8} + \sqrt{18} = \sqrt{(4.2)} + \sqrt{(9.2)} = 2\sqrt{2} + 3\sqrt{2} = 5\sqrt{2}$.

2. Add together $\sqrt{54}$, $\sqrt{\frac{1}{6}}$, and $\sqrt{\frac{8}{27}}$.

$\sqrt{54} = \sqrt{(9.6)} = \sqrt{9} \times \sqrt{6} = 3\sqrt{6}$
$\sqrt{\frac{1}{6}} = \sqrt{\frac{6}{36}} = \sqrt{\frac{1}{36}} \times \sqrt{6} = \frac{1}{6}\sqrt{6}$
$\sqrt{\frac{8}{27}} = \sqrt{(\frac{4.2}{27} \cdot \frac{3}{3})} = \sqrt{\frac{4.6}{81}} = \frac{2}{9}\sqrt{6}$

The sum of these is $(3 + \frac{1}{6} + \frac{2}{9})\sqrt{6} = 3\frac{7}{18}\sqrt{6}$.

3. $\sqrt{27 a^4 x} + \sqrt{3 a^2 x} = \sqrt{(9 a^4 . 3 x)} + (a^2 . 3 x) = 3 a^2 \sqrt{3 x} + a\sqrt{3 x} = (3 a^2 + a)\sqrt{3 x}$.

4. $8\sqrt[3]{a^3 b} + \sqrt[3]{a^6 b} = (8\sqrt[3]{a^3} \times \sqrt[3]{b}) + (\sqrt[3]{a^6} \times \sqrt[3]{b}) = 8 a \sqrt[3]{b} + a^2 \sqrt[3]{b} = (8 a + a^2)\sqrt[3]{b}$.

SUBTRACTION.

Ex. 1. $2\sqrt{50} - \sqrt{18} = 2\sqrt{(25 . 2)} - \sqrt{(9 . 2)} = 2 . 5\sqrt{2} - 3\sqrt{2} = (10 - 3)\sqrt{2} = 7\sqrt{2}$.

2. $\sqrt{\frac{3}{5}} - \sqrt{\frac{5}{27}} = \sqrt{(\frac{3}{5} \cdot \frac{5}{5})} - \sqrt{(\frac{5}{27} \cdot \frac{3}{3})} = \sqrt{\frac{15}{25}} - \sqrt{\frac{15}{81}} = \frac{1}{5}\sqrt{15} - \frac{1}{9}\sqrt{15} = \frac{4}{45}\sqrt{15}$.

3. $\sqrt[3]{\frac{2}{3}} - \sqrt[3]{\frac{9}{32}} = \sqrt[3]{(\frac{2}{3} \cdot \frac{9}{9})} - \sqrt[3]{\frac{9}{32} \cdot \frac{2}{2})} = \sqrt[3]{\frac{18}{27}} - \sqrt[3]{\frac{18}{64}} = \frac{1}{3}\sqrt[3]{18} - \frac{1}{4}\sqrt[3]{18} = \frac{1}{12}\sqrt[3]{18}$.

4. $\sqrt[3]{250 a^4 x} - \sqrt[3]{16 a^3 x} = \sqrt[3]{(125 a^3 . 2 x)} - \sqrt[3]{(8 a^3 . 2 x)} = 5 a \sqrt[3]{2 x} - 2 a \sqrt[3]{2 x} = 3 a \sqrt[3]{2 x}$.

5. $\sqrt{45 s^4 x} - \sqrt{20 s^2 x^3} = \sqrt{(9 s^4 . 5 x)} - \sqrt{(4 s^2 x^2 . 5 x)} = (3 s^2 - 2 s x)\sqrt{5 x}$.

6. $\left(\frac{a}{c}\right)^{\frac{1}{2}} - \left(\frac{c}{a}\right)^{\frac{1}{2}} = \frac{a}{1} \times \left(\frac{1}{a c}\right)^{\frac{1}{2}} - \frac{c^2}{a} \times \left(\frac{1}{a c}\right)^{\frac{1}{2}}$
$= \frac{a^2 - c^2}{a}\left(\frac{1}{a c}\right)^{\frac{1}{2}}$.

MULTIPLICATION.

Ex. 1. $\sqrt[3]{18} \times 5\sqrt[3]{4} = 5\sqrt[3]{(18 . 4)} = 5\sqrt[3]{(4 . 2 . 9)} = 5\sqrt[3]{(8 . 9)} = 5 . 2\sqrt[3]{9} = 10\sqrt[3]{9}$.

2. $\frac{2}{3}\sqrt{\frac{1}{8}} \times \frac{3}{4}\sqrt{\frac{7}{10}} = \frac{2}{3} \cdot \frac{3}{4}\sqrt{(\frac{1}{8} \cdot \frac{7}{10})} = \frac{1}{2}\sqrt{\frac{7}{80}} = \frac{1}{2}\sqrt{(\frac{7}{80} \cdot \frac{5}{5})} = \frac{1}{2}\sqrt{\frac{35}{400}} = \frac{1}{2} \cdot \frac{1}{20}\sqrt{35} = \frac{1}{40}\sqrt{35}$.

3. $a^{\frac{2}{3}} \times a^{\frac{1}{4}} = a^{\frac{2}{3} + \frac{1}{4}} = a^{\frac{8}{12} + \frac{3}{12}} = a^{\frac{11}{12}}$.

4. $(x + z)^{\frac{1}{2}} \times (x + z)^{\frac{1}{3}} = (x + z)^{\frac{1}{2} + \frac{1}{3}} = (x + z)^{\frac{5}{6}}$.

5. $(x + \sqrt{y}) \times (x - \sqrt{y}) = x^2 - y$.

6. $(x + \sqrt{y})^{\frac{1}{3}} \times (x - \sqrt{y})^{\frac{1}{3}} = (x^2 - y)^{\frac{1}{3}}$.

7. $z^{\frac{1}{n}} \times z^{\frac{1}{m}} = z^{\frac{1}{n} + \frac{1}{m}} = z^{\frac{m+n}{mn}}$.

8. $\sqrt{d} \times \sqrt[3]{ab} = \sqrt[6]{d^3} \times \sqrt[6]{a^2 b^2} = \sqrt[6]{a^2 b^2 d^3}$.

9. $\sqrt{a - \sqrt{b - \sqrt{3}}} \times \sqrt{a + \sqrt{b - \sqrt{3}}} = \sqrt{(a^2 - b + \sqrt{3})}$.

10. $a^m \times a^{-n} = a^m \times \frac{1}{a^n} = a^{m-n}$.

11. $\sqrt{-a} \times \sqrt{-a} = \sqrt{a}\sqrt{-1} \times \sqrt{a}\sqrt{-1} = a \times -1 = -a$.

12. $\sqrt{-a} \times \sqrt{-b} = \sqrt{a}\sqrt{-1} \times \sqrt{b}\sqrt{-1} = \sqrt{ab} \times -1 = -\sqrt{ab}$.

DIVISION.

Ex. 1. $\sqrt[3]{1000} \div 2\sqrt[3]{4} = \frac{1}{2}\sqrt[3]{\frac{1000}{4}} = 2\sqrt[3]{250} = 2\sqrt[3]{(125 \cdot 2)} = 10\sqrt[3]{2}$.

2. $\frac{4}{7}\sqrt[3]{\frac{8}{5}} \div \frac{2}{5}\sqrt[3]{\frac{3}{4}} = \frac{4}{7} \cdot \frac{5}{2}\sqrt[3]{(\frac{8}{5} \cdot \frac{4}{3})} = \frac{20}{14}\sqrt[3]{\frac{8}{9}} = \frac{20}{14}\sqrt[3]{(\frac{8}{9} \cdot \frac{3}{3})} = \frac{10}{14}\sqrt[3]{(\frac{8}{27} \cdot 3)} = \frac{20}{14} \cdot \frac{2}{3}\sqrt[3]{3} = \frac{20}{21}\sqrt[3]{3}$.

3. $x^{\frac{2}{3}} \div x^{\frac{4}{3}} = x^{\frac{2}{3} - \frac{4}{3}} = x^{-\frac{2}{3}} = 1 \div x^{\frac{2}{3}}$.

4. $x^{\frac{1}{m}} \div x^{\frac{1}{n}} = x^{\frac{1}{m} - \frac{1}{n}} = x^{\frac{n-m}{mn}}$.

5. $(x^2 - xd - b + d\sqrt{b}) \div (x - \sqrt{b}) = x + \sqrt{b} - d$.

6. $\frac{ac - ad}{2b}\sqrt{(a^2 x - ax^2)} \div \frac{a}{2b}\sqrt{(a - x)} = \frac{ac - ad}{2b} \times \frac{2b}{a} \times \sqrt{\frac{a^2 x - ax^2}{a - x}} = (c - d)\sqrt{ax}$.

INVOLUTION.

Ex. 1. $(\frac{2}{3}a^{\frac{1}{3}})^2 = \frac{2}{3} \cdot \frac{2}{3} \cdot a^{\frac{1}{3}+\frac{1}{3}} = \frac{4}{9}a^{\frac{2}{3}} = \frac{4}{9}\sqrt[3]{a^2}$.

2. $(\frac{1}{3}\sqrt{\frac{1}{2}})^3 = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \sqrt{(\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2})} = \frac{1}{27}\sqrt{\frac{1}{8}} = \frac{1}{27}\sqrt{(\frac{1}{8} \cdot \frac{2}{2})} = \frac{1}{27}\sqrt{(\frac{1}{16} \cdot 2)} = \frac{1}{27} \cdot \frac{1}{4}\sqrt{2} = \frac{1}{108}\sqrt{2}$.

3. $(3 + \sqrt{5})^2 = \{(3 + \sqrt{5})(3 + \sqrt{5})\} = 14 + 6\sqrt{5}$.

4. $(a - \sqrt{b})^3 = a^3 - 3a^2\sqrt{b} + 3ab - b\sqrt{b}$.

EVOLUTION.

Ex. 1. $\sqrt{10^3} = \sqrt{1000} = \sqrt{(100 \cdot 10)} = \sqrt{100} \times \sqrt{10} = 10\sqrt{10}$.

2. $\sqrt[4]{81\, a^4 y^6 z} = \sqrt[4]{(81\, a^4 y^4 \cdot y^2 z)} = 9a^2 y^2 \sqrt[4]{y^2 z}$.

3. $\sqrt{(a^2 - 4a\sqrt{b} + b^2)} = a - 2\sqrt{b}$, the operation being performed as in the arithmetical extraction of the square root.

Note.—The *square* root of a binomial or residual $a \pm b$, or even of a trinomial or quadrinomial, may often be conveniently extracted thus:—Take $d = \sqrt{(a^2 - b^2)}$; then $\sqrt{(a \pm b)} = \sqrt{\frac{a+d}{2}} \pm \sqrt{\frac{a-d}{2}}$. This is evident: for, if $\sqrt{\frac{a+d}{2}} \pm$

$\sqrt{\frac{a-d}{2}}$ be squared, it will give $a + \sqrt{(a^2 - d^2)}$ or $a + b$, as it ought: and, in like manner, the square of $\sqrt{\frac{a+d}{2}} - \sqrt{\frac{a-d}{2}}$, is $a - \sqrt{(a^2 - d^2)}$, or $a - b$.

Ex. 1. Find the square root of $3 + 2\sqrt{2}$.

Here $a = 3$, $b = 2\sqrt{2}$, $d = \sqrt{(9 - 8)} = 1$,

And $\sqrt{\frac{a+d}{2}} + \sqrt{\frac{a-d}{2}} = \sqrt{\frac{3+1}{2}} + \sqrt{\frac{3-1}{2}}$

$= \sqrt{\frac{4}{2}} + \sqrt{\frac{2}{2}} = \sqrt{2} + \sqrt{1} = 1 + \sqrt{2}$.

2. Find the square root of $6 - 2\sqrt{5}$.

Here $a = 6$, $b = 2\sqrt{5}$, $d = \sqrt{(36 - 20)} = \sqrt{16} = 4$,

And $\sqrt{\frac{a+d}{2}} - \sqrt{\frac{a-d}{2}} = \sqrt{\frac{6+4}{2}} - \sqrt{\frac{6-4}{2}}$

$= \sqrt{\frac{10}{2}} - \sqrt{\frac{2}{2}} = \sqrt{5} - 1$.

3. Find the square root of $6 + \sqrt{8} - \sqrt{12} - \sqrt{24}$.

Here $a = 6 + \sqrt{8}$, $b = \sqrt{12} + \sqrt{24}$, $d = \sqrt{(6 + \sqrt{8})^2 - (\sqrt{12} + \sqrt{24})^2} = \sqrt{(44 + 12\sqrt{8} - 36 - 2\sqrt{12} \cdot 24)} = \sqrt{(44 - 36 + 12\sqrt{8} - 12\sqrt{8})} = \sqrt{8}$.

Conseq. $\frac{a+d}{2} = \frac{6 + 2\sqrt{8}}{2} = 3 + \sqrt{8}$, and $\frac{a-d}{2} = \frac{6 + \sqrt{8} - \sqrt{8}}{2} = 3$.

But (*Ex.* 1), $\sqrt{(3 + 2\sqrt{2})} = \sqrt{(3 + \sqrt{8})} = 1 + \sqrt{2}$. Therefore the root required is $1 + \sqrt{2} - \sqrt{3}$*.

SECT. VIII. *Simple Equations.*

AN algebraic equation is an expression by which two quantities, called members (whether simple or compound), are indicated to be equal to each other, by means of the sign of equality (=) placed between them.

In equations consisting of known and unknown quantities, when the unknown quantity is expressed by a simple power, as x, x^2, x^3, &c., they are called *simple equations*, generally; and particularly, simple or *pure quadratics*, *cubics*, &c. according to the exponent of the un-

* For the cube and higher roots of binomials, &c. the reader may consult the treatises on Algebra, by Maclaurin, Emerson, Lacroix, Bonnycastle, J. R. Young, and Hine.

known quantity. But when the unknown quantity appears in two or more different powers in the same equation, it is named an *adfected* equation. Thus $x^2 = a + 15$, is a simple quadratic equation: $x^2 + a x = b$, an adfected quadratic.

It is the former class of equations that we shall first consider.

The reduction of an equation consists in so managing its terms, that, at the end of the process, the *unknown* quantity may stand alone, and in its first power, on one side of the sign =, with the *known* quantities, whether denoted by letters or figures, on the other. Thus, what was previously unknown is now affirmed to be *equal* to the aggregate of the terms in the second number of the equation.

"In general, *the unknown quantity is disengaged from the known ones, by performing upon both members the* REVERSE OPERATIONS,"* to those indicated by the equation, whatever they may be. Thus,

If any known quantity be found added to the unknown quantity, let it be subtracted from both members or sides of the equation; or if any such quantity be found subtracted, let it be added†.

If the unknown quantity have a multiplier, let the equation be divided by it.

If it be found divided by any quantity, let that become the multiplier.

If any power of the unknown quantity be given, take the corresponding root; or if any root, find the corresponding power.

If the unknown quantity be found in the terms of a proportion (*Arith.* Sect. 10), let the respective products of the means and extremes constitute an equation; and then apply the general principle, as above.

Examples.

1. Given $x - 3 + 5 = 9$, to find x.

First, by adding 3 to both sides, } we have $x + 5 = 9 + 3 = 12$.

Then, by subtracting 5, } $x + 5 - 5 = 12 - 5$, or $x = 7$.

Otherwise, in appearance only, not in effect,

By transposing the 3, and changing its sign, $x + 5 = 9 + 3$.

By transposing the 5, and changing its sign, $x = 9 + 3 - 5 = 7$.

2. Given $3x + 5 = 20$, to find x.

First, by transposing the 5, $3x = 20 - 5 = 15$.

by dividing by 3, $x = \frac{15}{3} = 5$.

* This simple direction, comprehending the seven or eight particular rules for the reduction of equations given by most writers on algebra, from the time of Newton down to the present day, is due to Dr. Hutton. It is obviously founded upon the mathematical axiom, that equal operations performed upon equal things produce equal results.

† These two operations constitute what is usually denominated *transposition*, because the operation of thus adding or subtracting any quantity from each side of the equation is most simply performed by *transposing* it from one side of the equation to the other, and changing its sign.

3. Given $\frac{x}{a} + d = 3b - 2c$, to find x.

First, transposing d, $\frac{x}{a} = 3b - 2c + d$.

Then, multiplying by a, $x = 3ab - 2ac + ad$.

4. Given $\sqrt[3]{(3x + 4)} + 2 = 6$, to find x.

First, transposing the 2, $\sqrt[3]{(3x + 4)} = 6 - 2 = 4$.

Then, cubing, $3x + 4 = 4^3 = 64$.

Then, transposing the 4, $3x = 64 - 4 = 60$.

Lastly, dividing by 3, $x = \frac{60}{3} = 20$.

5. Given $4ax - 5b = 3dx + 4c$, to find x.

First, transposing $5b$ and $3dx$, $4ax - 3dx = 5b +$

Then, by collecting the coefficients, $(4a - 3d)x = 5b +$

$\therefore$ by dividing by $4a - 3d$, $x = \frac{5b + 4c}{4a - 3d}$.

6. Given $\frac{1}{4}x + \frac{1}{5}x - \frac{1}{6}x = 3$, to find x.

Multiplying by $120 = 4 \times 5 \times 6$, we have $30x + 24x - 20x = 360$.

That is, collecting the coefficients, $34x = 360$.

$\therefore$ dividing by 34, $x = \frac{360}{34} = \frac{180}{17} = 10\frac{10}{17}$.

7. Given $\frac{2}{4}x : a :: 5b : 3c$, to find x.

Mult. means and extremes, $\frac{2}{4}cx = 5ab$,

Dividing by $\frac{2}{4}c$, $x = 5ab \div \frac{2}{4}c = \frac{20ab}{9c}$.

8. Given $a + x = \sqrt{a^2 + x\sqrt{(4b^2 + x^2)}}$, to find x.

First, by squaring, we have, $a^2 + 2ax + x^2 =$ $x\sqrt{(4b^2 + x^2)}$

Then, striking out a^2 from both sides, $2ax + x$ $x\sqrt{(4b^2 + x^2)}$

dividing by x, $2a + x = \sqrt{(4b^2 + x^2)}$

squaring, $4a^2 + 4ax + x^2 = 4b^2 + x^2$

striking out x^2, and transposing $4a^2$, $4ax = 4b^2 - 4a^2$

dividing by $4a$, $x = \frac{4b^2 - 4a^2}{4a} = \frac{b^2}{a} - a$.

9. Given $\sqrt{cx - ac} = b + \sqrt{x - a}$, to find x.

First, dividing by $\sqrt{x - a}$, we have $\sqrt{c} = \frac{b}{\sqrt{(x - a)}}$

transposing the 1, $\sqrt{c} - 1$, or $\frac{\sqrt{c} - 1}{1} = \frac{b}{\sqrt{(x - a)}}$

inverting and transposing the fractions, $\frac{\sqrt{(x - a)}}{b} = \frac{1}{\sqrt{c} - 1}$

multiplying by b, $\sqrt{(x - a)} = \frac{b}{\sqrt{c} - 1}$

squaring both sides, $x - a = \frac{b^2}{c - 2\sqrt{c} + 1}$

transposing a, $x = a + \frac{b^2}{c - 2\sqrt{c} + 1}$

10. Given $13 - \sqrt{3}x = \sqrt{13 + 3x}$, to find x.

Ans. $x = 12$.

11. Given $y + \sqrt{4 + y^2} = \frac{8}{\sqrt{(4 + y^2)}}$, to find y.

Ans. $y = \frac{2}{3}\sqrt{3}$.

12. Given $\frac{1}{3}(x + 1) + \frac{1}{4}(x + 3) = \frac{1}{5}(x + 4) + 16$, to find x.

Ans. $x = 41$.

13. Given $\sqrt{\frac{3x}{2}} : \sqrt{(x - 1)} :: 3 : 1$, to find x. *Ans.* $1\frac{1}{2}$.

14. Given $(b^4 + x^4)^{\frac{1}{2}} = (a^2 + x^2)^{\frac{1}{2}}$, to find x.

Ans. $x = \frac{1}{a}\sqrt{\frac{b^4 - a^4}{2}}$.

EXTERMINATION.

When two or more unknown quantities occur in the consideration of an algebraical problem, they are determinable by a series of given independent equations. In order, however, that specific and finite solutions may be obtained, this condition must be observed, that there be *given as many independent equations as there are unknown quantities*. For, if the number of independent equations be fewer than the unknown quantities, the question proposed will be susceptible of an indefinite number of solutions*: while, on the other hand, a greater number of independent equations than of unknown quantities, indicates the impossibility or the absurdity of the thing attempted.

Where two unknown quantities are to be determined from two independent equations, one or other of the following rules may be employed.

* This, though generally true, has one striking exception, namely, in the case of equations constituted partly of rational quantities and partly of quadratic surds; where two unknown quantities are determinable by one equation, four unknown quantities by two equations; and so on.

Thus, if $x + \sqrt{y} = a + \sqrt{b}$
and $x - \sqrt{v} = c - \sqrt{d}$
Then $x = a$, $y = b$, $x = c$, $v = d$.

Rule 1.—Find the value of one of the unknown letters in each of the given equations; make those two values equal to one another in a third equation, and from thence deduce the value of the other unknown letter. This substituted for it in either of the former equations, will lead to the determination of the first unknown quantity.

2. Find the value of either of the unknown quantities in one of the equations, and substitute this value for it in the other equation: so will the other unknown quantity become known, and then the first, as before.

3. Or, after due reduction when requisite, multiply the first equation by the coefficient of one of the unknown quantities in the second equation, and the second equation by the coefficient of *the same* unknown quantity in the first equation: then the addition or subtraction of the resulting equations (according as the signs of the unknown quantity whose coefficients are now made equal, are unlike or like) will exterminate that unknown quantity, and lead to the determination of the other by former rules.

Notes.—The third rule is usually the most commodious and expeditious in practice.

The same precepts may be applied, *mutatis mutandis*, to equations comprising three, four, or more unknown quantities: and they often serve to *depress* equations, or reduce them from a higher to a lower degree.

Examples.

1. Given $4x^3 + 3y = 41$, and $3x^3 - 4y = 12$, to find x and y.
1st equa. $\times$ by 3, gives $12x^3 + 9y = 123$
2nd equa. $\times$ by 4, gives $12x^3 - 16y = 48$.
The difference of these, $25y = 75$, whence $y = 3$.
Then, from 2nd equa., $3x^3 = 12 + 4y = 12 + 12 = 24$
Whence dividing by 3, $x^3 = 8$, or $x = 2$.

Ex. 2. Given $x + y + z = 53$, $x + 2y + 3z = 105$, and $x + 3y + 4z = 134$.

1. $x + y + z = 53$
2. $x + 2y + 3z = 105$
3. $x + 3y + 4z = 134$
4. 1st equa. taken from 2nd, gives $y + 2z = 52$
5. 2nd equa. taken from 3rd, $y + z = 29$
6. 5th equa. taken from 4th, $z = 23$
7. 6th equa. taken from 5th, $y = 6$
8. 5th equa. taken from 1st, $x = 24$.

Ex. 3. Given $x + y = a$, $x + z = b$, $y + z = c$, to find x, y, and z.

1. $x + y = a$
2. $x + z = b$
3. $y + z = c$

4. 1st + 2nd + 3rd, gives $2x + 2y + 2z = a + b + c$.
5. Half 4th equa. gives $x + y + z = \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}c$.
6. 3rd equa. taken from 5th, gives $x = \frac{1}{2}a + \frac{1}{2}b - \frac{1}{2}c$.
7. 2nd equa. taken from 5th, $y = \frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}c$.
8. 1st equa. taken from 5th, $z = -\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}c$.

Ex. 4. Given $ax + by = c$, and $a'x + b'y = c'$, to find x and y.

Ans. $x = \dfrac{cb' - bc'}{ab' - ba'}$ and $y = \dfrac{ac' - ca'}{ab' - ba'}$.

Ex. 5. Given $ax + by + cz = d$, $a'x + b'y + c'z = d'$ $a''x + b''y + c''z = d''$ to find x, y, and z.

Ans. $$x = \frac{db'c'' - dc'b'' + cd'b'' - bd'c'' + bc'd'' - cb'd''}{ab'c'' - ac'b'' + ca'b'' - ba'c'' + bc'a'' - cb'a''}.$$

$$y = \frac{ad'c'' - ac'd'' + ca'd'' - da'c'' + dc'a'' - cd'a''}{ab'c'' - ac'b'' + ca'b'' - ba'c'' + bc'a'' - cb'a''}.$$

$$z = \frac{ab'd'' - ad'b'' + da'b'' - ba'd'' + bd'a'' - db'a''}{ab'c'' - ac'b'' + ca'b'' - ba'c'' + bc'a'' - cb'a''}.$$

Ex. 6. Given $\left.\begin{aligned} x(x + y + z) &= 18 \\ y(x + y + z) &= 27 \\ z(x + y + z) &= 36 \end{aligned}\right\}$ to find x, y, and z.

Ans. $x = 2$, $y = 3$, $z = 4$.

Ex. 7. Given $(x + y)\dfrac{x}{y} = 60$, and $(x + y)\dfrac{y}{x} = 2\frac{2}{3}$, to find x and y.

Ans. $x = 10$, $y = 2$.

Ex. 8. Given $\left.\begin{aligned} \tfrac{1}{2}x + \tfrac{1}{3}y + \tfrac{1}{4}z &= 62 \\ \tfrac{1}{3}x + \tfrac{1}{4}y + \tfrac{1}{5}z &= 47 \\ \tfrac{1}{4}x + \tfrac{1}{5}y + \tfrac{1}{6}z &= 38 \end{aligned}\right\}$ to find x, y, and z.

Ans. $x = 24$, $y = 60$, $z = 120$.

SOLUTION OF GENERAL PROBLEMS.

A general algebraic problem is that in which all the quantities concerned, both known and unknown, are expressed by letters, or other general characters. Not only such problems as have their conditions *proposed in general terms* are here implied; but every *particular* numeral problem may be *made general*, by substituting letters for the known quantities concerned in it; when this is done, the problem which was originally proposed in a *particular* form becomes *general*.

In solving a problem algebraically, some letter of the alphabet must be substituted for an unknown quantity. And if there be more unknown quantities than one, the second, third, &c., must either be expressed by means of their dependence upon the first and one or other of the data conjointly, or by so many distinct letters. Thus, as many separate equations will be obtained, the resolution of which,

by some of the foregoing rules, will lead to the determination of the quantities required.

Examples.

1. Given the sum of two magnitudes, and the difference of their squares, to find those magnitudes separately.

Let the given sum be denoted by s, the difference of the squares by D; and let the two magnitudes be represented by x and y respectively.

Then, the first condition of the problem expressed algebraically is $x + y = s$.

And the second is $x^2 - y^2 = \text{D}$.

Equa. 2 divided by equa. 1, gives $x - y = \frac{\text{D}}{s}$

Equa. 1 added to equa. 3, gives $2x = \frac{\text{D}}{s} + s = \frac{s^2 + \text{D}}{s}$

Equa. 4 divided by 2, gives $x = \frac{s^2 + \text{D}}{2s}$

Equa. 5 taken from equa. 1, gives $y = s - \frac{s^2 + \text{D}}{2s} = \frac{s^2 - \text{D}}{2s}$

To apply this general solution to a particular example, suppose the sum to be 6, and the difference of the squares 12. Then $s = 6$ and $\text{D} = 12$,

$$\text{and } x = \frac{s^2 + \text{D}}{2s} = \frac{36 + 12}{12} = \frac{48}{12} = 4.$$

$$\text{and } y = \frac{s^2 - \text{D}}{2s} = \frac{36 - 12}{12} = \frac{24}{12} = 2.$$

Suppose, again, $s = 5$, $\text{D} = 5$:

$$\text{then } x = \frac{25 + 5}{10} = 3, \text{ and } y = \frac{25 - 5}{10} = 2$$

Ex. 2. Given the product of two numbers, and their quotient, to find the numbers.

Let the given product be represented by p, the quotient by q; and the required numbers by x and y, as before.

Then we have, 1. $xy = p$,

$$\text{and } 2. \quad \frac{x}{y} = q.$$

Equa. 2 $\times$ by y, gives, $x = qy$

Substituting this value of x for it in equa. 1 $\Big\}$ $qy^2 = p$

Dividing by q, $y^2 = \frac{p}{q}$

Extracting the square root, $y = \sqrt{\frac{p}{q}}$

Then, by substitution, $x = qy = q\sqrt{\frac{p}{q}} = \sqrt{\frac{pq^2}{q}} = \sqrt{pq}$.

Suppose the product were 50 and the quotient 2.

Then $y = \sqrt{\frac{p}{q}} = \sqrt{\frac{50}{2}} = \sqrt{25} = 5$, and $x = \sqrt{pq} = \sqrt{100} = 10$.

Again, suppose the product 36, and the quotient $2\frac{1}{4}$.

Then $y = \sqrt{\frac{p}{q}} = \sqrt{\frac{36}{2\frac{1}{4}}} = \sqrt{16} = 4$, and $x = \sqrt{pq} = \sqrt{81} = 9$.

Ex. 3. Given the sum (s) of two numbers, and the sum of their squares s, to find those numbers.

Ans. $x = \frac{1}{2}s + \frac{1}{2}\sqrt{2s - s^2}$, and $y = \frac{1}{2}s - \frac{1}{2}\sqrt{2s - s^2}$.

Ex. 4. The sum and product of two numbers are equal, and if to either sum or product the sum of the squares be added, the result will be 12. What are the numbers?—*Ans.* each = 2.

Ex. 5. The square of the greater of two numbers multiplied into the less, produces 75; and the square of the less multiplied into the greater produces 45. What are the numbers?

Ex. 6. A man has six sons whose successive ages differ by four years, and the eldest is thrice as old as the youngest. Required their several ages?—*Ans.* 10, 14, 18, 22, 26, and 30 years.

SECT. IX. *Quadratic Equations.*

QUADRATIC Equations are such as contain the square of the unknown quantity, and which, after due reduction, may be made to assume the general form $Ax^2 + Bx + c = 0$; then dividing by A, the coefficient of the first term, there results $x^2 + \frac{B}{A}x + \frac{C}{A} = 0$, or, making $p = \frac{B}{A}$, $q = \frac{C}{A}$, we have

$$x^2 + px + q = 0 \ldots\ldots (1)$$

an equation which may represent all those of the second degree, p and q being known numbers, either positive or negative.

Let a be a number or quantity which, when substituted for x, renders $x^2 + px + q = 0$; then $a^2 + pa + q = 0$, or $q = -a^2 - pa$. Consequently $x^2 + px + q$, is the same thing as $x^2 - a^2 + px - pa$, or as $(x + a)(x - a) + p(x - a)$, or, lastly, as $(x - a)(x + a + p)$.

The inquiry, then, is reduced to this, viz. to find all the values of x which shall render the product of the above two factors equal to nothing. This will evidently be the case when either of the factors

is $= 0$; but in no other case. Hence, we have $x - a, = 0$, and $x + a + p = 0$, or $x = a$, and $x = -a - p$*.

And hence we may conclude—

1. That every equation of the second degree whose conditions are satisfied by one value (a) of x, admits also of another value ($-a-p$). These values are called the *roots* of the quadratic equation.

2. The sum of the two roots a and $-a-p$ is $= -p$; their product is $-a^2 - ap$, which as appears above is $= q$. So that the *coefficient*, p, of the *second* term is the *sum* of the roots with a contrary sign; and the *known term*, q, is their *product*.

3. It is easy to constitute a quadratic equation whose roots shall be any given quantities b and d. It is evidently $x^2 - (b + d)\, x + b\, d = 0$.

4. The determination of the roots of the proposed equation (1) is equivalent to the finding two numbers whose sum is $-p$, and product q.

5. If the roots b and d are equal, then the factors $x - b$ and $x - d$ are equal; and $x^2 + px + q$ is the square of one of them.

To solve a quadratic equation of the form $x^2 + px + q = 0$, let it be considered that the square of $x + \frac{1}{2}p$ is a trinomial, $x^2 + px + \frac{1}{4}p^2$, of which the first two terms agree with the first two terms of the given equation, or with the first member of that equation when q is transposed.

That is, with $x^2 + px = -q$.

Let then $\frac{1}{4}p^2$ be added, we have

$x^2 + px + \frac{1}{4}p^2 = \frac{1}{4}p^2 - q$

of which the first member is a complete square.

Its root is $x + \frac{1}{2}p = \pm \sqrt{(\frac{1}{4}p^2 - q)}$

and consequently $x = -\frac{1}{2}p \pm \sqrt{(\frac{1}{4}p^2 - q)}$

otherwise, from number 2 above, we have

$x + x' = -p$ and $x\,x' = q$.

Taking 4 times the second of these equations from the square of the first, there remains $x^2 - 2\,x\,x' + x'^2 = p^2 - 4q$

Whence, by taking the root, $x - x' = \sqrt{(p^2 - 4q)}$

Half this *added* to half equa. 1, gives

$x = -\frac{1}{2}p + \frac{1}{2}\sqrt{(p^2 - 4q)} = -\frac{1}{2}p^2 + \sqrt{(\frac{1}{4}p^2 - q)}$

And the same *taken* from half equa. 1, gives

$x' = -\frac{1}{2}p - \frac{1}{2}\sqrt{(p^2 - 4q)} = -\frac{1}{2}p - \sqrt{(\frac{1}{4}p^2 - q)}$

which two values of a evidently agree with the preceding.

It would be easy to analyze the several cases which may arise, according to the different signs and different values, of p and q. But

* If it be affirmed that the given equation admits of another value of x, besides the above, b for instance, it may be proved as before that $x - b$ must be of the number of the factors of $x^2 + px + q$, or of $(x - a)\,(x + a + p)$. But $x - a$ and $x + a + p$ being prime to each other, or having no common factor, their product cannot have any other factor than they. Consequently b must either be equal to a or to $-a - p$; and the number of roots is restricted to two.

these need not here be traced. It is evident that whether there be given

1. $x^2 + px = q$
2. $x^2 - px = q$
3. $x^2 + px = -q$
4. $x^2 - px = -q$

The general method of solution is by *completing the square*, that is, adding the square of $\frac{1}{2}p$, to both members of the equation, and then extracting the root.

It may farther be observed that all equations in which there are two terms involving the unknown quantity or any function of it, and the index of one double that of the other, may be solved as quadratics, by completing the square. Thus:

$x^4 \pm px^2 = q$, $x^{2n} \pm px^n = q$, $x^{\frac{n}{2}} \pm px^{\frac{n}{4}} = q$, $(x^2 + px + q)^2 \pm (x^2 + px + q) = r$, $(x^{2n} - x^n)^2 \pm (x^{2n} - x^n) = q$, &c., are of the same form as quadratics, and admit of a like determination of the unknown quantity. Many equations, also, in which more than one unknown quantity are involved, may be reduced to lower dimensions by completing the square and reducing; such, for example, as $(x^2 + y^2)^2 \pm p(x^2 + y^2) = q$, $\frac{x^2}{y^2} \pm \frac{px}{y} = q$, and so on.

Note.—In some cases a quadratic equation may be conveniently solved without dividing by the coefficient of the square, and thus without introducing fractions. To solve the general equation $ax^2 \pm bx = c$, for example, multiply the whole by $4a$, whence $4a^2x^2 \pm 4abx = 4ac$, adding b^2 to complete the square, $4a^2x^2 \pm 4abx + b^2 = 4ac + b^2$ taking the square root, $2ax \pm b = \pm\sqrt{(4ac + b^2)}$; whence $x = \frac{\mp b \pm \sqrt{(4ac + b^2)}}{2a}$: which will serve for a general theorem.

Examples.

1. Given $x^2 - 8x + 10 = 19$, to find x.
   transposing the 10, $x^2 - 8x = 19 - 10 = 9$
   completing the square, $x^2 - 8x + 16 = 9 + 16 = 25$
   extracting the root, $x - 4 = \pm 5$
   consequently $x = 4 \pm 5 = 9$ or -1.

2. Given $\frac{10}{x} - \frac{14 - 2x}{x^2} = \frac{22}{9}$, to find the values of x.
   multiplying by x^2, $10x - 14 + 2x = \frac{22x^2}{9}$,
   transposing, $\frac{22}{9}x^2 - 12x = -14$,
   dividing by $\frac{22}{9}$, $x^2 - \frac{54}{11}x = -\frac{63}{11}$,
   complet. squ. $x^2 - \frac{54}{11}x + (\frac{27}{11})^2 = \frac{729}{121} - \frac{693}{121} = \frac{36}{121}$,
   extract. root, $x - \frac{27}{11} = \pm\frac{6}{11}$,
   transposing, $x = \frac{27}{11} \pm \frac{6}{11} = 3$ or $\frac{21}{11}$.

3. Given $x^2+2x+4\sqrt{x^2+2x+1}=44$, to find x.
adding 1, we have $(x^2+2x+1)+4\sqrt{(x^2+2x+1)}=45$
complet. squ. $(x^2+2x+1)+4\sqrt{(x^2+2x+1)}+4=49$
extract. root, $\sqrt{(x^2+2x+1)}+2=\pm 7$
transposing the 2, $\sqrt{(x^2+2x+1)}=\pm 7-2=5$ or -9
that is, $x+1=5$ or -9
hence $x=4$ or -10.

4. Given $x^n-2ax^{\frac{n}{2}}=c$, to find x.
complet. squ. $x^n-2ax^{\frac{n}{2}}+a^2=c+a^2$
extract. root, $x^{\frac{n}{2}}-a=\pm\sqrt{(c+a^2)}$
transposing, $x^{\frac{n}{2}}=a\pm\sqrt{(c+a^2)}$
consequently, $x=(a\pm\sqrt{c+a^2})^{\frac{2}{n}}$.

5. Given $\frac{x^2}{y^2}+\frac{4x}{y}=12$, and $x-y=2$, to find x and y.
complet. squ. in equa. 1, $\frac{x^2}{y^2}+4\frac{x}{y}+4=16$.
Extracting root $\frac{x}{y}+2=\pm 4$: whence $\frac{x}{y}=2$ or -6, and $x=2y$ or $-6y$.
Substituting the former value of x in the 2nd equa., it becomes $2y-y=2$, or $y=2$; whence $x=4$.
Again, substituting the 2nd value of x, in equa. 2, it becomes $-6y-y$ or $-7y=2$; whence $y=-\frac{2}{7}$, and $x=+\frac{12}{7}$.

6. Given $x^2y^2-5=4xy$, and $\frac{1}{2}xy=\frac{1}{2}y^2$, to find x and y.
equa. 1, by transposition, becomes $x^2y^2-4xy=5$
completing the square, $x^2y^2-4xy+4=9$
extracting the root, $xy-2=\pm 3$
whence $xy=5$ or -1.
Substituting the first of these values for xy in equa. 2, it becomes $\frac{5}{2}y^2=\frac{5}{2}$: whence $y=1$ and $x=5$.
Substituting the 2nd value in the same equation, it becomes $\frac{5}{2}y^2=-\frac{1}{2}$: whence $y=-\sqrt[3]{\frac{1}{5}}=-\frac{1}{5}\sqrt[3]{25}$, and $x=-1\div-\frac{1}{5}\sqrt[3]{25}=\sqrt[3]{5}$.

7. Given $\frac{x^2}{(x^2-4)^2}+\frac{6}{x^2-4}=\frac{351}{25x^2}$, to find x.
Ans. $x=\pm 3$, or $\pm\sqrt{\frac{25}{11}}$.

8. A man travelled 105 miles at a uniform rate, and then found that if he had not travelled so fast by two miles an hour, he would have been six hours longer in performing the same journey. How many miles did he travel per hour?
Ans. 7 miles per hour.

9. Find two such numbers that the sum, product, and difference of their squares may be equal.

Ans. $\frac{1}{2} + \frac{1}{2}\sqrt{5}$, and $\frac{3}{2} + \frac{1}{2}\sqrt{5}$.

10. A waterman who can row eleven miles an hour with the tide, and two miles an hour against it, rows five miles up a river and back again in three hours: now, supposing the tide to run uniformly the same way during these three hours, it is required to find its velocity?

Ans. $4\frac{1}{4}\frac{9}{3}$ miles per hour.

SECT. X. *Equations in General.*

EQUATIONS in general may be prepared or constituted by the multiplication of factors, as we have shown in quadratics. Thus, suppose the values of the unknown quantity x in any equation were to be expressed by a, b, c, d, &c., that is, let $x = a$, $x = b$, $x = c$, $x = d$, &c., disjunctively, then will $x - a = 0$, $x - b = 0$, $x - c = 0$, $x - d = 0$, &c., be the simple radical equations of which those of the higher orders are composed. Then, as the product of any two of these gives a *quadratic* equation, so the product of any three of them, as $(x - a)(x - b)(x - c) = 0$, will give a *cubic* equation, or one of three dimensions. And the product of four of them will constitute a *biquadratic equation*, or one of four dimensions; and so on. Therefore, in general, *the highest dimension of the unknown quantity* x *is equal to the number of simple equations that are multiplied together to produce it.*

When any equation equivalent to this biquadratic $(x - a)(x - b)(x - c)(x - d) = 0$ is proposed to be resolved, the whole difficulty consists in finding the simple equations $x - a = 0$, $x - b = 0$, $x - c = 0$, $x - d = 0$, by whose multiplication it is produced; for each of these simple equations gives one of the values of x, and one solution of the proposed equation. For, if any of the values of x deduced from those simple equations be substituted in the proposed equation, in place of x, then all the terms of that equation will vanish, and the whole be found equal to nothing. Because when it is supposed that $x = a$, or $x = b$, or $x = c$, or $x = d$, then the product $(x - a)(x - b)(x - c)(x - d)$ vanishes, because one of the factors is equal to nothing. There are therefore four suppositions that give $(x - a)(x - b)(x - c)(x - d) = 0$, according to the proposed equation; that is, there are four *roots* of the proposed equation. And after the same manner *any other* equation admits of as many solutions as there are simple equations multiplied by one another that produce it, or as many as there are units in the highest exponent of the unknown quantity in the proposed equation.

But as there are no other quantities whatsoever besides these four (a, b, c, d,) that, substituted in the proposed product in the place of x, will make that product vanish; therefore, the equation $(x - a)(x - b)(x - c)(x - d) = 0$, cannot possibly have *more* than these

four roots, and cannot admit of more solutions than four. If substitute in that product a quantity neither equal to a, nor b, nor nor d, which suppose e, then since neither $e - a$, $e - b$, $e - c$, n $e - d$, is equal to nothing; their product cannot be equal to nothir but must be some real product: and, therefore, there is no suppo tion beside one of the aforesaid four, that gives a just value of according to the proposed equation. So that it can have no mc than these four roots. And after the same manner it appears, th *no equation can have more roots than it contains dimensions of unknown quantity.*

To make all this still plainer by an example, in numbers, suppc the equation to be resolved to be $x^4 - 10x^3 + 35x^2 - 50x + 24 =$ and that we discover that this equation is the same with the prodi of $(x - 1)$ $(x - 2)$ $(x - 3)$ $(x - 4)$, then we certainly infer th the four values of x are 1, 2, 3, 4; seeing any of these numbe placed for x, makes that product, and consequently $x^4 - 10x^3$ $35x^2 - 50x + 24$, equal to nothing, according to the propos equation. And it is certain that there can be no other values of besides these four: for when we substitute any other number for in those factors $x - 1$, $x - 2$, $x - 3$, $x - 4$, none of them vani and therefore their product cannot be equal to nothing, according the equation.

A variety of rules, some of them very ingenious, for the soluti of equations, may be found in the best writers on Algebra*; I we shall simply exhibit the easy rule of Trial-and-Error, as it is giv by *Dr. Hutton*, in the 1st vol. of his "Course of Mathematics."

Rule for the general solution of Equations by Trial-and-Error.

"1. Find, by trial, two numbers, as near the true root as possib and substitute them in the given equation instead of the unknoi quantity; marking the errors which arise from each of them.

"2. Multiply the difference of the two numbers, found by tri by the least error, and divide the product by the difference of t errors, when they are alike, but by their sum when they are unlil Or say, as the difference or sum of the errors is to the differer of the two numbers, so is the least error to the correction of supposed number.

"3. Add the quotient, last found, to the number belonging to t least error, when that number is too little, but subtract it when t great, and the result will give the true root *nearly*.

"4. Take this root and the nearest of the two former, or a other that may be found nearer; and, by proceeding in like mann a root will be had still nearer than before; and so on to any degr of exactness required.

"*Note.*—It is best to employ always two assumed numbers th shall differ from each other only by unity in the last figure on t right; because then, the difference, or multiplier, is 1."

* See the treatises of Lacroix, Bonnycastle, Wood, J. R. Young, &c.

Example.

To find the root of the cubic equation $x^3 + x^2 + x = 100$, or the value of x in it.

Here it is soon found that x lies between 4 and 5. Assume, therefore, these two numbers, and the operation will be as follows:

1st Sup.		2nd Sup.
4	x	5
16	x^2	25
64	x^3	125
84	sums	155
−16	errors	+55

the sum of which is 71.

Then as 71 : 1 :: 16 : ·225.
Hence $x = 4{\cdot}225$ nearly.

Again, suppose 4·2 and 4·3, and repeat the work as follows:

1st Sup.		2nd Sup.
4·2	x	4·3
17·64	x^2	18·49
74·088	x^3	79·507
95·928	sums	102·297
−4·072	errors	+2·297

the sum of which is 6·369.

As 6·369 : ·1 :: 2·297 : 0·036
This taken from . . 4·300

leaves x nearly = 4·264

Again, suppose 4·264 and 4·265, and work as follows:

4·264	.....	x	4·265
18·181696	.....	x^2	18·190225
77·526752	.....	x^3	77·581310
99·972448	.....	sums	100·036535
−0·027552	.....	errors	+0·036535

the sum of which is ·064087.

Then as ·064087 : ·001 :: ·027552 : 0·0004299
To this adding ... 4·264

gives x very nearly = 4·2644299

When one of the roots of an equation has been thus found, then take for a dividend the given equation with the known term transposed to the unknown side, so as to make the equation equal to nothing; and for a divisor take x minus the root just determined: the quotient will be equal to nothing, and will be a new equation

depressed a degree lower than the former. From this a new val of x may be found: and so on, till the equation is reduced to a qu dratic, of which the roots may be found by the proper rules.

Example.

Given the *biquadratic* equation, $x^4 - 11x^3 + 28x^2 + 36x - 144 = 0$, to find the four roots.

First, by the above method of Trial-and-Error, we find one of roots to be 6; then,

$$
\begin{array}{l}
x - 6 \,)\, x^4 - 11x^3 + 28x^2 + 36x - 144 \,(\, x^3 - 5x^2 - 2x + \\
\quad x^4 - 6x^3 \\
\quad\quad\quad

differences or by successive equal products or quotients, they are said to form a *progression*.

If the quantities proceed by successive equi-differences they are said to be in *Arithmetical Progression*. But if they proceed in the same continued proportion, or by equal multiplications or divisions, they are said to be in *Geometrical Progression*.

If the terms of a progression successively increase, it is called an *ascending* progression: if they successively decrease, it is called a *descending* progression. Thus,

1, 3, 5, 7, 9, &c. form an *ascending arithmetical*
24, 22, 20, 18, 16, &c. form a *descending arithmetical*
1, 3, 9, 27, 81, &c. form an *ascending geometrical*
4, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, &c. form a *descending geometrical*
} Progression.

ARITHMETICAL PROGRESSION.

1. Let a be the first term of an arithmetical progression,
d the common difference of the terms,
z the last term,
n the number of terms,
s the sum of all the terms.

Then a, $a+d$, $a+2d$, $a+3d$, &c., is an ascending progression, and a, $a-d$, $a-2d$, $a-3d$, &c., a descending progression.

Hence, in an ascending progression, $a+(n-1)d$, is the last term; in a descending progression, $a-(n-1)d$, is the last term.

2. Let a series be $a + (a+d) + (a+2d) + (a+3d)$.
The same inverted $(a+3d) + (a+2d) + (a+d) + a$

The sum of the two $(2a+3d)+(2a+3d)+(2a+3d)+(2a+3d) = 2s$.

That is, $(2a+3d) \times 4$, in this case $(a+\overline{a+3d})n = 2s$.

Consequently, $s = \frac{1}{2}(a+\overline{a+3d})n$, or $= \frac{1}{2}(a+z)n$, since z is here $= a+3d$. The same would be obtained, if the progression were descending; and let the number of terms be what it may.

3. From the equations $z = a+(n-1)d$, $s = \frac{1}{2}n(a+z)$, and $s = \frac{1}{2}n\{a+a+(n-1)d\}$, we may readily deduce the following theorems applicable to ascending series. When the series is descending, either the signs of the terms affected with d must be changed, or s must be taken for z; and *vice versâ*.

(1.) $a = z - nd + d = \sqrt{(-2sd + z^2 + dz + \frac{1}{4}d^2)} + \frac{1}{2}d$
$= \frac{s}{n} + \frac{1}{2}d - \frac{1}{2}nd = \frac{2s}{n} - z.$

(2.) $d = \frac{z-a}{n-1} = \frac{z^2-a^2}{2s-z-a} = \frac{2zn-2s}{n^2-n} = \frac{2s-2na}{n^2-n}.$

(3.) $z = a + nd - d = \frac{s}{n} + \frac{1}{2}nd - \frac{1}{2}d =$
$\sqrt{(2sd + a^2 - ad + \frac{1}{4}d^2)} - \frac{1}{2}d = \frac{2s}{n} - a.$

(4.) $s = \frac{1}{2} n (a + z) = (a + \frac{1}{2} n d - \frac{1}{2} d) n = (z - \frac{1}{2} n d + \frac{1}{2} d) n$
$= \frac{z^2 - a^2 + d(z + a)}{2d}$.

(5.) $n = \frac{2s}{a + z} = \frac{z - a}{d} - 1$.

Examples.

1. Required the sum of 20 terms of the progression 1, 3, 5, 7, 9, &c.

Here $a = 1$, $d = 2$, $n = 20$; which being substituted in the theorem $s = (a + \frac{1}{2} n d - \frac{1}{2} d) n$, it becomes $s = (1 + 20 - 1)\ 20$ $= 20 \times 20 = 400$, the sum required.

Note.—In any other case the sum of a series of *odd* numbers beginning with unity, would be $= n^2$, the square of the number of terms.

2. The first term of an arithmetical progression is 5, the last term 41, the sum 299 Required the number of the terms, and the common difference.

Here $n = \frac{2s}{a + z} = \frac{598}{46} = 13$, the number of terms,

and $d = \frac{z - a}{n - 1} = \frac{41 - 5}{12} = 3$, the common difference.

3. There are 8 equidifferent numbers: the least is 4, the greatest 32. What are the numbers?

Here $d = \frac{z - a}{n - 1} = \frac{32 - 4}{7} = 4$, the common difference.

Whence 4, 8, 12, 16, 20, 24, 28, 32, are the numbers.

4. The first term of an arithmetical progression is 3, the number of terms 50, the sum of the progression 2600. Required the last term and the common difference.

Here $z = \frac{2s}{n} - a = \frac{5200}{50} - 3 = 104 - 3 = 101$, the last term,

and $d = \frac{z - a}{n - 1} = \frac{101 - 3}{49} = 2$, the common difference.

5. The sum of six numbers in arithmetical progression is 48; and if the common difference d be multiplied into the less extreme, the product equals the number of terms.—Required those terms.

Ans. 3, 5, 7, 9, 11, and 13.

GEOMETRICAL PROGRESSION.

Let a be the first term of a geometrical series;
r the common ratio;
z the last term;
n the number of terms;
s the sum of all the terms.

Then a, ra, r^2a, r^3a,.....$r^{n-1}a$, is a geometrical progression, which will be *ascending* or *descending*, according as r is an *integer* or a *fraction*.

Let the prog. $a + ra + r^2a + r^3a + r^4a = s$, be $\times$ by r, it becomes $ra + r^2a + r^3a + r^4a + r^5a = rs$.

The diff. of these is, $-a + r^5a = rs - s$.

But r^5a is the last term of the original progression multiplied by r, or in general terms $r^{n-1}a \times r$, that is r^na. Consequently $r^na - a = rs - s$.

Whence $s = \dfrac{r^na - a}{r - 1} = \dfrac{r^n - 1}{r - 1}a =$ the sum of the series.

A similar method will lead to a like expression for s, whatever be the value of n. If r be a fraction, the expression becomes transformed to $s = \dfrac{1 - r^n}{1 - r}a$.

Now from these values of z and s the following theorems may be deduced.

(1.) $a = \dfrac{z}{r^{n-1}} = \dfrac{(r-1)s}{r^n - 1} = s + rz - rs.$

(2.) $z = ar^{n-1} = \dfrac{rs - s + a}{r} = \dfrac{s(r^n - r^{n-1})}{r^n - 1}.$

(3.) $s = \dfrac{a(r^n - 1)}{r - 1} = \dfrac{a(1 - r^n)}{1 - r} = \dfrac{rz - a}{r - 1} = \dfrac{zr^n - z}{r^n - r^{n-1}}.$

(4.) $r = \dfrac{s - a}{s - z} = \dfrac{a - s}{z - s} = \left(\dfrac{z}{a}\right)^{\frac{1}{n-1}}.$

(5.) $n = \dfrac{\log. r + \log. z - \log. a}{\log. r}.$

And, if the logarithm of $\dfrac{z}{a} = N$, that of $\dfrac{sr - s + a}{a} = M$, and that of $r = R$: then

(6.) $n = \dfrac{N}{R} + 1 = \dfrac{M}{R}.$

Also, if when r is a fraction, n is infinite, then is $r^n = 0$, and the expression for s becomes

(7.) $s = \dfrac{a}{1 - r}$, which expression is often of use in the summation of infinite series.

Examples.

1. The least of ten terms in geometrical progression is 1, the ratio 2. Required the greatest term, and the sum.

Here $z = a r^{n-1} = 1 \times 2^9 = 512$, the greatest term;

and $s = \dfrac{rz - a}{r - 1} = \dfrac{2 \times 512 - 1}{1} = 1023$, the sum.

2. Find the sum of 12 terms of the progression $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}$, &c.

Here $s = \dfrac{1 - r^n}{1 - r} = \dfrac{1 - \frac{1}{531441}}{1 - \frac{1}{3}} = \dfrac{265720}{177147}$, the sum.

3. Find the sum of the series $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$, &c., carried to infinity.

Here by Theor. (7.) $s = \dfrac{a}{1 - r}$, becomes $s = \dfrac{1}{1 - \frac{1}{2}} = 2$, the sum required.

4. Find the vulgar fraction equivalent to the circulating decimal ·36363636.

This decimal, expressed in the form of a series, is, $\frac{36}{100} + \frac{36}{10000} + \frac{36}{1000000} +$ &c., where $a = \frac{36}{100}$, and $r = \frac{1}{100}$.

Consequently, $s = \dfrac{a}{1 - r} = \frac{36}{100} \div \frac{99}{100} = \frac{4}{11}$, the fraction sought.

5. Find the sum of the descending infinite series $1 - x + x^2 - x^3 + x^4$, &c.

Here $a = 1$, $r = -x$, and $s = \dfrac{a}{1 - r} = \dfrac{1}{1 + x}$, the sum req.

And, by way of proof, it will be found that if 1 be divided by $1 + x$, the quotient will be the above series.

6. Of four numbers in geometrical progression the product of the two least is 8, and of the two greatest 128. What are the numbers?

Ans. 2, 4, 8, and 16.

SECT. XII. *Fractional and Negative Exponents.*

IN the preceding sections on powers and surds, we have only spoken of such powers as have *positive* and *integral* quantities for their exponents, and which are termed *direct* powers. And we explained that the exponent of the quantity, expressed the *number* of times that that quantity was multiplied by itself; so that to denote the division of any power by its root, we have only to subtract 1 from the index or exponent of that power; thus $a^{n-1} = \dfrac{a^n}{a}$; and by continuing to divide by a, we have $a^{n-1-1} = a^{n-2} = \dfrac{a^n}{a^2}$, and $a^{n-2-1} = a^{n-3} = \dfrac{a^n}{a^3}$; or generally, $a^{n-m} = \dfrac{a^n}{a^m}$; that is, *to express the division of one power of a quantity by any other power of the same quantity, we subtract the exponent of the divisor from that of the dividend.*

If n is greater than m, then will the quotient $\frac{a^n}{a^m} = a^p$ have for its exponent (p) a *positive* number, and is then called a *direct* power of a; but if m is greater than n, p will be *negative*, and a^{-p} is then called an *inverse* power of a; thus $\frac{a^4}{a^2} = a^{4-2} = a^2$, is a direct power, but $\frac{a^2}{a^4} = a^{2-4} = a^{-2}$, is an inverse power.

Now, to arrive at a just idea of the value of a quantity with a negative exponent or of an *inverse* power, let us successively subtract unity from the index of any *direct* power; or, in other words, divide that power successively by its root; thus, let a^n be the direct n^{th} power of a, then

$$a^{n-1} = \frac{a^n}{a}, \ldots\ldots$$

$$\ldots\ldots a^{3-1} = \frac{a^3}{a} = a^2,$$

$$a^{2-1} = \frac{a^2}{a} = a^1 = a,$$

$$a^{1-1} = \frac{a^1}{a} = a^0 = 1,$$

$$a^{0-1} = \frac{a^0}{a} = \frac{1}{a} = a^{-1},$$

$$a^{-1-1} = \frac{1}{a} \div a = \frac{1}{a^2} = a^{-2},$$

$$a^{-2-1} = \frac{1}{a^2} \div a = \frac{1}{a^3} = a^{-3},$$

Or, generally, $a^{-n} = \frac{1}{a^n}$. *That is, the value of an* inverse *power is equal to unity divided by the same* direct *power.*

As the properties of inverse powers are the same as those of direct powers, all that has been stated in Section V. regarding the latter, equally applies to the former, and it is therefore unnecessary to recapitulate it here.

We have, however, yet to mention another kind of exponent, viz., the *fractional*, used to express the *roots* of quantities, in a similar manner to that employed for expressing powers; and which extends to them also, all the properties already stated as belonging to inverse and direct powers.

This method of expressing any root of a quantity, consists in attaching to that quantity as an exponent, a fraction having unity for its numerator, and the index of the root for its denominator.

Thus $\sqrt[3]{a}$ is written $a^{\frac{1}{3}}$

$\sqrt{a}$,, $a^{\frac{1}{2}}$

$\sqrt[n]{x}$,, $x^{\frac{1}{n}}$

And the roots thus expressed may be employed in algebraical calculations in the same manner as powers with integral exponents. Thus, if we wish to express the square of the cube of x, we do so by $x^{3\times 2} = x^6$; and in like manner, to express the square of the cube *root* of x, we should have $x^{\frac{1}{3}\times 2} = x^{\frac{2}{3}}$; or, the square of the square *root* of $x = x^{\frac{1}{2}\times 2} = x^{\frac{2}{2}} = x$ itself, which is obvious. And in general, $x^{\frac{n}{m}}$ may be employed to express the n^{th} power of the m^{th} root of x, or what is the same thing, the m^{th} root of the n^{th} power of x.

If the fractional exponent has a negative sign, its value will be found by the rule already given for integral inverse powers; thus, $a^{-\frac{n}{m}} = \frac{1}{a^{\frac{n}{m}}} = \frac{1}{\sqrt[m]{a^n}}$ = unity divided by the m^{th} root of the n^{th} power of a.

It only remains to state, that in using fractional exponents, we may substitute any equivalent fraction, or a decimal, in place of the original exponent; thus $a^{\frac{1}{2}}$, $a^{\frac{2}{4}}$, $a^{\frac{3}{6}}$, $a^{\cdot 5}$, are only so many different ways of expressing the square root of a; advantage may frequently be taken of this circumstance, to facilitate the working with surds.

By way of recapitulation, then, there are four different kinds of exponents employed in analytical operations, viz.:—

The *positive integral* exponent, as x^n, which denotes the direct nth power of x, and is equal to x multiplied n times by itself.

The *negative integral* exponent, as x^{-n}, which denotes the inverse n^{th} power of x, and is equal to unity divided n times by x.

The *positive fractional* exponent, as $x^{\frac{1}{n}}$, which denotes the direct n^{th} root of x, and is equal to a quantity, which being multiplied n times into itself will equal x.

The *negative fractional* exponent, as $x^{-\frac{1}{n}}$, which denotes the inverse n^{th} root of x, and is equal to unity divided by the direct n^{th} root of x.

SECT. XIII. *Logarithms.*

LOGARITHMS are a series of numbers in *arithmetical* progression, answering to another series in *geometrical* progression; so taken that 0 in the former corresponds with 1 in the latter.

Thus, 0, 1, 2, 3, 4, 5, are the Logs. or *arithmetical* series;

and 1, 2, 4, 8, 16, 32, { are the Nos. or *geometrical* series, answering thereto.

Or, 0, 1, 2, 3, 4, 5, the logarithms;
and 1, 5, 25, 125, 625, 3125, the corresponding numbers.

Or, 0, 1, 2, 3, 4, 5, the logarithms;
and 1, 10, 100, 1000, 10000, 100000, the corresponding numbers.

In which it will be seen, that by altering the common ratio of the geometrical series, the same arithmetical series may be made to serve as the logarithms of any series of numbers. As above, where the common ratios of the several geometrical series are 2, 5, and 10, respectively.

Or, the logarithms of a number may be considered as the indices or exponents of the powers of some root or radix, which when involved to the power indicated by the logarithm, is then equal to the number. Thus, taking the foregoing series as an illustration, in which the roots or radices are 2, 5, and 10, respectively, we have

Nos.	*Logs.*	Nos.	*Logs.*	Nos.	*Logs.*
$1 = 2^{0}$		$1 = 5^{0}$		$1 = 10^{0}$	
$2 = 2^{1}$		$5 = 5^{1}$		$10 = 10^{1}$	
$4 = 2^{2}$		$25 = 5^{2}$		$100 = 10^{2}$	
$8 = 2^{3}$		$125 = 5^{3}$		$1000 = 10^{3}$	
$16 = 2^{4}$		$625 = 5^{4}$		$10000 = 10^{4}$	
$32 = 2^{5}$		$3125 = 5^{5}$		$100000 = 10^{5}$	

in which the exponents in the columns headed *Logs.*, are the logarithms of the numbers in the first column headed Nos.

The last of these series, or that in which the common ratio of the geometrical series, or (what is the same thing) the radix of the system of logarithms is 10, is that usually employed for the purpose of facilitating calculations; and is the same as the series of logarithms contained in the Logarithmic Tables in the Appendix to this work.

As in this system of logarithms the numbers whose logarithms are integral increase as the powers of 10, and it is requisite in calculations to use the logarithms of the intermediate numbers, it becomes necessary to interpolate with fractional logarithms; thus the series then becomes

Nos.	*Logs.*	Nos.	*Logs.*
$1 = 10^{0\cdot}$		$8 = 10^{0\cdot90309}$	
$2 = 10^{0\cdot30103}$		$9 = 10^{0\cdot954243}$	
$3 = 10^{0\cdot477121}$		$10 = 10^{1\cdot}$	
$4 = 10^{0\cdot60206}$		$11 = 10^{1\cdot041393}$	
$5 = 10^{0\cdot69897}$		$12 = 10^{1\cdot079181}$	
$6 = 10^{0\cdot778151}$		$13 = 10^{1\cdot113943}$	
$7 = 10^{0\cdot845098}$		&c.	&c.

in which the exponents of the powers of 10, answering to any given number, will be found to correspond with the logarithm of that number as given in Table II. in the Appendix.

In order to explain the properties of logarithms, and the reason of the rules given in the section on Logarithmic Arithmetic, let us assume any series of numbers, N, N_1, N_2, N_3, &c., to which correspond the logarithms, l, l_1, l_2, l_3, &c., to the same root or radix (r).

1. Then we have, $N = r^l$, $N_1 = r^{l_1}$, $N_2 = r^{l_2}$, &c.; and by the rule for multiplying powers, (page 56,) $r^l \times r^{l_1} = r^{l \times l_1} = N \times N_1$; and $N . N_1 . N_2 = r^{l+l_1+l_2}$. *Or the logarithm of the* product *of two or more numbers, is equal to the* sum *of the logarithms of those numbers; and conversely, the logarithm of the* quotient *of two numbers, is equal to the* difference *of the logarithms of the* dividend *and* divisor; which correspond with the rules given at page 45, for the multiplication and division of numbers by logarithms.

2. If $N = N_1 = N_2 = N_3$; then we have, by the foregoing, $\log (N . N . N . N) = \log (N^4) = r^{l+l+l+l} = r^{4l}$; and in general, $\log (N^n) = r^{nl}$. *Or the logarithm of the* n^{th} *power of any quantity is equal to* n *times the logarithm of that quantity, whether* n *is integral, fractional, negative, or positive.*

For let A be the quantity, and let n be negative; then $A^{-n} = \frac{1}{A^n}$, and $\log \left(\frac{1}{A^n}\right) =$ log of $1 - n (\log A)$; and as log of $1 = 0$, $\log 1 - n \log A = -n (\log A) = \log (A^{-n})$.

If $n = \frac{p}{q}$; then, let $A^{\frac{p}{q}} = K$; raising both sides to the power q, we have $A^p = K^q$, and their logarithms, $p (\log A) = q (\log K)$; then dividing both sides by q, we obtain $\frac{p}{q} (\log A) = \log K = \log (A^{\frac{p}{q}})$.

Lastly, if $n = -\frac{p}{q}$, then will $A^{-\frac{p}{q}} = \frac{1}{A^{\frac{p}{q}}}$; and since $\log (A^{\frac{p}{q}}) = \frac{p}{q} (\log A) . \therefore \log \left(\frac{1}{A^{\frac{p}{q}}}\right) = \log 1 - \frac{p}{q} (\log A) = -\frac{p}{q} (\log A) = \log (A^{-\frac{p}{q}})$.

3. In Section XI., page 41, it was stated that the decimal portion of a logarithm was always the same for the same digits, and that it was only the characteristic which was altered, by the multiplication or division of the quantity by 10, or any power of 10; the reason of this is obvious, since all the powers of 10 have integers *only* for their logarithms.

4. It was also stated, (page 41,) that it was only the characteristic of the logarithm of a fraction, which was negative, the decimal portion being always positive; to illustrate this, let us take the fraction $\cdot 05 = \frac{5}{100} = \frac{10^{0 \cdot 698970}}{10^2} = 10^{-2} \times 10^{0 \cdot 698970} = 10^{-2 + \cdot 698970}$; and therefore $\bar{2} \cdot 698970$ is the log. of $\cdot 05$.

5. Suppose there be two systems of logarithms whose roots or

bases are r and s. Let any number N have p for its logarithm in the first system, and q for its logarithm in the second: we shall have $N = r^p$ and $N = s^q$; which gives $r^p = s^q$, and $s = r^{\frac{p}{q}}$. Therefore, taking the logarithms for the system r, we shall have $\log s = \frac{p}{q}(\log r)$; or, if in the system r we have $\log r = 1$, then $\log s = \frac{p}{q}$, or $q = \frac{p}{\log s} = p \times \frac{1}{\log s}$. Thus, knowing the logarithm p of any number N, for the system whose base is r, we may obtain the logarithm q of the same number for the system s, by multiplying p by a fraction whose numerator is unity and denominator the logarithm of s taken in the system r.

6 In the system of logarithms first constructed by Baron Napier, the great inventor, $s = 2{\cdot}718281828459$, &c., and the exponents are usually denominated *Napierian*, or *Hyperbolic* logarithms; the latter name being given because of the relation between these logarithms and the lines and asymptotic spaces in the equilateral hyperbola: so that in this system n is always the hyperbolic logarithm of $(2{\cdot}71828, \&c.)^n$. But in the system constructed by Mr. Briggs (corresponding with the spaces in a hyperbola whose asymptotes make an angle of 25° 44′ 25″ 28‴), called *common* or *Briggean* logarithms, $r = 10$; so that the common logarithm of any number is, as already stated, the index of that power of 10 which is equal to the said number.

7. Although the Briggean logarithms are those usually employed for the purpose of computation, the hyperbolic are always used in the differential calculus, and the higher branches of analysis; it therefore becomes sometimes requisite to find the hyperbolic logarithm of a quantity, which may be obtained from the common logarithm as follows: from § 5, above, we have $q = p \frac{1}{\log s}$; then, putting h. log for hyperbolic logarithm, log for common logarithm, and substituting for s its proper value, we have $\text{h. log } n = \log n \frac{1}{\log 2{\cdot}718281} = \log n \frac{1}{0{\cdot}43429448} = 2{\cdot}3025851 (\log n)$; or *to obtain the* hyperbolic *logarithm of any number, multiply its* common *logarithm by* 2·3025851.

In practice the following method will be found more convenient than multiplying by 2·3025851.

To convert common logs. into hyperbolic.

Write the common log, as shown in the following Examples, and then take from Table I. the equivalent value of each figure in hyperbolic logs, taking care that the latter are each moved as many places to the right as the corresponding numbers in the common logarithm. The sum of the whole will then be the hyperbolic logarithm required.

To convert hyperbolic logs. into common.

Proceed in the same manner, only using Table II. instead of Table I.

TABLE I.		TABLE II.	
Com. Logs.	Hyp. Logs.	Hyp. Logs.	Com. Logs.
1·	2·3025851	1·	·4342945
2·	4·6051702	2·	·8685890
3·	6·9077553	3·	1·3028834
4·	9 2103404	4·	1·7371779
5·	11·5129255	5·	2·1714724
6·	13·8155106	6·	2·6057669
7·	16·1180957	7·	3·0400614
8·	18·4206807	8·	3·4743559
9·	20·7232658	9·	3·9086503

Examples.

1. What is the hyp. log. of 1662?

By reference to Table III. in the Appendix, we find that 3·220631 is the common log. of 1662; then

Com. Log.		Hyp. Log.	
3·000000	=	6·907755	3
·200000	=	·460517	0
·020000	=	·046051	7
·000600	=	·001381	5
·000030	=	·000069	0
·000001	=	·000002	3
3·220631	=	7·415778	

2. What is the common log. of the number whose hyp. log. is 7·415778?

Hyp. Log.		Com. Log.	
7·000000	=	3·040061	4
·400000	=	·173717	7
·010000	=	·004342	9
·005000	=	·002171	4
·000700	=	·000304	0
·000070	=	·000030	4
·000008	=	·000003	4
7·415778	=	3·220631	

SECT. XIV. *Computation of Formulæ.*

SINCE the comprehension, and the numerical computation of formulæ expressed algebraically, are of the utmost consequence to practical men, enabling them to avail themselves advantageously of the theoretical results of men of science, as well as to express in scientific language the results of their own experimental or other researches; it has appeared expedient to present brief treatises of Arithmetic and Algebra. The thorough understanding of these two initiatory departments of science will serve essentially in the application of all that follows in the present volume; and that application may probably be facilitated by a few examples, as below :—

Ex. 1. Let $b = 5$, $c = 12$, $d = 13$, and $s = \frac{b + c + d}{2}$; then what is the numerical value of the expression $\sqrt{s\,(s-b)\,.\,(s-c)\,.\,(s-d)}$ from page 127, which denotes the area of the triangle whose sides are 5, 12, and 13?

Here $s = \frac{b + c + d}{2} = \frac{5 + 12 + 13}{2} = 15$; then $s - b = 15 - 5 = 5$; $s - c = 15 - 12 = 3$; and $s - d = 15 - 13 = 2$.

Consequently, by substituting the numerical values of the several quantities between the parentheses for them, we shall have

$$\sqrt{(15 \times 10 \times 3 \times 2)} = \sqrt{900} = 30, \text{ the value required.}$$

The same values being given to b, c, and d, we may, as a verification of the above, compute the area of the same triangle from the equivalent expression, $\sqrt{c^2 - \left(\frac{c^2 - b^2}{2\,d} + \frac{d}{2}\right)^2} \cdot \frac{d}{2}$.

Here $c^2 = 12^2 = 144$; $b^2 = 5^2 = 25$; $2\,d = 26$; and $\frac{d}{2} = 6\frac{1}{2}$.

Substituting these, the expression becomes

$$\sqrt{144 - \left(\frac{144 - 25}{26} + 6\tfrac{1}{2}\right)^2} \times 6\tfrac{1}{2} = 30,$$

the same result as before.

Ex. 2. Suppose $g = 32\frac{1}{6}$, $t = 6$: required the value of $\frac{1}{2}\,g\,t^2$, an expression denoting the space in feet which a heavy body would fall vertically from quiescence in six seconds, in the latitude of London.

Here $\frac{1}{2}\,g\,t^2 = 16_{\frac{1}{12}} \times 6^2 = 96\frac{1}{12} \times 6 = 579$ feet.

Ex. 3. Given $\mathrm{D} = 6$, $d = 4$, $h = 12$, $\pi = 3{\cdot}141593$; required the value of $\frac{1}{12}\,\pi\,h\,(\mathrm{D}^2 + \mathrm{D}\,d + d^2)$, a theorem for the solid content of a conic frustum whose diameters of the two ends are D, d, and height h.

Here $D^2 = 36$, $D\,d = 6 \times 4 = 24$, $d^2 = 16$, $\frac{\pi}{12} = \cdot 2618$ nearly.

Hence $\frac{1}{12}\,\pi\,h\,(D^2 + D\,d + d^2) = \cdot 2618\,(36 + 24 + 16)\,12 = \cdot 2618 \times 76 \times 12 = 3 \cdot 141593 \times 76 = 238 \cdot 761068$.

Ex. 4. Let $a = 1$, $h = 25$, $g = 193$ inches: what is the value of $2\,a\,\sqrt{g\,h}$? This being the expression for the cubic inches of water discharged in a second, from an orifice whose area is a, and depth below the upper surface of water in the vessel, or reservoir, h, both in inches.

Here $2\,a\,\sqrt{g\,h} = 2\,\sqrt{(25 \times 193)} = 10\,\sqrt{193} = 10 \times 13 \cdot 89244 = 138 \cdot 9244$ cubic inches.

Ex. 5. Suppose the velocity of the wind to be known in miles per hour; required short approximative expressions for the yards per minute, and for the feet per second.

First $1760 \div 60 = \frac{88}{3} = 29\frac{1}{3} = 30$ nearly.

Also $5280 \div (60 \times 60) = \frac{528}{360} = \frac{88}{60} = \frac{44}{30} = 1\frac{1}{2}$ nearly.

If, therefore, n denote the number of miles per hour:

$30\,n$ will express the yards per minute; and $1\frac{1}{2}\,n$, the feet per second.

These are approximative results: to render them correct, where complete accuracy is required, subtract from each result its 45th part, or the *fifth* part of its *ninth* part.

Thus, suppose the wind blows at the rate of 20 miles per hour:

Then $30\,n = 30 \times 20 = 600$ yards per minute, or more correctly $600 - \frac{600}{45} = 600 - 13\frac{1}{3} = 586\frac{2}{3}$ yards.

Also $1\frac{1}{2}\,n = 30$ feet per second;

or, correctly $30 - \frac{30}{45} = 30 - \frac{2}{3} = 29\frac{1}{3}$ feet.

Conversely, $\frac{2}{3}$ of the feet per second will indicate the miles per hour, correct within the 45th part, which is to be *added* to obtain the true result.

Ex. 6. To find a theorem by means of which it may be ascertained when a general law exists, and what that law is.

Suppose, for example, it were required to determine the law which prevailed between the resistances of bodies moving in the air and other resisting media, and the velocities with which they move. Let V_1, V_2, denote any two velocities, and R_1, R_2, the corresponding resistances experienced by a body moving with those velocities: we wish to ascertain what power of V_1 it is to which R_1 is proportional. Let x denote the index or exponent of the power: then will $V_1^x : V_2^x :: R_1 : R_2$, if a law subsist.

Div. the consequents by the antecedents, we have $1 : \left(\frac{V_2}{V_1}\right)^x :: 1 : \frac{R_2}{R_1}$.

Consequently $\left(\frac{V_2}{V_1}\right)^x = \frac{R_2}{R_1}$. This, expressed logarithmically, gives

$$x \times \log \frac{V_2}{V_1} = \log \frac{R_2}{R_1};$$

$$\text{or } x = \frac{\log R_2 - \log R_1}{\log V_2 - \log V_1}.$$

Hence the quotient of the differences of the logs. of the resistances, divided by the difference of the corresponding velocities, will express the exponent x required.

This theorem is of very frequent application in reference to the motion of cannon balls, of barges on canals, of carriages on rail-roads, &c., and may indeed be applied to the planetary motions.

When two or more values of any quantities occur in a formula, it is usual to denote both of them by the same letter, distinguishing between them by the attachment of a small number below it, as in the last example above, where the two *velocities* are both denoted by the letter V, but distinguished by the numerals, as V_1 and V_2, and the two *resistances* by R_1 and R_2.

CHAP. III.

PLANE AND SOLID GEOMETRY.

SECT. I. *Definitions.*

1. GEOMETRY is that department of science, by means of which we demonstrate the properties, affections, and measures of all sorts of *magnitude*.

2. *Magnitude* is a term used to denote the extension of any thing, and is of three kinds; as the magnitude of a *line*, which is only in *one* direction, viz. *length;* the magnitude of a *surface*, which is in *two* directions, viz. *length* and *breadth;* and the magnitude of a *solid*, which is in *three* directions, viz. *length*, *breadth* and *depth*.

3. A *point* has no parts or magnitude; neither length, breadth, nor thickness, and serves only to assign *position*.

4. A *line* has length without breadth or thickness, and indicates direction and distance.

Cor. The extremities of a line are, therefore, points.

5. A *right line* is that which lies evenly, or in the same direction, between two points. A *curve line* continually changes its direction.

Cor. Hence there can only be one species of right lines, but there is an infinite variety in the species of curves.

6. An *angle* is the inclination of two lines to one another, meeting in a point, called the *angular point*. When it is formed by two *right lines*, it is a *plane* angle, as A; if by *curve* lines, it is a *curvilineal* angle, as B. (Fig. 1.)

7. A *right angle* is that which is made by one right line A B falling upon another C D, and making the angles on each side equal, that is, A B C = A B D; so that the line A B does not incline more to one side than another: it is then said to be *perpendicular* to the line C D. All other angles are called *oblique* angles. (Fig. 2.)

8. An *obtuse angle* is *greater* than a right angle, as R. (Fig. 3.)

9. An *acute angle* is *less* than a right angle, as S. (Fig. 3.)

10. *Contiguous* or *adjacent angles*, are the two angles formed by one line falling upon another, as R, S. (Fig. 3.)

11. *Vertical* or *opposite* angles, are those made on contrary sides of two lines intersecting one another, as A E C, D E B. (Fig. 4.)

12. A *surface* has length and breadth, but no thickness, and shows extension. The area of a figure is the quantity of space which its surface occupies.

Cor. The boundaries or limits of a surface are lines.

13. A *plane* is that surface which lies perfectly even between its extremes; or which, being cut by another plane in any direction, its section would be a straight line.

14. A *solid* is a magnitude extended every way, or which has length, breadth, and depth.

Cor. The boundaries or extremes of a solid are surfaces.

15. The *square* of a right line is the space included by four right lines equal to it, set perpendicular to one another.

16. The *rectangle* of two lines is the space included by four lines equal to them, set perpendicular to one another, the opposite ones being equal.

17. One right line is said to be *parallel* to another, when both lines, being extended to any length, will never meet, but always preserve the same distance between them.

SECT. II. *Of Angles, and Right Lines, and their Rectangles.*

PROP. I. If to any point C in a right line A B, several other right lines D C, E C are drawn on the same side; all the angles formed at the point C, taken together, are equal to two right angles; thus, A C D + D C E + E C B = the two right angles A C P + P C B. (Fig. 5.) [Euclid, Book I. Prop. 13.]

Cor. 1. All the angles made about one point in a plane, being taken together, are equal to four right angles.

Cor. 2. If all the angles at C, on one side of the line A B, are found to be equal to two right angles; then A C B is a straight line. (Fig. 5.) [Euc. B. I. Prop. 14.]

II. If two right lines, A B, C D, cut one another, the opposite angles C E B and A E D will be equal. (Fig. 4.) [Euc. B. I. Prop. 15.]

III. A right line, B I, which is perpendicular to one of two parallels, is perpendicular to the other. (Fig. 6.)

IV. If a right line C G, intersects two parallels A D, F H; the alternate angles, A B E, and B E H, will be equal. (Fig. 6.) [Euc. B. I. Prop. 29.]

Cor. 1. The two internal angles D B E and B E H on the same side are equal to two right angles. (Fig. 6.)

Cor. 2. The external angle C B D, is equal to the internal angle on the same side B E H. (Fig. 6.) [Euc. B. I. Prop. 28.]

V. Right lines, parallel to the same right line, are parallel to one another. [Euc. B. I. Prop. 30.]

VI. If a right line A C be divided into two parts A B, B C; the square of the whole line is equal to the squares of both the parts, and twice the rectangle of the parts; or $AC^2 = AB^2 + BC^2 + 2AB \times BC$. (Fig. 7.) [Euc. B. II. Prop. 4.]

VII. The square of the difference of two lines A C, B C, is equal to the sum of their squares, wanting twice their rectangle; or $AB^2 = AC^2 + BC^2 - 2AC \times BC$. Or $AB^2 + 2AC \times BC = AC^2 + BC^2$. (Fig. 7.) [Euc. B. II. Prop. 7.]

VIII. The square of the sum of two lines is equal to the sum of their squares, together with their rectangle.

IX. The rectangle of the sum and difference of two lines is equal to the difference of their squares.

X. The square of the sum, together with the square of the difference of two lines, is equal to twice the sum of their squares.

SECT. III. *Of Triangles.*

DEFINITIONS.

1. A *triangle* is a plane figure bounded by three right lines, called the *sides* of the triangle.

2. An *equilateral triangle* is one in which all the three sides are equal.

3. An *isosceles triangle* has only two sides equal.

4. A *scalene triangle* has all its three sides unequal.

5. An *equiangular triangle* is one which has three equal angles; and *two* triangles are said to be *equiangular*, when the angles in the one are respectively equal to those in the other.

6. A *right-angled triangle* is that which has one of its angles a right angle. The side opposite to the right angle is called the *hypotenuse*, and the other two sides the *legs*.

7. An *oblique angled triangle* has all its angles oblique.

8. An *obtuse angled triangle* has one of its angles obtuse.

9. An *acute angled triangle* has all its angles acute.

10. In a triangle, the lowest side, as A B, is called the *base*, and the opposite angle C the *vertex*: the *altitude* of a triangle is the perpendicular height (C D) from the base, or its extension to the vertex. (Fig. 8.)

11. *Similar triangles* are those whose angles or sides are respectively equal, each to each. And *homologous sides* are those lying between equal angles.

PROPOSITIONS.

PROP. I. In any triangle A B C, if one side B C be produced or drawn out; the external angle A C D will be equal to the two internal opposite angles A B C and B A C. (Fig. 9.) [Euc. B. I. Prop. 32.]

II. In any triangle, the sum of the three internal angles is equal to two right angles. [Euc. B. I. Prop. 32.]

Cor. 1. If two angles in one triangle be equal to two angles in another: the third will also be equal to the third.

Cor. 2. If one angle of a triangle be a right angle, the sum of the other two will be equal to a right angle.

III. The angles at the base of an isosceles triangle, are equal. [Euc. B. I. Prop. 5.]

Cor. 1. An equilateral triangle is also equiangular; and the contrary.

Cor. 2. The line which is perpendicular to the base of an isosceles triangle, bisects both it and the vertical angle.

IV. In any triangle, the greatest side is opposite to the greatest angle, and the least to the least. [Euc. B. I. Prop. 18.]

V. In any triangle A B C, the sum of any two sides B A, A C, is greater than the third B C, and their difference is less than the third side. (Fig. 10.) [Euc. B. I. Prop. 20.]

VI. If two triangles A B C, *a b c*, have two sides, and the included angle equal in each; these triangles, and their correspondent parts, shall be equal. (Fig. 11.) [Euc. B. I. Prop. 4.]

VII. If two triangles A B C and *a b c*, have two angles and an included side equal, each to each; the remaining parts shall be equal, and the whole triangles equal. (Fig. 11.) [Euc. B. I. Prop. 26.]

VIII. Triangles of equal bases and heights are equal. [Euc. B. I. Prop. 37.]

IX. Triangles of the same height, are in proportion to one another as their bases. [Euc. B. VI. Prop. 1.]

X. If a line D E be drawn parallel to one side B C, of a triangle; the segments of the other sides will be proportional; that is, A D : D B : : A E : E C. (Fig. 12.) [Euc. B. VI. Prop. 2.]

Cor. 1. If the segments be proportional, A D : D B : : A E : EC; then the line D E is parallel to the side B C. (Fig. 12.)

Cor. 2. If several lines be drawn parallel to one side of a triangle, all the segments will be proportional.

Cor. 3. A line drawn parallel to any side of a triangle, cuts off a triangle similar to the whole.

XI. In similar triangles, the homologous sides are proportional; that is, A B : A C : : D E : D F. (Fig. 13.) [Euc. B. VI. Prop. 4.]

XII. Like triangles are in the duplicate ratio, or as the squares of, their homologous sides. [Euc. B. VI. Prop. 19.]

XIII. In any triangle, the difference of the squares of the two legs A B and A C, is equal to twice the rectangle contained by the base B C, and the distance D O, of its middle point from the perpendicular D A. (Fig. 14.)

XIV. In a right-angled triangle B A C, if a perpendicular be let fall from the right angle upon the hypothenuse, it will divide it into two triangles, similar to one another and to the whole, A B D, A D C. (Fig. 14.) [Euc. B. VI. Prop. 8.]

Cor. 1. The rectangle of the hypothenuse and either segment is equal to the square of the adjoining side.

XV. The distance A O of the right angle, from the middle of the hypothenuse is equal to half the hypothenuse. (Fig. 14.)

XVI. In a right-angled triangle, the square of the hypothenuse is equal to the sum of the squares of the two sides. [Euc. B. I. Prop. 47.]

XVII. If the square of one side of a triangle be equal to the sum of the squares of the other two sides; then the angle comprehended by them is a right angle. [Euc. B. I. Prop. 48.]

XVIII. I fan angle A, of a triangle B A C be bisected by a right line A D, which cuts the base; the segments of the base will be proportional to the adjoining sides of the triangle; that is, B D : D C : : A B : A C. (Fig. 10.) [Euc. B. VI. Prop. 3.]

XIX. If the sides be as the segments of the base, the line A D bisects the angle A. (Fig. 10.) [Euc. B. VI. Prop. A.]

XX. Three lines drawn from the three angles of a triangle to the middle of the opposite sides, all meet in one point.

XXI. Three perpendicular lines erected on the middle of the three sides of any triangle, all meet in one point.

XXII. The point of intersection of the three perpendiculars, will be equally distant from the three angles; or, it will be the centre of the circumscribing circle.

XXIII. Three perpendiculars drawn from the three angles of a triangle, upon the opposite sides, all meet in one point.

XXIV. Three lines bisecting the three angles of a triangle, all meet in one point.

XXV. If D be any point in the base of a scalene triangle, A B C: then is $A\ B^2 \times D\ C + A\ C^2 \times B\ D = AD^2 \times B\ C + B\ C \times B\ D \times D\ C$. (Fig. 10.)

SECT. IV. *Of Quadrilaterals and Polygons.*

DEFINITIONS.

1. A *quadrangle* or *quadrilateral*, is a plane figure bounded by four right lines.

2. A *parallelogram* is a quadrangle whose opposite sides are parallel, as A G B H. The line A B drawn to the opposite corners is called the *diameter* or *diagonal*. And if two lines be drawn parallel to the two sides, through any point of the diagonal, they divide it into several others, and then C and D are called parallelograms about the diameter; and E and F the *complements;* and the figure E D F a *gnomon*. (Fig. 15.)

3. A *rectangle* is a parallelogram whose sides are perpendicular to one another.

4. A *square* is a rectangle of four equal sides and four equal angles.

5. A *rhombus* is a parallelogram, whose sides are equal, and all its angles oblique.

6. A *rhomboid* is a parallelogram in which only the opposite sides are equal, and all its angles oblique.

7. A *trapezoid* is a quadrangle, having only two sides parallel, as Fig. 16.

8. A *trapezium* is a quadrangle that has no two sides parallel, as Fig. 17.

9. A *polygon* is a plane figure enclosed by many right lines. If all the sides and angles are equal, it is called a *regular* polygon, and denominated according to the number of sides or angles, as a *pentagon* having five sides, a *hexagon*, having six sides, a *heptagon*, having seven sides, &c.

10. The *diagonal* of a quadrangle or polygon is a line drawn between any two opposite corners of the figure, as A B. (Fig. 15.)

11. The *height* of a figure is a line drawn from its *vertex*, perpendicular to the base, or opposite side on which it stands.

12. *Like* or *similar* figures, are those whose several angles are equal to one another, and the sides about the equal angles proportional.

13. *Homologous* sides of two like figures are those between two angles, respectively equal.

14. The *perimeter* or circumference of a figure, is the compass of it, or sum of all the lines that enclose it.

15. The *internal angles* of a figure are those on the inside, made by the lines that bound the figure, as A B C, D C B, &c. (Fig. 18.)

16. The *external angles* of a figure are the angles made by each side of a figure, and the adjoining side drawn out, as B A F, A E G. (Fig. 18.)

PROPOSITIONS.

PROP. I. In any parallelogram the opposite sides and angles are equal; and the diagonal divides it into two equal triangles. [Euc. B. I. Prop. 34.]

II. The diagonals of a parallelogram intersect each other in the middle point of both.

III. Any line B C passing through the middle of the diagonal of a parallelogram P, divides the area into two equal parts. (Fig. 19.)

IV. Any right line B C drawn through the middle point P of the diagonal of a parallelogram, is bisected in that point; or B P = P C. (Fig. 19.)

V. In any parallelogram A G H B, the complements E and F are equal. (Fig. 15.) [Euc. B. I. Prop. 43.]

VI. Parallelograms of equal bases and heights are equal. [Euc. B. I. Prop. 36.]

VII. The area of a parallelogram is double the area of a triangle having the same base and height. [Euc. B. I. Prop. 41.]

VIII. Parallelograms of the same height are to one another as their bases. [Euc. B. VI. Prop. 1.]

IX. Parallelograms of equal bases are as their heights.

X. Parallelograms are to one another, as their bases and heights.

XI. In any parallelogram the sum of the squares of the diagonals is equal to the sum of the squares of all the four sides.

XII. The sum of the four internal angles of any quadrilateral figure, is equal to four right angles.

XIII. If two angles of a quadrangle be right angles, the sum of the other two amounts to two right angles.

XIV. The sum of all the internal angles of a polygon is equal to twice as many right angles, abating four, as the polygon has sides.

Cor. Hence all right-lined figures of the same number of sides, have the sum of all the internal angles equal.

XV. The sum of the external angles of any polygon is equal to four right angles.

Cor. All right-lined figures have the sum of their external angles equal.

XVI. In two similar figures A C, P R; if two lines B E, Q T, be drawn after a like manner, as suppose, to make the angle C B E = R Q T; then these lines have the same proportion as any two homologous sides of the figure; viz.,

B E : Q T :: B C : Q R :: A B : P Q :: A D : P S. (Fig. 20.)

XVII. All similar figures are to one another as the squares of their homologous sides.

XVIII. Any figure described on the hypothenuse of a right-angled triangle, is equal to two similar figures described the same way upon the two legs; that is, B F C = A L C + A G B. (Fig. 21.) [Euc. B. VI. Prop. 31.]

XIX. Any regular figure A B C D E, is equal to a triangle whose base is the perimeter A B C D E A; and height the line O P, drawn from the centre, perpendicular to one side. (Fig. 22.)

XX. Only three sorts of regular figures can fill up a plane surface, that is, the whole space round an assumed point, and these are six triangles, four squares, or three hexagons.

SECT. V. *Of the Circle, and Inscribed and Circumscribed Figures.*

DEFINITIONS.

1. A *circle* is a plane figure described by a right line moving about a fixed point, as A C about C: or it is a figure bounded by a curved line, every part of which is equidistant from a fixed point. (Fig. 23.)

2. The *centre* of a circle is the fixed point about which the line moves, as C. (Fig. 23.)

3. The *radius* is the line that describes the circle, as C A. (Fig. 23.)

Cor. All the radii of a circle are equal.

4. The *circumference* is the line described by the extreme end of the moving line, as A B D E A. (Fig. 23.)

5. The *diameter* is a line drawn through the centre, from one side to the other, as A D. (Fig. 23.)

6. A *semicircle* is half the circle, cut off by the diameter, as A B D.

7. A *quadrant*, or quarter of a circle, is the part between two radii perpendicular to one another, as C D E. (Fig. 23.)

8. An *arc* is any part of the circumference, as A B. (Fig. 24.)

9. A *sector* is a part bounded by two radii, and the arc between them, as A C B. (Fig. 24.)

10. A *chord* is a right line drawn through the circle, as D F.

11. A *segment* is a part cut off by a right line, or chord, as D E F, or D A B F. (Fig. 24.)

12. *Angle at the centre* is that whose angular point is at the centre A C B. (Fig. 24.)

13. *Angle at the circumference* is when the angular point is in the circumference, as B A D, or B C D. (Fig. 25.)

14. *Angle in a segment*, is the angle made by two lines drawn

from some point of the arc of that segment to the ends of the base; as B C D is an angle in the segment B C D. (Fig. 25.)

15. *Angle upon a segment* is the angle made in the opposite segment, whose sides stand upon the base of the first; as B A D, which stands upon the segment B C D. (Fig. 25.)

16. A *tangent* is a line touching a circle, which, produced, does not cut it, as G A F. (Fig. 23.)

17. Circles are said to touch one another, which meet, but do not cut one another.

18. *Similar arcs*, or *similar sectors*, are those bounded by radii that make the same angle.

19. *Similar segments* are those which contain similar triangles, alike placed.

20. A figure is said to be *inscribed in a circle*, or a circle *circumscribed about a figure*, when all the angular points of the figure are in the circumference of the circle. (Fig. 26.)

21. A circle is said to be *inscribed in a figure*, or a figure *circumscribed about a circle*, when the circle touches all the sides of the figure. (Fig. 27.)

22. *One figure is inscribed in another*, when all the angles of the inscribed figure are in the sides of the other. (Fig. 28.)

PROPOSITIONS.

PROP. I. The radius C R, bisects any chord at right angles, which does not pass through the centre, as A B. (Fig. 29.) [Euc. B. III. Prop. 3.]

Cor. 1. If a line bisects a chord at right angles, it passes through the centre of the circle.

Cor. 2. The radius that bisects the chord also bisects the arc.

II. In a circle equal chords are equally distant from the centre. [Euc. B. III. Prop. 14.]

III. If several lines be drawn through a circle, the greatest is the diameter, and those that are nearest the centre are greater than those that are farther off. [Euc. B. III. Prop. 15.]

IV. If from any point three equal right lines can be drawn to the circumference, that point is the centre. [Euc. B. III. Prop. 9.]

V. No circle can cut another in more than two points. [Euc. B. III. Prop. 10.]

VI. There can be only two equal lines drawn from any exterior point, to the circumference of a circle.

VII. In any circle, if several radii be drawn making equal angles, the arcs and sectors comprehended thereby will be equal; that is, if the angle A C B = B C D, then, the arc A B = B D, and the sector A C B = B C D. (Fig. 30.) [Euc. B. III. Prop. 26.]

VIII. In the same or equal circles, the arcs, and also the sectors, are proportional to the angles intercepted by the radii.

IX. The circumferences of circles are to one another as their diameters.

X. A right line, perpendicular to the diameter of a circle, at the

extreme point, touches the circle in that point, and lies wholly without the circle; or is a tangent to the circle.

XI. If two circles touch one another, either inwardly or outwardly, the line passing through their centres shall also pass through the point of contact. [Euc. B. III. Prop 11 and 12.]

XII. In a circle the angle at the centre is double the angle at the circumference, standing upon the same arc; or B D C = 2 B A C. (Fig. 31.) [Euc. B. III. Prop. 20.]

XIII. All angles in the same segment of a circle are equal, as D A C = D B C, and D G C = D H C. (Fig 32.) [Euc. B. III. Prop. 21.]

XIV. If two right lines D C, A B, be drawn from the extremities of two equal arcs D A, B C, they will be parallel. (Fig. 32.)

XV. The angle A B C in a semicircle is a right angle. (Fig. 33.) [Euc. B. III. Prop. 31.]

XVI. The angle F B G, in a greater segment F A B C G, is less than a right angle; and the angle D B E, in a less segment D B E, is greater than a right angle. (Fig. 33.) [Euc. B. III. Prop. 31.]

XVII. If two lines cutting a circle, intersect one another in A; and there be made at the centre, $\angle$ E C F = $\angle$ B A D; then the arc B D + G H = 2 E F, if A is within the circle; or the arc B D — G H = 2 E F, if A is without. (Fig. 34.)

XVIII. If from a point without, two lines, A B, A D be drawn to cut a circle; the angle made by them is equal to the angle at the centre, standing on half the difference of the two arcs of the circumference G H, B D. (Fig. 34.)

XIX. The angle A = $\angle$ B H D + H D G, when A is within; or A = B H D — H D G, when A is without the circle. (Fig. 34.)

XX. In a circle, the angle made at the point of contact between the tangent and any chord, is equal to the angle in the alternate or opposite segment; E C F = E B C, and E C A = E G C. (Fig. 35.) [Euc. B. III. Prop. 32.]

XXI. A tangent to the middle point of an arc, is parallel to the chord of it.

XXII. If from any point B in a semicircle, a perpendicular B D be let fall upon the diameter, it will be a mean proportional between the segments of the diameter; that is, A D : D B :: D B : D C. (Fig. 36.) [Euc. B. VI. Prop. 13.]

XXIII. The chord is a mean proportional between the adjoining segment and the diameter, from the similarity of the triangles: that is, A D : A B :: A B : A C; and C D : C B :: C B : C A. (Fig. 36.)

XXIV. In a circle, if the diameter A D be drawn, and from the ends of the chords A B, A C, perpendiculars be drawn upon the diameter; the squares of the chords will be as the segments of the diameter; that is, A E : A F :: AB^2 : AC^2. (Fig. 37.)

XXV. If two circles touch one another in P, and the line P D E be drawn through their centres; and any line P A B is drawn through that point to cut the circles, that line will be divided in proportion to the diameters; that is P A : P B :: P D : P E. (Fig. 38.)

XXVI. If through any point F in the diameter of a circle, any chord, C F D be drawn, the rectangle of the segments of the chord is equal to the rectangle of the segments of the diameter; C F × F D = A F × F B = also G F × F E. (Fig. 39.) [Euc. B. III. Prop. 35.]

XXVII. If through any point F out of the circle in the diameter B A produced, any line F C D be drawn through the circle: the rectangle of the whole line and the external part is equal to the rectangle of the whole line passing through the centre, and the external part; D F × F C = A F × F B = also F E × F G. (Fig. 40.)

XXVIII. Let H F be a tangent at H; then the rectangle C F × F D = square of the tangent F H. (Fig. 40.) [Euc. B. III. Prop. 36.]

XXIX. If from the same point F, two tangents be drawn to the circle, they will be equal; that is, F H = F I. (Fig. 40.)

XXX. If a line P F C be drawn perpendicular to the diameter A D of a circle; and any line drawn from A to cut the circle and the perpendicular; then the rectangle of the distances of the sections from A, will be equal to the rectangle of the diameter and the distance of the perpendicular from A; that is, A B × A C = A P × A D. (Fig. 41.)

Also, A B × A C = A K². (Fig. 41.)

XXXI. In a circle E D F whose centre is C, and radius C E, if the points B, A, be so placed in the diameter produced, that C B, C E, C A be in continual proportion, then two lines B D, A D drawn from these points to any point in the circumference of the circle will always be in the given ratio of B E to A E. (Fig. 42.)

XXXII. In a circle, if a perpendicular D B be let fall from any point D, upon the diameter C I, and the tangent D O drawn from D, then A B, A C, A O, will be in continual proportion. (Fig. 43.)

XXXIII. If a triangle B D F be inscribed in a circle, and a perpendicular D P let fall from D on the opposite side B F, and the diameter D A drawn; then, as the perpendicular is to one side including the angle D, so is the other side to the diameter of the circle; that is, D P : D B :: D F : D A. (Fig. 44.)

XXXIV. The rectangle of any two sides of an inscribed triangle is equal to the rectangle of the diameter, and the perpendicular on the third side; that is, B D × D F = A D × D P. (Fig. 44.)

XXXV. If a triangle B A C be inscribed in a circle, and the angle A bisected by the right line A E D, then as one side is to the segment of the bisecting line within the triangle, so is the whole bisecting line to the other side; that is, A B : A E :: A D : A C; and A B × A C = B E . E C + A E². (Fig. 45.)

XXXVI. If a quadrilateral A B C D be inscribed in a circle, the sum of two opposite angles is equal to two right angles; that is, A D C + A B C = two right angles. (Fig. 46.) [Euc. B. III. Prop. 22.]

XXXVII. If a quadrangle be inscribed in a circle, the rectangle of the diagonals is equal to the sum of the rectangles of the opposite sides; or C A × B D = C B × D A + C D × A B. (Fig. 46.)

XXXVIII. A circle is equal to a triangle whose base is the circumference of the circle; and height, its radius.

XXXIX. The area of a circle is equal to the rectangle of half the circumference and half the diameter.

XL. Circles (that is, their areas) are to one another as the squares of their diameters, or as the squares of the radii, or as the squares of the circumferences. [Euc. B. XII. Prop. 2.]

XLI. Similar polygons inscribed in circles, are to one another as the circles wherein they are inscribed.

XLII. A circle is to any circumscribed rectilineal figure, as the circle's periphery to the periphery of the figure.

XLIII. If an equilateral triangle A B C be inscribed in a circle; the square of the side thereof is equal to three times the square of the radius; that is, $A B^2 = 3 A D^2$. (Fig. 47.)

XLIV. If from any point D in the circumference of a circle, having inscribed in it an equilateral triangle, chords be drawn to the three angles A, B, C; the longest chord A D, is equal to the sum of the two lesser chords, B D and C D. (Fig. 48.)

XLV. A square inscribed in a circle, is equal to twice the square of the radius.

XLVI. The side of a regular hexagon inscribed in a circle, is equal to the radius of the circle.

XLVII. If two chords in a circle mutually intersect at right angles, the sum of the squares of the segments of the chords is equal to the square of the diameter of the circle; that is, $A P^2 + P B^2 + P C^2 + P D^2 = \text{diam.}^2$. (Fig. 49.)

XLVIII. If the diameter P Q be divided into two parts at any point R, and if R S be drawn perpendicular to P Q; also R T applied equal to the radius, and T R produced to the circumference at V: then,

$$R T = \frac{P R + R Q}{2};$$ or R T is the *arithmetical* mean,

$$R S = \sqrt{P R \times R Q};$$ or R S is the *geometrical* mean,

$$R V = \frac{2 \times P R \times R Q}{P R + R Q};$$ or R V is the *harmonical* mean.

} between the two segments of the diameter P R, R Q. (Fig. 50.)

XLIX. If the arcs P Q, Q R, R S, &c., be equal, and there be drawn the chords P Q, P R, P S, &c., then it will be P Q : P R :: P R : P Q + P S :: P S : P R + P T :: P T : P S + P V, &c. (Fig. 51.)

L. If the arcs P Q, Q R, R S, &c. be equal, the angles Q P R, R P S, S P T, &c., will be equal; or in equal circles, equal angles stand upon equal arcs, whether they be at the centres or circumferences. (Fig. 51.) [Euc. B. III. Prop. 26.]

LI. The centre of a circle being O, and P a point in the radius, or in the radius produced; if the circumference be divided into as many equal parts A B, B C, C D, &c., as there are units in $2n$, and lines be drawn from P to all the points of division; then shall the continual product of all the alternate lines, viz. P A × P C × P E, &c.,

be $= r^n - x^n$ when P is within the circle, or $= x^n - r^n$ when P is without the circle; and the product of the rest of the lines, viz., P B × P D × P F, &c., $= r^n + x^n$: where $r =$ A O the radius, and $x =$ O P the distance of P from the centre. (Fig. 52.)

SECT. VI. *Of Planes and Solids.*

DEFINITIONS.

1. THE *common section* of two planes, is the line in which they meet, or cut each other.

2. A line is *perpendicular* to a plane, when it is perpendicular to every line in that plane which meets it.

3. One plane is *perpendicular* to another, when every line of the one, which is perpendicular to the line of their common section, is perpendicular to the other.

4. The *inclination* of one plane to another, or the angle they form between them, is the angle contained by two lines, drawn from any point in the common section, and at right angles to the same, one of these lines in each plane.

5. *Parallel planes* are such as being produced ever so far in any direction, will never meet, or which are everywhere at an equal perpendicular distance.

6. A *solid angle* is that which is made by three or more plane angles, meeting each other in the same point.

7. *Similar solids*, contained by plane figures, are such as have all their solid angles equal, each to each, and are bounded by the same number of similar planes, alike placed.

8. A *prism* is a solid whose ends are parallel, equal, and like plane figures, and its sides, connecting those ends, are parallelograms. (Fig. 53 and 54.)

9. A prism takes particular names according to the figure of its base or ends, whether triangular, square, rectangular, pentagonal, hexagonal, &c.

10. A *right* or *upright prism*, is that which has the planes of the sides perpendicular to the planes of the ends or base. (Fig. 53.) When such is not the case it is called an *oblique* prism. (Fig. 54.)

11. A *parallelopiped*, or parallelopipedon, is a prism bounded by six parallelograms, every opposite two of which are equal, alike, and parallel. (Fig. 55.)

12. A *rectangular parallelopipedon* is that whose bounding planes are all rectangles, which are perpendicular to each other. (Fig. 56.)

13. A *cube* is a square prism, being bounded by six equal square sides or faces, which are perpendicular to each other. (Fig. 57.)

14. A *cylinder* is a round prism having circles for its ends; and is conceived to be formed by the rotation of a right line about the circumferences of two equal and parallel circles, always parallel to the axis. (Fig. 58.)

15. The *axis* of a cylinder is the right line A B joining the centres of the two parallel circles, about which the figure is described.

16. A *pyramid* is a solid whose base is any right-lined plane figure, and its sides triangles, having all their vertices meeting together in a point above the base, called the *vertex* of the pyramid. (Fig. 59.)

17. Pyramids, like prisms, take particular names from the figure of their base.

18. A *cone* is a round pyramid having a circular base, and is conceived to be generated by the rotation of a right line about the circumference of a circle, one end of which is fixed at a point above the plane of that circle. (Fig. 60.)

19. The *axis* of a cone is the right line, A B, joining the vertex, or fixed point, and the centre of the circle about which the figure is described.

20. When the axis of a cone or pyramid is perpendicular to the base, it is called a *right* cone or pyramid; but if inclined it is called *oblique*.

21. *Similar cones and cylinders*, are such as have their altitudes and the diameters of their bases proportional.

22. A *sphere* is a solid bounded by one curve surface, which is everywhere equally distant from a certain point within, called the centre. It is conceived to be generated by the rotation of a semicircle about its diameter, which remains fixed. (Fig. 61.)

23. The *axis* of a sphere is the right line about which the semicircle revolves, and the centre is the same as that of the revolving semicircle.

24. The *diameter* of a sphere is any right line passing through the centre, and terminated both ways by the surface.

25. The *altitude* of a solid is the perpendicular drawn from the vertex to the opposite side or base.

PROPOSITIONS.

PROP. I. If any prism be cut by a plane parallel to its base, the section will be equal and like to the base.

II. If a cylinder be cut by a plane parallel to its base, the section will be a circle, equal to the base.

III. All prisms and cylinders, of equal bases and altitudes, are equal to each other. [Euc. B. XI. Prop. 31.]

IV. Rectangular parallelopipedons, of equal altitudes, are to each other as their bases. [Euc. B. XI. Prop. 32.]

V. Rectangular parallelopipedons, of equal bases, are to each other as their altitudes.

VI. Because prisms and cylinders are as their altitudes, when their bases are equal: and, as their bases when their altitudes are equal. Therefore, universally, when neither are equal, they are to one another as the product of their bases and altitudes: hence, also, these products are the proper numeral measures of their quantities or magnitudes.

VII. Similar prisms and cylinders are to each other as the cubes of their altitudes, or of any like linear dimensions. [Euc. B. XI. Prop. 33.]

VIII. In any pyramid a section parallel to the base is similar to the base; and these two planes are to each other as the squares of their distances from the vertex.

IX. In a right cone, any section parallel to the base is a circle; and this section is to the base as the squares of their distances from the vertex.

X. All pyramids and cones of equal bases and altitudes, are equal to one another.

XI. Every pyramid is the third part of a prism of the same base and altitude. [Euc. B. XII. Prop. 10.]

XII. If a sphere be cut by a plane, the section will be a circle.

XIII. Every sphere is two-thirds of its circumscribing cylinder.

XIV. A cone, hemisphere, and cylinder of the same base and altitude, are to each other as the numbers 1, 2, 3. [Euc. B. XII. Prop. 10.]

XV. All spheres are to each other as the cubes of their diameters; all these being like parts of their circumscribing cylinders. [Euc. B. XII. Prop. 18.]

XVI. There are only three sorts of regular plane figures which can be joined together, so as to form a solid angle; viz. three, four, or five *triangles*, three *squares*, and three *pentagons*: thus giving five different species of solid angles, which are those of the five *regular* or *Platonic bodies*; viz.:—

1. The *tetraedron*, (Fig. 62,) bounded by four *equilateral triangles*, each solid angle of which is formed by three *triangles*.
2. The *hexaedron*, or *cube*, (Fig. 57,) contained by six *squares*, each solid angle of which is formed by three *squares*.
3. The *octaedron*, (Fig. 63,) bounded by eight *triangles*, each solid angle of which is formed by four *triangles*.
4. The *dodecaedron*, (Fig. 64,) bounded by twelve *pentagons*, each solid angle of which is formed by three *pentagons*.
5. The *icosaedron*, (Fig. 65,) bounded by twenty triangles, each solid angle of which is formed by five *triangles*.

XVII. Only one sort of the foregoing five regular bodies, joined at their angles, can completely fill a solid space, without leaving any vacuity; viz. eight hexaedrons, or cubes.

XVIII. A sphere is to any circumscribing solid B F, (all whose planes touch the sphere,) as the surface of the sphere to the surface of the solid. (Fig. 66.)

XIX. All bodies circumscribing the same sphere, are to one another as their surfaces.

XX. The sphere is the greatest or most capacious of all bodies of equal surface.

SECT. VII. *Practical Geometry.*

IT is not intended in this place to present a complete collection of Geometrical Problems, but merely a selection of the most useful, especially in reference to the employments of Mechanics and Engineers.

The instruments for the purposes of geometrical construction, are too well known to require any description here; and their use is much easier learned by an examination of the instruments themselves, than by any written explanation.

PROB. I. *From a given point* B *in a given straight line* A C, *to draw a line perpendicular to the same.*

From B as a centre, with any radius, describe arcs cutting A C in *a* and *c*, then from *a* and *c* with any larger radius describe arcs cutting each other in *d* and *e*; then the straight line *d* B *e* will be perpendicular to A C. (Fig. 67.)

PROB. II. *To erect a perpendicular at the end of a given line.*

With any radius, and from a point somewhere above A B, describe a semicircular arc passing through the point B, at which the perpendicular is to be erected, and also cutting A B in C; then, through C and the centre of the arc produce the right line C D, cutting the arc in D, and a line joining D and B will be the perpendicular required. (Fig. 68.)

PROB. III. *To bisect any given angle* A C B.

From C as a centre, with any radius, describe an arc cutting the sides in D and E; then from D and E, as centres with the same radius, describe arcs cutting each other in F; then the straight line joining C and F will bisect the angle A C B. (Fig. 69.)

PROB. IV. *To bisect a given angle* B A P; *then to bisect its half; and so on.*

Through any point B draw B E parallel to A P, and upon B E set off the distance B C equal to B A; then join A C, and it will bisect the angle B A P.

Again, set off, upon B E, from C, C D = C A; join A D, and it will bisect C A P, or quadrisect B A P.

Again, set off, upon B E, D E = D A; join E A; and E A P will be $\frac{1}{8}$ of B A P: and so on. (Fig. 70.)

PROB. V. *At a given point* A *in a given line* A B, *to make an angle equal to a given angle* C.

From the centres A and C, with the same radius, describe the arcs D E, F G. Then, with radius D E, and centre F, describe an arc, cutting F G in G. Through G draw the line A G; and it will form the angle required. (Fig. 71.)

PROB. VI. *To divide any given angle* A B C *into three equal parts.*

From B, with any radius, describe the circle A C D A. Bisect the angle A B C by B E, and produce A B to D. On the edge of a ruler

mark off the length of the radius A B. Lay the ruler on D, and move it till one of the marks on the edge intersects B E, and the other the arc A C in G. Set off the distance C G from G to F: and draw the lines B F, and B G, they will trisect the angle A B C. (Fig. 72.)

PROB. VII. *To divide a given line* A B *into any proposed number of equal parts.*

1st *Method.* Draw any other line A C, forming any angle with the given line A B; on which set off as many of any equal parts, A D, D E, E F, F C, as the line A B is to be divided into. Join B C; parallel to which draw the other lines F G, E H, D I: then these will divide A B in the manner required. (Fig. 73.)

2nd *Method*, without drawing parallel lines. Let A B be the line which is to be divided into n equal parts. Through one extremity A draw any right line A D, upon which set off $n + 1$ equal parts, the point D being at the termination of the $(n + 1)$th part. Join D B and produce it until the prolongation B E = B D. Let F be the termination of the $(n - 1)$th part. Join F E, and the right line of junction will cut the given line A B in the point P, such that $B P = \frac{1}{n} A B$; and of course n distances each equal to B P set off upon B A, will divide it, as required*. (Fig. 74.)

PROB. VIII. *To cut off from a given line* A B, *supposed to be very short, any proportional part.*

Suppose, for example, it were required to find the $\frac{1}{12}$, $\frac{2}{12}$, $\frac{3}{12}$, &c. of the line A B, fig. 75. From the ends A and B draw A D, B C, perpendicular to A B, and divide A D into twelve equal parts; then through these divisions 1, 2, 3, &c., draw lines 1*f*, 2*g*, &c., parallel to A B. Draw the diagonal A C, and 1*d* will be the $\frac{1}{12}$ of A B; 2*c*, $\frac{2}{12}$, and so on. The same method is applicable to any other part of a given line.

PROB. IX. *To make a diagonal scale, say, of feet, inches, and tenths of an inch.*

Draw an indefinite line A B, on which set off from A to B the given length for one foot, any required number of times; and from these divisions A, C, H, B, draw A D, C E, &c., perpendicular to A B. On A D and B F set off any length ten times, and through these divisions draw lines parallel to A B; then divide A C and D E into twelve equal parts, each of which will be one inch. Draw the lines A 1, *g* 2, &c., and they will form the scale required; viz., each of the larger divisions from E to G, G to F, &c., will represent a foot; each of the twelve divisions between D and E, an inch; and the

* The truth of this method is easily demonstrated. Through I the intermediate point of division, on A D, between F and D, draw I B. Then, because D B = B E, and D I = I F, I B is parallel to F P. Consequently, B P : B A :: I F : I A :: 1 : n, by construction.

several horizontal lines parallel to R C in the triangle E C R, will be equal to $\frac{1}{10}$, $\frac{2}{10}$, $\frac{3}{10}$, &c., of an inch. (Fig. 76.)

Note.—If the scale be meant to represent feet, or any other unit, and *tenths* and *hundredths*, then D E must be divided into *ten* instead of *twelve* equal parts.

PROB. X. *To find the centre of a circle.*

Draw any chord A B, and bisect it perpendicularly with the line R D. Then bisect R D in C, which will be the centre required. (Fig. 29.)

PROB. XI. *To divide a given circle into any number of concentric parts, equal to each other in area.*

Draw the radius A B, and on it describe the semicircle A *f e d* B. Divide A B into the proposed number of equal parts, 1, 2, 3, &c., and erect the perpendiculars 1 *d*, 2 *e*, 3 *f*, &c., meeting the semicircle in *d*, *e*, *f*, &c. Then from the centre B, and with radii B *d*, B *e*, &c. describe circles; so shall the given circle be divided into the proposed number of equal concentric parts. (Fig. 77.)

PROB. XII. *To divide a given circle into any number of parts, equal both in area and perimeter.*

Divide the diameter Q R into the proposed number of equal parts at the points S, T, V, &c.; then, on one side of the diameter describe semicircles on the diameters Q S, Q T, Q V, and on the other side of it describe semicircles on R V, R T, R S; so shall the parts 17, 35, 53, 71, be all equal, both in area and perimeter. (Fig. 78.)

PROB. XIII. *To describe the circumference of a circle through three given points*, A, B, C.

From the middle point B draw chords B A, B C, to the two other points, and bisect these chords perpendicularly by lines meeting in O, which will be the centre. Then from the centre O, at the distance of any of the points, as O A, describe a circle, and it will pass through the two other points B, C, as required. (Fig. 79.)

The same method may be employed for finding the centre of a circular arc, by taking any three points in the same, as A, B, and C.

PROB. XIV. *To describe mechanically the circumference of a circle through three given points*, A, B, C, *when the centre is inaccessible; or the circle too large to be described with compasses.*

Place two rulers M N, R S, cross ways, touching the three points A B C. Fix them in V by a pin, and by a trnasverse piece T. Hold a pencil in A, and describe the arc B A C, by moving the angle R A N so as to keep the outside edges of the rulers against the pins B C. Remove the instrument R V N, and on the arc described mark two

points, D, E, so that their distance shall be equal to the length B C. Apply the edges of the instrument against D E, and with a pencil in G describe the arc B C, which will complete the circumference of the circle required. (Fig. 80.)

Otherwise.—Let an axle of 12 or 15 inches long carry two unequal wheels A and B, of which one, A, shall be fixed, while the other, B, shall be susceptible of motion along the axle, and being placed at any assigned distance, A B, upon the paper or plane on which the circle is to be described. Then will A and B be analogous to the ends of a conic frustum, the vertex of the complete cone being the centre (O) of the circle (C D E F) which will be described by the rim, or edge, of the wheel A, as it rolls upon the proposed plane. Then it will be, as the diameter of the wheel A is to the difference of the diameters of A and B, so is the radius of the circle proposed to be described by A, to the distance, A B, at which the two wheels must be asunder, measured upon the plane on which the circle is to be described.

The wheel B will evidently describe, simultaneously, another circle (G H I K) whose radius will be less than that of the former by A B. (Fig. 81.)

PROB. XV. *On a given chord* A B *to describe mechanically an arc of a large circle that shall contain any number of degrees.*

Place two rulers, forming an angle A C B, equal to the supplement of half the given number of degrees, and fix them in C. Place two pins at the extremities of the given chord, and hold a pencil in C; then move the edges of this instrument against the pins, and the pencil will describe the arc required. (Fig. 82.)

Suppose it is required to describe an arc of 50 degrees on the given chord A B; subtract 25 degrees (which is half the given angle) from 180, and the difference, 155 degrees, will be the *supplement.* Then form an angle A C B of 155 degrees with the two rulers, and proceed as has been shown above.

PROB. XVI. *To describe the segment of a circle of large radius, of which the chord* A B *and versine* C D *are given.*

Through D draw H I parallel to A B; then join A D and D B, and draw A H perpendicular to A D and B I perpendicular to B D, also draw A *d* and B *d* perpendicular to A B. Then divide A C, B C, H D, and I D, into any number of equal parts, 1, 2, 3, &c., and draw lines joining the corresponding numbers in H I and A B; also divide A *d*, B *d*, into the same number of equal parts, *a*, *b*, *c*, &c., and draw lines from the point D to these last divisions, then will the points where the lines *a* D, *b* D, *c* D, &c. cut the lines 11, 22, 33, &c., be so many points in the required segment. (Fig. 83.)

PROB. XVII. *To find the length of any given arc of a circle,* A B.

From A and B, as centres, with radius equal to a quarter of the chord of the arc A B, describe arcs cutting the given arc in C, and its chord in D; join C D, which will equal half the length of the arc A B nearly. (Fig. 84.)

PROB. XVIII. *To draw a straight line equal in length to any given portion of the circumference of a circle.*

Let A B C D be the circle, and let A *a*, *a b*, *b* B, be the portions of the circumference of which the length is required; draw the two diameters B D and A C at right angles to each other, and at the end of the latter draw the tangent line A *e*; then divide the radius E C into four equal parts, and set off three of them from C to F on the diameter produced; then draw lines from F through the points *a*, *b*, B, cutting the tangent line A *e*, in *c*, *d*, *e*; so shall the lines A *c*, *c d*, *d e*, be equal in length to the arcs A *a*, *a b*, *b* B, and the whole line A *e* equal to the quadrant A B. (Fig. 85.)

PROB. XIX. *To bisect any given triangle* A B C.

Upon any one of the sides, as A B, describe a semicircle, which bisect in the point D; then from B as a centre with radius B D describe an arc cutting A B in E, through which point draw the line E F parallel with the side A C; then will the line E F bisect the given triangle A B C. (Fig. 86.)

PROB. XX. *To reduce a given rectilinear figure*, A B C D E F G A, *of any number of sides, to a triangle of equal area.*

Join any two alternate angles, as A C, and through the intermediate angle B, draw B H parallel to A C, cutting one of the adjoining sides in H, and join C H, then will the triangle C B *a*, added to the figure, be equal to the triangle *a* A H taken out of the same; in like manner join H D, and draw G I parallel to the same through the intermediate point G, producing it to cut the side A G (also produced) in I, and join I D. Next join D F, and through E draw a line parallel thereto, cutting the side G F produced in K, and join D K; lastly, join D G, and draw parallel to the same through the point K a line cutting the side A G produced in L, then join D L, and the triangle I D L will be equal in area to the given rectilinear figure A B C D E F G A. (Fig. 87.)

PROB. XXI. *To form a rectangle of the largest area, in a given triangle*, A B C.

Bisect any two sides A B and B C in D and E, and from those points draw lines D F and E G perpendicular to the third side A C, and join D E, then will the rectangle D E F G be the largest which can be inscribed in the given triangle. (Fig. 88.)

PROB. XXII. *To form a square equal in area to a given triangle*, A B C.

On the longest side produced set off C D, equal half the perpendicular height B E; and at C erect the perpendicular C F; then on A D describe a semicircle cutting C F in G, and on C G form the square C G H I, which will be equal in area to the given triangle. (Fig. 89.)

PROB. XXIII. *To find the side of a square equal in area to a given rectangle*, A B C D.

Produce the lesser side A B of the rectangle till A E equal the longer side A D; then describe upon A E a semicircle cutting B C in F, and join A F, which will be the side of the square required. (Fig. 90.)

PROB. XXIV. *To find the side of a square equal to the difference between two given squares.*

Let the sides of the two given squares be A and B. Then draw the line C D equal to the lesser line B, and at one of its extremities erect the indefinite perpendicular D E; then with radius equal to A, and from C as a centre, describe an arc cutting this perpendicular in E; then D E will equal the side of the square required. (Fig. 91.)

PROB. XXV. *To find the side of a square equal in area to any number of given squares.*

Let the lines A, B and C be the sides of the given squares. Draw D E equal to A, and at the end of it erect the perpendicular E F equal to B; join D F and perpendicular to it, from the point F, erect the perpendicular G F equal to C; then join D G, which will be equal to the side of a square equal in area to the three squares on A, B, and C. (Fig. 92.)

PROB. XXVI. *To find the side of a square nearly equal in area to a given circle*, A B C D.

Draw the two diameters, A C, B D, at right angles to each other; then bisect the radius E C in F, and through D and F produce the straight line D G cutting the circle in G, then will D G equal the side of the square required. (Fig. 93.)

PROB. XXVII. *Given the side of a regular polygon of any number of sides (not exceeding twelve), to find the radius of the circle in which it may be inscribed.*

Multiply the given side of the polygon by the number which stands in column *b*, opposite its proper name in the annexed Table of Polygons; the product will be the radius required.

Thus, suppose the polygon was to be an octagon, and each side 12, then $1{\cdot}3065628 \times 12 = 15{\cdot}6687536$ would be the radius sought. Take 15·67 as a radius from a diagonal scale, describe a circle, and from the same scale, taking off 12, it may be applied as the side of an octagon in that circle.

PROB. XXVIII. *Given the radius of a circle, to find the side of any regular polygon (not having more than twelve sides) inscribed in it.*

Multiply the given radius by the number in column *c*, standing opposite the number of sides of the proposed polygon: the product is the length of the side required.

Thus, suppose the radius of the circle to be 5, then $5 \times 1{\cdot}732051 = 8{\cdot}66025$, will be the side of the inscribed equilateral triangle.

TABLE OF POLYGONS.

No. of sides.	Names.	Multipliers for areas. (*a*)	Radius of circum. cir. (*b*)	Fac for s (
3	Trigon	0·4330127	0·5773503	1·73
4	Tetragon, or Square	1·0000000	0·7071068	1·41
5	Pentagon	1·7204774	0·8506508	1·17
6	Hexagon	2·5980762	1·0000000	1·00
7	Heptagon	3·6339124	1·1523824	0·86
8	Octagon	4·8284271	1·3065628	0·76
9	Nonagon	6·1818242	1·4619022	0·68
10	Decagon	7·6942088	1·6180340	0·61
11	Undecagon	9·3656399	1·7747324	0·56
12	Dodecagon	11·1961524	1·9318517	0·51

PROB. XXIX. *To reduce a simple rectilinear figure to a one upon either a smaller or a larger scale.*

Pitch upon a point P any where about the given figure A B either within it, or without it, or in one side or angle; but n middle is best. From that point P draw lines through all the upon one of which take P *a* to P A in the proposed proportion scales, or linear dimensions; then draw *a b* parallel to A B, B C, &c.; so shall *a b c d e* be the reduced figure sought, greater or smaller than the original. (*Hutton's Mens.*) (Fig.

Otherwise to Reduce a Figure by a Scale.—Measure all the and diagonals of the figure, as A B C D E, by a scale; and lay the same measures respectively from another scale, in the pro required.

To Reduce a Map, Design, or Figure, by Squares.—Divide t ginal into a number of little squares, and divide a fresh paper, dimensions required, into the same number of other squares, greater or smaller, in the proportion required. Then, in every of the second figure, draw what is found in the corresponding of the first or original figure.

The cross lines forming these squares may be drawn with a and rubbed out again after the work is finished. But a more and convenient way, especially when such reductions are wanted, would be to keep always at hand frames of squares made, of several sizes; for, by only just laying them down up papers, the corresponding parts may be readily copied. These may be made of four stiff or inflexible bars, strung across with hairs, or fine catgut.

When figures are rather complex, the reduction to a differen will be best accomplished by means of such an instrument as I sor Wallace's *Eidograph*, or by means of a *Pantograph*, an i ment which is now considerably improved by simply changi place of the fulcrum. See the *Mechanics' Oracle*, Part II. p.

CHAP. IV.

MENSURATION.

SECT. I. *Weights and Measures.*

MENSURATION is the application of arithmetic to geometry, by which we are enabled to discover the magnitude and dimensions of any geometrical figures, whether solid or superficial. To enable us to express this magnitude in determinate terms, it is necessary to assume some magnitude of the same kind as the *unit*, and then, by stating how many times the given magnitude contains that unit, we obtain its *measure*.

The different species of magnitude which have most frequently to be determined, are distinguishable into six kinds, viz.:—

1. Length.
2. Surface.
3. Solidity, or Capacity.
4. Force of Gravity, commonly called Weight.
5. Angles.
6. Time.

The several *units* assumed as the *standards* of measurement of each of these particular species of magnitude, are entirely arbitrary, and, consequently, vary among different nations. In this kingdom they have been fixed by Act of Parliament*, and are as follows, viz.:—

The standard of
- Length is a yard.
- Surface is a square yard, the $\frac{1}{4840}$th of an acre.
- Solidity is a cubic yard.
- Capacity is a gallon.
- Weight is a pound.
- Angular measurement is a degree, the 360th part of the circumference of the circle.
- Time is a day.

The values of the whole of the above are determined directly or indirectly by comparison with the length of a pendulum, which in the latitude of London, placed at the level of the sea, and in a

* 5 Geo. IV. c. 74; which took effect 1st January, 1826.

vacuum, would vibrate seconds of mean time. The length of such a pendulum being, to the length of the standard yard, as 39·1393 inches are to 36.

Since the passing of this act, however, some very elaborate and scientific experiments of Mr. Francis Baily have shown that errors of sufficient moment to be taken into the account in an inquiry of this kind, render the above proportion inaccurate*. We do not, in fact, yet know the length of a seconds' pendulum at London, vibrating in the circumstances proposed.

The following standard yards, made with great accuracy, give the annexed results:—

	Inches.
General Lambton's scale, used in India ...	35·99934
Sir George Shuckburgh's scale	35·99998
General Ray's scale	36·00088
Royal Society's standard	36·00135
Ramsden's bar	36·00249
Its copy, at Marischal College, Aberdeen...	36·00244

1. MEASURES OF LENGTH.

Inches.		Feet.		Yards.		Rods or Poles.		Furlongs.		Mile.
12	=	1								
36	=	3	=	1						
198	=	$16\frac{1}{2}$	=	$5\frac{1}{2}$	=	1				
7920	=	660	=	220	=	40	=	1		
63360	=	5280	=	1760	=	320	=	8	=	1

The mean length of a degree of latitude measured on the terrestrial meridian, is 69·0444 imperial miles; the 60th part of which, or 6075·6 feet, is the length of a nautical or geographical mile, three of which are equal to a league. And the length of a degree of longitude, measured upon the equator, is 69·1555 imperial miles.

An inch is the smallest lineal measure to which a name is given; but subdivisions are used for many purposes. Among mechanics the inch is commonly divided into *eighths*. By the officers of the revenue, and by scientific persons, it is divided into *tenths*, *hundredths*, &c. Formerly it was made to consist of 12 parts, called *lines*, but these have properly fallen into disuse.

Particular Measures of Length.

A Nail	=	$2\frac{1}{4}$	Inches	Used for measuring cloth of all kinds
Quarter	=	4	Nails	
Yard	=	4	Quarters	
Ell	=	5	Quarters	
Hand	=	4	Inches	Used for the height of horses.
Fathom	=	6	Feet	Used in measuring depths.
Link	=	7	Inc., 92 hdths.	Used in Land Measure, to facilit computation of the content,
Chain	=	100	Links	square chains being equal to an a

* See foot-note, p. 223.

2. MEASURES OF SURFACES.

Square Inches.	Sq. Feet.	Sq. Yards.	Perches or Rods.	Roods.	Acres.	Sq. Mile.
144 =	1					
1296 =	9 =	1				
39204 =	272¼ =	30¼ =	1			
1568160 =	10890 =	1210 =	40 =	1		
6272640 =	43560 =	4840 =	160 =	4 =	1	
40144896000 =	27878400 =	3097600 =	102400 =	2560 =	640 =	1

3. MEASURES OF SOLIDITY AND CAPACITY.

Division I. Measures of Solidity.

Cubic Inches.	Cubic Feet.	Cubic Yard.
1728 =	1	
46656 =	27 =	1

Division II.

Measures of Capacity for all liquids, and for all dry goods, except such as are comprised in the third division.

The imperial gallon (the standard for all measures of capacity) contains 10 pounds imperial avoirdupois weight of distilled water, weighed in air at 62° Fahr. (the barometer being at 30 inches); consequently, its capacity is 277·274 imperial cubic inches.

4 Gills	= 1 Pint	=	34·659 Cubic Inches.
2 Pints	= 1 Quart	=	69·318 „ „
4 Quarts	= 1 Gallon	=	277·274 „ „
2 Gall.	= 1 Peck	=	554·548 „ „
8 Gall.	= 1 Bushel	=	2218·192 „ „
8 Bush.	= 1 Quarter	=	10·269 Cubic Feet.
5 Qrs.	= 1 Load	=	51·345 „ „

The four last denominations are used for dry goods only. For liquids several denominations have been heretofore adopted, viz.:— for beer, the firkin of 9 gallons, the kilderkin of 18, the barrel of 36, the hogshead of 54, and the butt of 108 gallons. These will probably continue to be used in practice. For wine and spirits, there are, the anker, runlet, tierce, hogshead, puncheon, pipe, butt, and tun; but these may be considered rather as the names of the casks in which such commodities are imported, than as expressing any definite number of gallons. It is the practice to gauge all such vessels, and to charge them according to their actual content.

Flour is sold, nominally, by measure, but actually by weight, reckoned at 7 lb. avoirdupois to a gallon.

Division III.

Imperial Measures of Capacity for coals, culm, lime, fish, potatoes, fruit, and other goods, commonly sold by HEAPED MEASURE:

2 Gallons	= 1 Peck	= 704	}	Cubic Inches, nearly.
8 Gallons	= 1 Bushel	= 2815½	}	
3 Bushels	= 1 Sack	= 4⅝	}	Cubic Feet, nearly.
12 Sacks	= 1 Chald.	= 58⅔	}	

The goods are to be heaped up in the form of a cone, to a height above the rim of the measure of at least ¼ of its depth. The outside diameter of measures used for heaped goods are to be at least double the depth; consequently not less than the following dimensions:—

Bushel	19½ inches.	Peck	12¼ inches.
Half-bushel	15½ inches.	Gallon	9¾ inches.
		Half-gallon	7¾ Inches.

The imperial measures described in the second and third divisions were established by Act 5 Geo. IV. c. 74. Before that time there were four different measures of capacity used in England, viz.:—

1. For wine, spirits, cyder, oils, milk, &c., the gallon of which contained ..... } 231 cubic inches.
2. For malt liquors, the gallon of which contained ..... } 282 „ „
3. For corn and all other dry goods, not heaped, the gallon of which contained ... } 268·8 „ „
4. For coals, which did not differ sensibly from the imperial measure.

Hence, with respect to wine, ale, and corn, it will be expedient to possess a

Table of Factors,

For converting old measures into new, and the contrary.

	By Decimals.			By Vulgar Fractions nearly.		
	Corn Measure.	Wine Measure.	Ale Measure.	Corn Measure.	Wine Measure.	Ale Measure.
To convert old measures to new.	·96943	·83311	1·01704	31/32	5/6	60/59
To convert new measures to old.	1·03153	1·20032	·98324	32/31	6/5	59/60

N.B. For reducing the *prices*, these numbers must all be reversed.

4. MEASURES OF WEIGHT.

Division I. Avoirdupois Weight.

The standard of weight is the avoirdupois pound, the value of which is determined by its being the weight of 27·7274 cubic inches of distilled water, weighed in air at 62° Fahr., with the barometer at 30 inches.

Avoirdupois weight is used in almost all commercial transactions, and in the common dealings of life; its divisions are as follows:—

Drachms.		Ounces.		Pounds.		Quarters.		Hundred-weights.		Ton.
16	=	1								
256	=	16	=	1						
7168	=	448	=	28	=	1				
28672	=	1792	=	112	=	4	=	1		
573440	=	35840	=	2240	=	80	=	20	=	1

Particular weights belonging to this Division.

			cwt.	qr.	lb.	
8 Pounds	= 1 Stone	=	0	0	8	Used for meat.
14 „	= 1 „	=	0	0	14	Used in the wool trade.
2 Stone	= 1 Tod	=	0	1	0	
6½ Tod	= 1 Wey	=	1	2	14	
2 Weys	= 1 Sack	=	3	1	0	
12 Sacks	= 1 Last	=	39	0	0	

Division II. Troy Weight.

For weighing gold, silver, and precious stones (except diamonds), troy weight is employed, its divisions are—

Grains (grs.)		Penny-weights (dwts.)		Ounces (oz.)		Pound (lb.)
24	=	1				
480	=	20	=	1		
5760	=	240	=	12	=	1

But troy weight is also used by apothecaries in compounding medicines; and is then divided as follows, viz.:—

Grains (gr.)	Scruples (℈.)		Drachms (ʒ.)		Ounces (℥)		Pound (℔.)
20	= 1						
60	= 3	=	1				
480	= 24	=	8	=	1		
5760	= 288	=	96	=	12	=	1

The troy pound is equal to the weight of 22·8157 cubic inches of distilled water, weighed in air at 62° Fahr., and the barometer at 30 inches; and is, therefore, less than the avoirdupois pound, in the proportion of 144 to 175.

		oz.	dwts.	grs.			
1 lb. avoirdupois	=	14	11	$15\frac{1}{2}$	troy	=	7000 troy grains.
1 oz. ,,	=	0	18	$5\frac{1}{2}$	,,	=	437·5 ,,
1 dr. ,,	=	0	1	$3\frac{1}{2}$	,,	=	27·343 ,,

1 troy lb. = 0·822857 avoirdupois lb.
1 avoir. lb. = 1·215271 troy lb.

For scientific purposes the grain only is used; and sets of weights are constructed in decimal progression, from 10,000 grs. downwards to $\frac{1}{10}\frac{1}{0}$th of a grain.

The *carat*, used for weighing diamonds, is $3\frac{1}{6}$ grains. The term, however, when used to express the fineness of gold, has a relative meaning only. Every mass of alloyed gold is supposed to be divided into 24 equal parts; thus the standard for coin is 22 carats fine, that is, it consists of 22 parts of pure gold, and 2 parts of alloy. What is called the *new standard*, used for watch-cases, &c. is 18 carats fine.

5. ANGULAR MEASURE; OR, DIVISIONS OF THE CIRCLE.

60 Seconds ″	=	1 Minute, denoted by ′
60 Minutes	=	1 Degree, ,, °
30 Degrees	=	1 Sign ,, ˢ
90 Degrees	=	1 Quadrant.
360 Degrees, or 12 Signs	=	1 Circumference.

Formerly the subdivisions were carried on by sixties; thus, the second was divided into 60 thirds, the third into 60 fourths, &c. At present, the second is more generally divided decimally into 10ths, 100ths, &c. The degree is frequently so divided.

6. MEASURE OF TIME.

60 Seconds	=	1 Minute.
60 Minutes	=	1 Hour.
24 Hours	=	1 Mean Solar Day.
23 H. 56 M. 3·5 S.	=	1 Sidereal Day.
7 Days	=	1 Week.
28 Days	=	1 Lunar Month.
28, 29, 30, or 31 Days	=	1 Calendar Month.
12 Calendar Months	=	1 Year.
365 Days	=	1 Common Year.
366 Days	=	1 Leap Year.
$365\frac{1}{4}$ Days	=	1 Julian Year.
365 D. 5 H. 48 M. $45\frac{1}{2}$ S.	=	1 Solar Year.
365 D. 6 H. 9 M. 11 S.	=	1 Sidereal Year.

A *solar* day is the time that elapses between two successive transits of the sun over the same meridian, and is not always of equal duration, being longer at some seasons of the year than at others; the difference between the *actual* length of a solar day and the *mean* length, is called the *equation of time*. A *sidereal* day is the interval of time that elapses between two successive transits of any fixed star

over the same meridian, and is the most uniform of all astronomical periods, neither theory nor observation having detected the slightest variation in its length. A *solar* year is the time in which the earth passes through the twelve signs of the zodiac, and is the natural year, because it always keeps the same seasons in the same months. The *sidereal* year is the time that elapses between the earth leaving any fixed star and returning to it again.

In 400 years, 97 are leap years, and 303 common.

The same remark, as in the case of angular measure, applies to the mode of subdividing the second of time.

COMPARISON OF ENGLISH AND FRENCH WEIGHTS AND MEASURES.

The following is a comparative Table of the Weights and Measures of England and France, which were published by the Royal and Central Society of Agriculture of Paris, in the Annuary for 1829, and founded on a Report, made by Mr. Mathieu, to the Royal Academy of Sciences of France, on the bill passed the 17th of May, 1824, relative to the Weights and Measures termed "Imperial," which are now used in Great Britain.

Measures of Length.

ENGLISH.		FRENCH.
1 Inch (1–36th of a yard)	=	2·539954 centimetres.
1 Foot (1–3rd of a yard	=	3·0479449 decimetres.
Yard imperial	=	0·91438348 metre.
Fathom (2 yards)	=	1·82876696 metre.
Pole, or perch (5½ yards)	=	5·02911 metres.
Furlong (220 yards)	=	201·16437 metres.
Mile (1760 yards)	=	1609·3149 metres.

FRENCH.		ENGLISH.
1 Millimetre	=	0·03937 inch.
1 Centimetre	=	0·393708 inch.
1 Decimetre	=	3·937079 inches.
1 Metre	=	39·37079 inches. 3·2808992 feet. 1·093633 yard.
Myriametre	=	6·2138 miles.

Square Measures.

ENGLISH.		FRENCH.
1 Yard square	=	0·836097 metre square.
1 Rod (square perch)	=	25·291939 metres square.
1 Rood (1210 yards square)	=	10·116775 ares.
1 Acre (4840 yards square)	=	0·404671 hectare.

FRENCH.		ENGLISH.
1 Metre square	=	1·196033 yard square.
1 Are	=	0·098845 rood.
1 Hectare	=	2·473614 acres.

Solid Measures.

ENGLISH.		FRENCH.
1 Pint (1–8th of a gallon)	=	0·567932 litre.
1 Quart (1–4th of a gallon)	=	1·135864 litre.
1 Gallon imperial	=	4·54345794 litres.
1 Peck (2 gallons)	=	9·0869159 litres.
1 Bushel (8 gallons)	=	36·347664 litres.
1 Sack (3 bushels)	=	1·09043 hectolitre.
1 Quarter (8 bushels)	=	2·907813 hectolitres.
1 Chaldron (12 sacks)	=	13·08516 hectolitres.

FRENCH.		ENGLISH.
1 Litre	=	1·760773 pint. 0·2200967 gallon.
1 Decalitre	=	2·2009667 gallons.
1 Hectolitre	=	22·009667 gallons.

Weights.

ENGLISH TROY.		FRENCH.
1 Grain (1-24th of a pennyweight)	=	0·06477 gramme.
1 Pennyweight (1-20th of an ounce)	=	1·55456 gramme.
1 Ounce (1-12th of a pound troy)	=	31·0913 grammes.
1 Pound troy imperial	=	0·3730959 kilogramme.

ENGLISH AVOIRDUPOIS.		FRENCH.
1 Drachm (1-16th of an ounce)	=	1·7712 gramme.
1 Ounce (1-16th of a pound)	=	28·3384 grammes.
1 Pound avoirdupois imperial	=	0·4534148 kilogramme.
1 Hundred-weight (112 pounds)	=	50·78246 kilogrammes.
1 Ton (20 hundred-weight)	=	1015·649 kilogrammes.

FRENCH.		ENGLISH.
1 Gramme	=	15·438 grains troy. 0·643 pennyweight. 0·03216 ounce troy.
1 Kilogramme	=	2·68027 pounds troy. 2·20548 pounds avoirdupois

Angular Measure.

In France, the centesimal division of the circle is frequently employed, in which the whole circumference is divided into 400 degrees, each degree into 100 minutes, each minute into 100 seconds, &c.

CENTESIMAL.		ENGLISH.
1 Degree	=	54 minutes.
1 Minute	=	32·4 seconds.
1 Second	=	·324 second.

ENGLISH.		CENTESIMAL.
1 Degree	=	$1\frac{1}{9}$ degree.
1 Minute	=	1·85185 minute.
1 Second	=	3·08641 seconds.

SECT. II. *Mensuration of Superficies.*

TRIANGLES.

LET b, c, and d, represent the three sides of a triangle (see fig. 95); β, γ, and δ, the angles opposite those sides respectively; and h the perpendicular height from the vertex to the base b; then the

$$\text{area} = \tfrac{1}{2} b h,$$

$$\text{or} = \tfrac{1}{2} b c \,.\, \sin \delta = \tfrac{1}{2} c d \,.\, \sin \beta = \tfrac{1}{2} d b \,.\, \sin \gamma.$$

$$\text{Or,} = \sqrt{d^2 - \left(\frac{d^2 - c^2}{2b} + \frac{b}{2}\right)^2} \cdot \frac{b}{2}$$ where b is the greatest side and c the least.

If half the sum of the three sides or $\frac{b + c + d}{2} = s$, then the

$$\text{area} = \sqrt{s\,(s - b) \,.\, (s - c) \,.\, (s - d)},$$ and the

$$\log \text{ of area} = \tfrac{1}{2} \{\log s + \log (s - b) + \log (s - c) + \log (s - d)\}.$$

For the method of obtaining the unknown sides or angles of triangles from those which are known, see Chap. V., Sect. IV.

QUADRILATERALS, OR FOUR-SIDED FIGURES.

Square, Rectangle, Rhombus and Rhomboid:—To obtain the area, multiply the perpendicular height (h, fig. 96) by the base (b) on which it falls.

The area of a *Trapezium* is best found by dividing it into two triangles (as fig. 17), the areas of which may be found by the foregoing rules.

The area of a *Trapezoid* (fig. 16) may either be found in the same way, or by multiplying half the sum of the two parallel sides ($a + b$) by the perpendicular distance between them (h).

POLYGONS.

To obtain the area of any regular polygon, having less than twelve sides, multiply the square of one of the sides by the number found in column a, of the Table of Polygons, page 118. Or, generally, if l = the length of one of the equal sides, and n the number of them; then the

$$\text{area} = l^2 \frac{n}{4} \tan \,.\, \left(\frac{90\,n - 180}{n}\right).$$

The area of an irregular polygon may be determined by dividing it into triangles, as in fig. 97. Or by forming one triangle equal in area to it, by Prob. XX., page 116.

CIRCLES.

Let d represent the diameter, c the circumference, a the area, and $p = 3{\cdot}14159$ (see Table V. in the Appendix); then the

$$\text{Diameter} = d = \frac{c}{p} = \frac{4a}{c} = 2\sqrt{\frac{a}{p}}.$$

$$\text{Circumference} = c = pd = \frac{4a}{d} = 2\sqrt{pa}.$$

$$\text{Area} = a = \frac{pd^2}{4} = \frac{c^2}{4p} = \frac{dc}{4} = \cdot 7854\, d^2.$$

$$\text{and } p = \frac{c}{d} = \frac{4a}{d^2} = \frac{c^2}{4a}.$$ *

Circular Arcs.—If r represents the radius, and d the diameter of the circle; s the sine, and v_1 the versed sine of the arc; c the chord of half the arc, v_2 the versed sine of half the arc; and m the measure in degrees of the whole arc; then

$$\text{the length of the arc} = \cdot 0174533\, r\, m;$$

$$\text{or} = 2d\sqrt{\frac{3v_1}{3d - v_2}} \text{ nearly};$$

$$\text{or} = \tfrac{2}{3}\left\{5d\sqrt{\frac{5v_1}{5d - 3v_2}} + 4\sqrt{dv_2}\right\} \text{ nearly};$$

$$\text{or} = \frac{8c - 2s}{3} \text{ nearly}.$$

And the following relations between the several quantities, from which any one of them may be obtained, will be found useful, viz.:—

$$v_1 = r - \sqrt{r^2 - s^2} \quad \ldots\ldots\ldots\ldots \quad (1.)$$

$$c = \sqrt{s^2 + {v_1}^2} \quad \ldots\ldots\ldots\ldots \quad (2.)$$

$$s = \sqrt{r^2 - (r - v_1)^2} \quad \ldots\ldots\ldots\ldots \quad (3.)$$

$$d = \frac{s^2}{v_1} + v_1 \quad \ldots\ldots\ldots\ldots \quad (4.)$$

Circular Sectors.—Let d represent the diameter of the circle, l the length of the arc of the sector, and m its measure in degrees; then the

$$\text{area} = \tfrac{1}{2} r\, l.$$
$$\text{or} = \cdot 00218\, d^2\, m.$$

Circular Segments.—Let d represent the diameter of the circle, v the versed sine, c_1 the chord of the whole arc, and c_2 the chord of half the arc; then the

$$\text{area} = \tfrac{4}{5}\left\{\sqrt{(dv - v^2)} + \tfrac{2}{3}\sqrt{dv}\right\} \text{ nearly};$$
$$\text{or} = \tfrac{4}{3} v\,(dv - \tfrac{3}{5} v^2) \text{ nearly};$$
$$\text{or} = \tfrac{4}{10} v\,(c_1 + \tfrac{4}{3} c_2) \text{ nearly};$$
$$\text{or} = \tfrac{4}{3} v\sqrt{(\tfrac{1}{4}{c_1}^2 + \tfrac{2}{5} v^2)} \text{ nearly}.$$

* Table VI. in the Appendix contains the diameter, circumference, area, and length of the side of an equal square to circles from 1 to 100·75 in diameter.

Or the area of any segment, as fig. 98, may be obtained by finding the area of the sector A B C D, and subtracting from it the area of the triangle A C D.

The area of a *circular zone*, as E F G H, may be found by subtracting the area of the segment G H I from the area of the segment EGIHF.

PARABOLA.

The area of a parabola is equal to two-thirds of the product of the base × the perpendicular height.

For the relations between the abscissæ and ordinates in the parabola, and the method of deriving one from the other, see Chap. VI. page 172.

Parabolic Arcs.—Let x be the abscissa $a\,b$, fig. 99, measured from the vertex a, and y the corresponding right ordinate $c\,b$; then the length of the arc $c\,a\,d$ is

$$= 2\sqrt{(y^2 + \tfrac{4}{3}x^2)} \text{ nearly;}$$

$$\text{or} = \tfrac{2}{5}\left\{\sqrt{(y^2 + \tfrac{4}{3}x^2)} - \frac{y^2 + \frac{9}{3}x^2}{\frac{1}{4}y}\right\}.$$

Parabolic Frustum, or Zone.—The word *zone* is here used to denote the space $c\,d\,e\,g$ (fig. 99), contained between the two parallel double ordinates $c\,d$ and $e\,g$. Let y_1 = the length of the double ordinate $c\,d$, and y_2 = the length of $e\,g$, and d = the perpendicular distance between them $b\,f$; then the

$$\text{area of the zone} = \tfrac{2}{3}\,d\,\frac{{y_2}^3 - {y_1}^3}{{y_2}^2 - {y_1}^2}.$$

ELLIPSE.

The *area* is equal to the product of the transverse diameter × the conjugate diameter × ·785398.

The *periphery* or circumference is equal to the *sum* of the transverse and conjugate diameters × 1·57079.

The foregoing is only an approximation to the periphery, although sufficiently near the truth for ordinary purposes; but where greater accuracy is required the following series may be employed. Let t represent the transverse axis, c the conjugate, $p = 3{\cdot}14159$, &c., and $d = 1 - \frac{c^2}{t^2}$; then

$$p\,t\left\{1 - \frac{d}{2^2} - \frac{3\,d^2}{2^2 \cdot 4^2} - \frac{3^2 \cdot 5\,d^3}{2^2 \cdot 4^2 \cdot 6^2} - \frac{3^2 \cdot 5^2 \cdot 7\,d^4}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} - \text{&c.}\right\}$$

will be the periphery.

Elliptic Segments.—To find the area of the elliptic segment $a\,b\,c$ (fig. 100), find the area of the corresponding circular segment, $d\,b\,e$, described on the same axis ($b\,f$) to which the cutting line or base of

the segment $a\,c$ is perpendicular. Then, as this axis $(b\,f)$: other axis $(g\,h)$ :: the circular segment $(d\,b\,e)$: the elliptic ment $(a\,b\,c)$.

Elliptic Arcs.—Let t represent the semi-transverse, and c semi-conjugate diameters of the ellipse, and d the distance of ordinate from the centre, then the length of the arc conta between the ordinate and the parallel semi-axis will be

$$= d\left\{1 + \frac{c^2}{6t^4}d^2 + \frac{4t^2c^2 - c^4}{40t^8}d^4 + \frac{8t^4c^2 - 4t^2c^4 + c^6}{112t^{12}}d^6 + \&c\right.$$

or make $\frac{t^2 - c^2}{t^2} = r$; then the

$$\text{length of the arc} = d\sqrt{\frac{t^2 - \frac{1}{3}rd^2}{t^2 - \frac{1}{3}d^2}} \text{ nearly.}$$

HYPERBOLA.

The *area* of an hyperbola or hyperbolic segment may be found follows:—let t and c represent the semi-transverse and semi-conjugate diameters, $2y$ the double ordinate which cuts off the segment and x its abcissa; also $q = \frac{x}{2t + x}$; then the

$$\text{area} = 2xy\left\{\frac{1}{3} - \frac{q}{3 \cdot 5} - \frac{q^2}{3 \cdot 5 \cdot 7} - \frac{q^3}{3 \cdot 5 \cdot 7 \cdot 9} - \&c.\right\}$$

$$\text{or} = \frac{4cx}{15t}\left\{4\sqrt{(2tx + \tfrac{3}{4}x^2)} + \sqrt{2tx}\right\} \text{ nearly.}$$

Hyperbolic Arcs.—The notation being the same, the follow approximation may be employed to obtain the length of an hyperbolic arc.

$$\text{Arc} = \frac{120c^2t + (19t^2 + 21c^2)4x}{120c^2t + (9t^2 + 21c^2)4x} + y, \text{ nearly.}$$

SECT. III. *Mensuration of Solids.*

PARALLELOPIPEDON, PRISM, OR CYLINDER.

Surface.—Multiply the perimeter of one end by the length perpendicular height, to which add the area of the two ends, sum will be the surface.

Solidity = the area of the base × the perpendicular height.

PYRAMID OR CONE.

Surface.—Multiply half the perimeter by the slant height, to w add the area of the base, the sum is the surface.

Solidity = the area of the base × one-third the perpendic height.

Frustum of a Pyramid or Cone. Surface.—Multiply half the sum of the perimeters of the two ends by the perpendicular height.

Solidity.—Add a diameter or side of the greater end to one of the less, and from the square of the sum subtract the product of the said two diameters or sides; then multiply the remainder by one-third of the height, and this product by ·785398 for circles, or by the proper multiplier for polygons; the last product will be the capacity.

That is, let D equal the greater diameter, d the less, h the perpendicular height, and $p = ·785398$ for cones, or for any pyramid the proper multiplier from column (a) in the table at page 118, then

$$\text{Solidity} = \tfrac{1}{3} p h (D^2 + D d + d^2).$$

SPHERE.

Surface = the diameter × the circumference, = the square of the diameter × 3·14159, = the square of the circumference × ·3183.

Solidity = the cube of the diameter × ·5236, = the cube of the circumference × ·01688.

Spherical Segment.—Let d equal the diameter of the sphere, r the radius of the base of the segment, and h its height; then the *curved surface* $= 3·14159\, d\, h$.

$$\textit{Solidity} = 0·5236\, h^2 (3\, d - 2\, h);$$

$$\text{or} = 0·5236\, h\, (3\, r^2 + h^2).$$

The surface and solidity of a *spherical zone* may be obtained by taking the difference between the two segments.

CONOIDS.

A *conoid* is the solid generated by the revolution of a conic section about one of its axes, and is called a *spheroid*, *paraboloid*, or *hyperboloid*, according to the section from which it is produced.

Spheroids.—When the *ellipse* revolves about its transverse axis, it is called an *oblong* or *prolate spheroid;* when about its conjugate axis, an *oblate spheroid;* and when about any other of its diameters, a *universal spheroid*, in which latter case its figure is somewhat resembling a heart. To obtain the *solidity* of a spheroid, multiply the square of the revolving axis by the fixed axis, and the product by 0·5236.

Paraboloid.—Let y be the radius of the circular base, and x the altitude of the solid; then

$$\textit{Surface} = \frac{0·5236\, y}{x^2} \{(y^2 + 4\, x^2)^{\frac{3}{2}} - y^3\}.$$

$$\textit{Solidity} = 1·5708\, y^2\, x.$$

Hyperboloid.—Let t equal the transverse axis, r the radius of the base, a the altitude, and p the parameter; then the

$$\textit{Solidity} = \tfrac{1}{2} p^2 \frac{t + \frac{2}{3} a}{t + a}.$$

THE REGULAR OR PLATONIC SOLIDS.

The regular or Platonic bodies are five in number, and have already been described at page 111.

1. *To find either the surface or the solid content of any of the regular bodies.*—Multiply the proper tabular area or surface (taken from column (*a*) in the following table) by the square of the linear edge of the solid, for the *superficies*. And

Multiply the tabular solidity in column (*b*) of the table by the cube of the linear edge for the *solid content*.

Surfaces and Solidities of Regular Bodies, the side being unity or 1

No. of sides.	Name.	Surface. (*a*)	Solidity. (*b*)
4	Tetraedron	1·7320508	0·1178513
6	Hexaedron	6·0000000	1·0000000
8	Octaedron	3·4641016	0·4714045
12	Dodecaedron	20·6457288	7·6631189
20	Icosaedron	8·6602540	2·1816950

2. *The diameter of a sphere being given, to find the side of one of the Platonic bodies, that may be either inscribed in the sphere, circumscribed about the sphere, or that is equal to the sphere.*—Multiply the given diameter of the sphere by the proper or corresponding number, in the following table, answering to the thing sought, and the product will be the side of the Platonic body required.

The diam. of a sphere being 1; the side of a	That may be inscribed in the sphere, is	That may be circumscribed about the sphere, is	That is equal to the sphere, is
Tetraedron	0·8164966	2·4494897	1·6439480
Hexaedron	0·5773503	1·0000000	0·8059958
Octaedron	0·7071068	1·2247447	1·0356300
Dodecaedron	0·3568221	0·4490279	0·4088190
Icosaedron	0·5257309	0·6615845	0·6214433

3. *The side of any of the five Platonic bodies being given, to find the diameter of a sphere, that may either be inscribed in that body, or circumscribed about it, or that is equal to it.*—As the respective number in the table above, under the title *inscribed*, *circumscribed*, or *equal*, is to 1, so is the side of the given Platonic body to the diameter of its inscribed, circumscribed, or equal sphere.

4. *The side of any one of the five Platonic bodies being given, to find the side of the other four bodies, that may be equal in solidity to that of the given body.*—As the number under the title *equal* in the

last column of the table above, against the given Platonic body, is to the number under the same title, against the body whose side is sought, so is the side of the given Platonic body to the side of the body sought.

Besides these there are *thirteen* demiregular bodies, called *Solids of Archimedes*. They are described in the Supplement to *Lidonne's Tables de tous les Diviseurs des Nombres*, &c., Paris, 1808; twelve of them were described by Abraham Sharp, in his Treatise on Polyedra.

TO FIND THE CONTENTS OF SURFACES AND SOLIDS NOT REDUCIBLE TO ANY KNOWN FIGURE, BY THE EQUIDISTANT ORDINATE METHOD.

The general rule is included in this proposition, viz.:—If any right line be divided into any *even* number of equal parts, 1, 2, 3, 4, &c., (fig. 101,) and at the points of division be erected perpendicular ordinates 1 A, 2 B, 3 C, &c., terminated by any curve A C G: then, if s be put for the sum of the first and last ordinates, 1 A, 7 G, e for the sum of the *even* ordinates, 2 B, 4 D, 6 F, &c., viz., the second, fourth, sixth, &c., and o for the sum of all the rest, 3 C, 5 E, &c., viz., the third, fifth, &c., or the *odd* ordinates, excepting the first and last: then, the common distance 1 2, 2 3, &c., of the ordinates being multiplied into the sum arising from the addition of s, four times e, and twice o, *one third* of the product will be the area 1 A G 7, very nearly.

That is, $\frac{s + 4e + 2o}{3}$. D = area, D being = A C = C E, &c.

The same theorem will equally serve for the contents of all solids, by using the sections perpendicular to the axis instead of the ordinates. The proposition is quite accurate, for all parabolic and right lined areas, as well as for all solids generated by the revolutions of conic sections or right lines about axes, and for pyramids and their frustums. For other areas and solidities it is an excellent approximation.

The greater the number of ordinates, or of sections, that are taken, the more accurately will the area or the capacity be determined. But in a great majority of cases *five* equidistant ordinates, or sections, will lead to a very accurate result.

CHAP. V.

TRIGONOMETRY.

SECT. I. *Definitions and Trigonometrical Formulæ.*

1. *Plane Trigonometry* is that branch of mathematics by which we learn how to determine or compute the unknown parts of a plane, or rectilinear triangle, from those which are known, when that is possible.

Every triangle consists of six parts, viz., three sides, and three angles opposite those sides. And any three of these being given (excepting only when the three angles are given) the others may always be determined from them.

The determination of the mutual relation of the sines, tangents, secants, &c., of the sums, differences, multiples, &c., of arcs or angles; or the investigation of the connected formulæ, is also usually classed under plane trigonometry.

2. Let A C B (fig. 102) be a rectilinear angle: if about C as a centre, with any radius C A, a circle be described, intersecting C A, C B, in A, B, the arc A B is called the *measure* of the angle A C B.

3. The circumference of a circle is supposed to be divided or to be divisible into 360 equal parts, called *degrees;* each degree into 60 equal parts, called *minutes;* each of these into 60 equal parts, called *seconds;* and so on to the minutest possible subdivisions. Of these, the first is indicated by a small circle, the second by a single accent, the third by a double accent, &c. Thus, 47° 18′ 34″ 45‴, denotes 47 degrees, 18 minutes, 34 seconds, and 45 thirds. The number of degrees, minutes, seconds, &c., contained in the arc A B of the circle described from the angular point C, and which is contained between the two legs A C, A B, is called the measure of the angle A C B, which is then said to be an angle of so many degrees, minutes, seconds, &c. Thus, since a quadrant, or quarter of a circle, contains 90 degrees, and a quadrantal arc is the measure of a right angle, a right angle is said to be one of 90 degrees.

4. The *complement of an arc* is its difference from a quadrant, as B E; and the *complement of an angle* is its difference from a right angle, as E C B.

5. The *supplement of an arc* is its difference from a semicircle, as A′ E B; and the *supplement of an angle* is its difference from two right angles, as A′ C B.

6. The *sine* of an arc is a perpendicular let fall from one extremity upon a diameter passing through the other, as B D.

7. The *versed sine* or *versine* of an arc is that part of the diameter which is intercepted between the foot of the sine and the arc, as D A.

8. The *tangent* of an arc is a right line which touches it in one extremity, and is limited by a right line drawn from the centre of the circle through the other extremity, as A T.

9. The *secant* of an arc is the radial line which thus limits the tangent, as C T.

10. These are also, by way of accommodation, said to be the sine, tangent, &c., of the angle measured by the aforesaid arc, to its determinate radius.

11. The *cosine* of an arc or angle, is the sine of the complement of that arc or angle, as G B: the *cotangent* of an arc or angle is the tangent of the complement of that arc or angle, as E M. And the *co-versed sine* E G, and *cosecant* C M, are similarly the versed sine and secant of the complement.

12. The *suversine* of an arc is the versed sine of its supplement, as A D.

Note.—The following contractions are employed to express the foregoing terms, viz.:—

For the *radius* of the arc	A B	we write *rad*	A B,
sine	ditto	..... *sin*	A B,
tangent	ditto	..... *tan*	A B,
secant	ditto	..... *sec*	A B,
versine	ditto	..... *versin*	A B,
cosine	ditto	..... *cos*	A B,
cotangent	ditto	..... *cot*	A B,
cosecant	ditto	..... *cosec*	A B,
coversine	ditto	..... *covers*	A B.

13. *The following Corollaries may be drawn from the above Definitions:—*

(A.) When the arc is evanescent, the sine, tangent, and versed sine, are evanescent also, and the secant becomes equal to the radius, that being its minimum limit. As the arc increases from this state, the sines, tangents, secants, and versed sines increase; thus they continue till the arc becomes equal to a quadrant A E, and then the sine is in its maximum state, being equal to the radius, and is then called the *sine total;* the versed sine is also then equal to the radius; and the secant and tangent becoming incapable of mutually limiting each other, are regarded as infinite.

In employing these lines for the purposes of calculation, they are all considered as having *positive* values for any arc not exceeding 90°; but in the second quadrant, the cosine falling on the opposite side of the diameter, and being measured in an opposite direction, is considered *negative*, but the sine remains *positive;* in the third quadrant

the cosine is still *negative*, and the sine, having now changed its direction, is *negative* also; in the fourth quadrant, the cosine having again returned to that side of the diameter on which it was in the first quadrant, again becomes *positive*, but the sine remains *negative*. The signs of the others are determined by the ordinary rules of algebra, from the formulæ at page 137. The following table exhibits both the value of trigonometrical lines at the commencement of each quadrant, and also the signs with which they are affected in passing through the same.

	Value at 0°.	Sign in 1st Quad.	Value at 90°.	Sign in 2nd Quad.	Value at 180°.	Sign in 3rd Quad.	Value at 270°.	Sign in 4th Quad.	Value at 360°.
Sin ...	0	+	R	+	0	—	R	—	0
Tan ...	0	+	∞	—	0	+	∞	—	0
Sec ...	R	+	∞	—	R	—	∞	+	R
Versin	0	+	R	+	2 R	+	R	+	0
Cos ...	R	+	0	—	R	—	0	+	R
Cot ...	∞	+	0	—	∞	+	0	—	∞
Cosec	∞	+	R	+	∞	—	R	—	∞

R signifies equal to rad; ∞ — infinite; 0 — evanescent.

(B.) An arc and its supplement have the same sine, tangent, and secant.

(C.) Of any arc less than a quadrant, the arc is less than its corresponding tangent; and of any arc whatever, the chord is less than the arc, and the sine less than the chord; but the smaller the arc, the nearer they all approach to equality.

(D.) The sine B D of an arc A B, is half the chord B F of the double arc B A F.

(E.) The versed sine of an arc, together with its cosine, are equal to the radius. Thus, A D + B G = A D + D C = A C.

(F.) The radius, tangent, and secant, constitute a right-angled triangle C A T; and the cosine, sine, and radius, constitute another right-angled triangle C D B, similar to the former. So, again, the cotangent, radius, and cosecant, constitute a third right-angled triangle M E C, similar to both the preceding. Hence, when the sine and radius are known, the cosine is determined by the property of the right-angled triangle. The same may be said of the determination of the secant, from the tangent and radius, &c. &c. &c.

(G.) Further, since the triangles C A T and M E C are similar, A T : C A :: B D : D C :: C E : E M, or tan : rad :: sin : cos :: rad : cot.

Also C T : C A :: C B : C D :: C M : E M, or sec : rad :: rad : cos : : cosec : cot.

And C M : C E :: C B : D B or cosec : rad :: rad : sin.

(H.) Also, by Geom. Sect. V. Prop. XXII., as A D : D B : : D B : HD, or $\sin^2 = HD \cdot \text{versin}$; or since H D = rad + cos, then $\text{versin} = \dfrac{\sin^2}{\text{rad} + \cos}$.

(I.) From Arts. (F.) (G.) and (H.) we deduce the following formulæ :—

1. $\sin = \sqrt{\text{rad}^2 - \cos^2} = \dfrac{\text{rad}^2}{\text{cosec}}$.

2. $\tan = \sqrt{\sec^2 - \text{rad}^2} = \dfrac{\text{rad} \cdot \sin}{\cos} = \dfrac{\text{rad}^2}{\cot}$.

3. $\sec = \sqrt{\text{rad}^2 + \tan^2} = \dfrac{\text{rad}^2}{\cos}$.

4. $\text{versin} = \dfrac{\sin^2}{\text{rad} + \cos}$.

5. $\cos = \sqrt{\text{rad}^2 - \sin^2} = \dfrac{\text{rad}^2}{\sec}$.

6. $\cot = \sqrt{\text{cosec}^2 - \text{rad}^2} = \dfrac{\text{rad} \cdot \cos}{\sin} = \dfrac{\text{rad}^2}{\tan}$.

7. $\text{cosec} = \sqrt{\text{rad}^2 + \cot^2} = \dfrac{\text{rad}^2}{\sin}$.

8. $\text{covers} = \dfrac{\cos^2}{\text{rad} + \sin}$.

(K.) If unity be regarded as the radius of the circle, the above formulæ become :—

1. $\sin = \sqrt{1 - \cos^2} = \dfrac{1}{\text{cosec}}$.

2. $\tan = \sqrt{\sec^2 - 1} = \dfrac{\sin}{\cos} = \dfrac{1}{\cot}$.

3. $\sec = \sqrt{1 + \tan^2} = \dfrac{1}{\cos}$.

4. $\text{versin} = \dfrac{\sin^2}{1 + \cos}$.

5. $\cos = \sqrt{1 - \sin^2} = \dfrac{1}{\sec}$.

6. $\cot = \sqrt{\text{cosec}^2 - 1} = \dfrac{\cos}{\sin} = \dfrac{1}{\tan}$.

7. $\text{cosec} = \sqrt{1 + \cot^2} = \dfrac{1}{\sin}$.

8. $\text{covers} = \dfrac{\cos^2}{1 + \sin}$.

14. The following are some of the most useful formulæ relating to trigonometrical lines, in all of which radius is regarded as unity.

(*a.*) *Expressions for the sine and cosine:*—Let a equal the length of the arc, then

$$\sin a = a - \frac{a^3}{1.2.3} + \frac{a^5}{1.2.3.4.5} - \frac{a^7}{1.2.3.4.5\ 6.7} + \&c. \quad (1)$$

$$\cos a = 1 - \frac{a^2}{1.2} + \frac{a^4}{1.2.3.4} - \frac{a^6}{1.2.3.4.5.6} + \&c. \quad \ldots\ldots (2).$$

(*b.*) *Expressions for the sum and difference of two arcs:*—Let a and b be the two given arcs, then

$$\sin(a + b) = \sin a \cos b + \cos a \sin b \quad \ldots\ldots (3)$$

$$\sin(a - b) = \sin a \cos b - \cos a \sin b \quad \ldots\ldots (4)$$

$$\cos(a + b) = \cos a \cos b - \sin a \sin b \quad \ldots\ldots (5)$$

$$\cos(a - b) = \cos a \cos b + \sin a \sin b \quad \ldots\ldots (6)$$

$$\tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b} \quad \ldots\ldots (7)$$

$$\tan(a - b) = \frac{\tan a - \tan b}{1 + \tan a \tan b} \quad \ldots\ldots (8)$$

$$\cot(a + b) = \frac{\cot a \cot b - 1}{\cot b + \cot a} \quad \ldots\ldots (9)$$

$$\cot(a - b) = \frac{\cot a \cot b + 1}{\cot b - \cot a} \quad \ldots\ldots (10).$$

(*c.*) *Expressions for the sine and cosine of multiple arcs:*—Let n be any integral number, then

$$\sin na = \cos a \left\{(2\sin a)^{n-1} - \frac{n-2}{1}(2\sin a)^{n-3} + \frac{(n-3)(n-4)}{1.2}(2\sin a)^{n-5} - \frac{(n-4)(n-5)(n-6)}{1.2.3}(2\sin a)^{n-7} + \&c.\right\} \ldots\ldots (11)$$

$$\cos na = \tfrac{1}{2}\left\{(2\sin a)^n - n(2\sin a)^{n-2} + \frac{n(n-3)}{1.2}(2\sin a)^{n-4} - \frac{n(n-4)(n-5)}{1.2.3}(2\sin a)^{n-6} + \&c.\right\} \ldots\ldots (12$$

(*d.*) *Table of multiple arcs:*—Let s be the sin, t the tan, and c the cot of the arc a, then

1. $\sin\frac{a}{2} = \left(\frac{1-\sqrt{1-s^2}}{2}\right)^{\frac{1}{2}}$,
2. $\sin a = s$,
3. $\sin 2a = 2s\sqrt{1-s^2}$,
4. $\sin 3a = 3s - 4s^3$,
5. $\sin 4a = (4s - 8s^3)\sqrt{1-s^2}$,
6. $\sin 5a = 16s^5 - 20s^3 + 5s$,
7. $\cos\frac{a}{2}$,
8. $\cos a = \sqrt{1-s^2}$,
9. $\cos 2a = 1 - 2s^2$,
10. $\cos 3a = (1 - 4s^2)\sqrt{1-s^2}$,
11. $\cos 4a = 1 - 8s^2 + 8s^4$,
12. $\cos 5a = (1 - 12s^2 + 16s^4)\sqrt{1-s^2}$.

13. $\tan\frac{a}{2} = \frac{t}{1+\sqrt{1+t^2}}$,
14. $\tan a = t$,
15. $\tan 2a = \frac{2t}{1-t^2}$,
16. $\tan 3a = \frac{3t - t^3}{1 - 3t^2}$,
17. $\tan 4a = \frac{4t - 4t^3}{1 - 6t^2 + t^4}$,
18. $\tan 5a = \frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$,
19. $\cot\frac{a}{2} = \frac{1}{\sqrt{1+c^2} - c}$,
20. $\cot a = c$,
21. $\cot 2a = \frac{c^2 - 1}{2c}$,
22. $\cot 3a = \frac{c^3 - 3c}{3c^2 - 1}$,
23. $\cot 4a = \frac{c^4 - 6c^2 + 1}{4c^3 - 4c}$,
24. $\cot 5a = \frac{c^5 - 10c^3 + 5c}{5c^4 - 10c^2 + 1}$.

(*e.*) *Table of Powers of the sine and cosine.*

1. $\sin a = \sin a$,
2. $2\sin^2 a = 1 - \cos 2a$,
3. $4\sin^3 a = 3\sin a - \sin 3a$,
4. $8\sin^4 a = 3 - 4\cos 2a + \cos 4a$,
5. $16\sin^5 a = \sin 5a - 5\sin 3a + 10\sin a$,
6. $\cos a = \cos a$,
7. $2\cos^2 a = \cos 2a + 1$,
8. $4\cos^3 a = \cos 3a + 3\cos a$,
9. $8\cos^4 a = \cos 4a + 4\cos 2a + 3$,
10. $16\cos^5 a = \cos 5a + 5\cos 3a + 10\cos a$.

(*f.*) *Expressions for the arc in terms of the sine or tangent:*—Let s be the sin, and t the tan of the arc a, then

$$\text{arc } a = s + \frac{s^3}{2.3} + \frac{1.3s^5}{2.4.5} + \&c. \quad \ldots\ldots\ldots (13)$$

$$\text{arc } a = t - \frac{t^3}{3} + \frac{t^5}{5} - \&c. \quad \ldots\ldots\ldots (14).$$

SECT. II. *Trigonometrical Tables.*

From the foregoing, and other properties, and theorems, mathematicians have computed the lengths of the sines, tangents, secants,

and versed sines (assuming unity for the radius) corresponding to ar from 1 second of a degree, through all the gradations of magni[illegible]ud up to a quadrant, or 90°; and the results of the computation[illegible] a arranged for use in tables called *Trigonometrical Tables*. As, [illegible]ow ever, these quantities have to be carried to several places of dec[illegible] ma in order to obtain sufficiently accurate results, their use in cal[illegible]cul tions is attended with much labour, and therefore it is usual to em[illegible]ploy their logarithms instead; but in this case the assumed radius is t[illegible]ake as 10,000000000 instead of unity, since with the latter most o[illegible] th quantities would be *fractional*, and therefore have *negative* ch[illegible]arac teristics, the use of which would be inconvenient, and is superseded by taking the radius as above.

Table IV. in the Appendix is such a table of the logarithms of the sines, cosines, tangents, cotangents, secants, and cosecants, fo every minute from 1 minute to 90 degrees, calculated to a radius of 10,000000000 as above. It will be observed that the headings of the columns run along the *tops* of the pages as far as the 45th degree, after which they return along the *bottoms* of the pages in contrary order, as below:—

sin	D.	cosec	tan	D.	cot	sec	D.	cos
cos	D.	sec	cot	D.	tan	cosec	D.	sin

The reason of this will be apparent, if we only consider that the co cot, or cosec of an arc, is the sin, tan, or sec of the *complement* of that arc. The intermediate columns, headed D, contain the differences of the consecutive logarithms in the contiguous columns on either side; it will be seen that the same difference is common to the sin and cosec, the tan and cot, and the sec and cos; since from Art. (I.) page 137, $\sin = \frac{\text{rad}^2}{\text{cosec}}$, or $\text{rad}^2 = \sin \cdot \text{cosec}$; $\tan = \frac{\text{rad}^2}{\cot}$, or $\text{rad}^2 = \tan \cdot \cot$; $\sec = \frac{\text{rad}^2}{\cos}$, or $\text{rad}^2 = \sec \cdot \cos$; and consequently, $\log \sin + \log \text{cosec} = \log \tan + \log \cot = \log \sec + \log \cos = 2 \log \text{rad} = 20$; therefore as the sin, tan, or sec increases, so must the corresponding cosec, cot, or cos, diminish, and their differences must be equal.

USE OF TABLE IV.

To find the logarithmic sine, tangent, &c. of a given arc.

If the arc contains only degrees and minutes, its sin, tan, &c., will be found simply by inspection, by looking along the top or *bottom* of the tables for the degrees, and then in the first or *last* vertical column for the minutes, according as the number of degrees is less or *greater* than 45; and on the same line, in the column having for its title

er at the top or *bottom*, according as the degrees were found) the e of the trigonometrical quantity required, its log will be found.

f the arc contains seconds as well, the logarithm must be found as re for the degrees and minutes; then take the number in the ntiguous column headed D on the same line, multiply it by the mber of seconds, and divide by 100 (which is done by cutting off e two last figures); the quotient must then be added to or *subtracted* om the log already taken out, according as the same would be in-reased or *decreased* by an increase in the arc.

Examples.

1. Find the log sin of 37° 47′.

As the arc is *less* than 45°, by looking along the *top* of the table for the degrees, and in the *first* column for the minutes, we find in the column having at its *top* the word sin, the figures 9·787232, which is the log sin of the arc required.

2. Find the log tan of 75° 34′.

Here, as the arc is *greater* than 45°, looking at the *bottom* of the tables for the degrees, and in the *last* column for the minutes, we find in the column having tan at the *bottom*, 10·589431, the tan of 75° 34′.

3. Find the log sin of 31° 45′ 5″.

The log sin of 31° 45′ is	9·721162	
The No. in col. D is 3402 × 5″ ÷ 100 = +	170	10
∴ The log sin of 31° 45′ 5″ = .	9·721332	

4. Find the log cos of 25° 1′ 47″.

The log cos of 25° 1′ is	9·957217	
The No. in col. D is 983 × 47″ ÷ 100 = −	462	01
∴ The log cos of 25° 1′ 47″ = . .	9·956755	

To find the arc corresponding to any given log sin, tan, &c.:— Look in Table IV. for the given log sin, &c., or the next less log thereto, and on the same line will be found the minutes, and at the top or bottom of the page the degrees of the arc required; if the log thus found is less than the given log, subtract the former from the latter, add two cyphers to the right of the remainder, and divide it by the number found in the contiguous column headed D; the quotient will be the number of seconds to be added to the degrees and minutes in the arc already obtained.

Examples.

1. Find the arc whose log tan is 10·577537.

Here the arc is found by inspection to be 75° 11′.

2. Find the arc corresponding to the log sin 9·395401.

Given log = 9·395401
Next less log = 9·395166 = log tan 14° 23′

23500 ÷ 492 = 48 seconds;

∴ 9·395401 is the log tan of 14° 23′ 48″.

In the sines and tangents of arcs less than about 2°, the differences between any two successive values are so great (as will be seen by an inspection of column D in the table), that the method above given for finding the intermediate values for seconds will not be sufficiently correct; and the same remark applies to the cosines and tangents of arcs greater than about 88°. It will also be observed, that in the cosines and secants of arcs less than 2°, and in the sines and cosecants of arcs greater than 88°, the differences are too small to enable us to calculate accurately the value of any arc from them.

The first of these difficulties may be removed by the rules given below for determining the values of the sines and tangents of small arcs, and the tangents of large arcs, and conversely the arcs from the sines and tangents. The second difficulty, however, could only be got over by extending the tables to more decimal places, but as this would also require all other quantities employed in the same calculations to be taken to an equal number of decimals, much additional trouble would be occasioned; and it is therefore better for determining the value of an arc when near 90°, to employ some other function than its sine, as, for instance, its cosine. In order to render this clearer to those who are not familiar with the use of logarithms, we subjoin an example of such a substitution of the cosine for the sine.

For instance, let it be desired from the formula,

$$P_1 : P_2 :: \sin\beta : \sin\delta$$

to determine the value of the angle δ, when $P_1 = 600{\cdot}1$; $P_2 = 669$; and the angle $\beta = 63° 45'$.

First by multiplying the means and extremes, and dividing both sides by P_1, we obtain

$$\sin\delta = \frac{P_2 \sin\beta}{P_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (a).$$

Then from Art. (I.), page 137,

$$\sin\delta = \sqrt{1 - (\cos\delta)^2}$$

$$\therefore \frac{P_2 \sin\beta}{P_1} = \sqrt{1 - (\cos\delta)^2};$$

squaring both sides

$$\left(\frac{P_2 \sin\beta}{P_1}\right)^2 = 1 - (\cos\delta)^2,$$

transposing, and extracting the square root

$$\cos \delta = \sqrt{1 - \left(\frac{P_2 \sin \beta}{P_1}\right)^2} \quad \ldots\ldots\ldots\ldots (b).$$

We have, therefore, two equations (a) and (b), from either of which we can obtain the value of the angle δ, by substituting the values of the known quantities; but in doing so we shall find that the second equation will give the value of δ much more exactly than the first.

Thus, by substituting the known quantities in equation (a), we have

$$\frac{669 \times \sin 63° 45'}{600{\cdot}1} = \sin \delta.$$

Whence by logarithms,

Log sin 63° 45′	=	9·952731
Log 669	=	2·825426
		12·778157
Log 600·1	=	2·778224
Log sin δ	=	9·999933

$$\therefore \delta = 88° 59' 25''.$$

Proceeding in a similar manner with equation (b), we have

$$\sqrt{1 - \left(\frac{669 \times \sin 63° 45'}{600{\cdot}1}\right)^2} = \cos \delta.$$

Whence by logarithms, taking the radius as unity, for the reasons explained at page 146,

Log sin 63° 45′	=	1·952731
Log 669	=	2·825426
		2·778157
Log 600·1	=	2·778224
		$\bar{1}$·999933
		2
Log ·99969	=	$\bar{1}$·999866

Then 1 − ·99969 = ·00031, the log of which = $\bar{4}$·491362,

and $\bar{4}$·491362 ÷ 2 = $\bar{2}$·245681 = cos δ;

or restoring the radius of the tables,

cos δ = 8·245681 = sin of the complement of δ;

therefore, by the rule given below for finding a small arc from its sin, we have

$$\begin{array}{r} 8{\cdot}245681 \\ 5{\cdot}314425 \\ \quad 22 \\ \hline 3{\cdot}560128 \end{array} = 3631{\cdot}86 \text{ seconds} = 1^\circ\ 0'\ 31''{\cdot}86\,;$$

$$\therefore\ \delta = 88^\circ\ 59'\ 28''{\cdot}14,$$

whence we see that the former value of δ obtained from equation (a) is upwards of three seconds too small.

To find accurately the log sin of an arc less than 2°.

Reduce the arc to seconds, and find the log of that number from Table III., to which add 4·685575 (the log sin of 1′), and subtract one-third of the decimal portion of the log sec of the arc taken from Table IV.; the remainder is the log sin of the arc required.

Examples.

Find the log sines of 13′ and of 1° 3′.

Log (13′ × 60) = 780″ = 2·892095	Log (63′ × 60) = 3780″ = 3·577492
4·685575	4·685575
7·577670	8·263067
Log sec of 13′ = ·000003 ÷ 3 = ·000001	Log sec of 1° 3′ = ·000073 ÷ 3 = ·000024
Log sin of 13′ = 7·577669	Log sin of 1° 3′ = 8·263043

To find accurately the log tan of an arc less than 2°.

To the log of the number of seconds in the arc add 4·685575, and two-thirds of the decimal portion of its log sec; the sum is the log tan of the arc required.

Examples.

Find the log tans of 24′ and of 1° 15′.

Log (24′ × 60) = 1440″ = 3·158363	Log (75′ × 60) = 4500″ = 3·653213
4·685575	4·685575
Log sec of 24′ = ·000011 × ⅔ = ·000007	Log sec of 1° 15′ = ·000103 × ⅔ = ·000068
Log tan of 24′ = 7·843945	Log tan of 1° 15′ = 8·338856

To find accurately the log tan of an arc greater than about 88°.

Add to the log of the number of seconds that the arc is less than 90°, two-thirds of the decimal portion of the log cosec, and subtract the sum from 15·314425; the remainder will be the log tan required.

Examples.

1. Find the log tan of 89° 5′ 13″.

$$\begin{array}{rrr} 90^\circ & 0' & 0'' \\ 89 & 5 & 13 \\ \hline 0^\circ & 54' & 47'' \end{array} = 3287 \text{ seconds}$$

Constant log . . . . . = 15·314425
Log of 3287 . . . . . = 3·516800
Log cosec of 89° 5′ 13″ = ·000057 × $\frac{2}{3}$ = ·000038
3·516838

Log tan of 89° 5′ 13″ = 11·797587

2. Find the log tan of 88° 51′ 10″.

90° 0′ 0″
88 51 10
1° 8′ 50″ = 4130 seconds.

Constant log . . . . . = 15·314425
Log of 4130 . . . . . = 3·615950
Log cosec of 88° 51′ 10″ = ·000087 × $\frac{2}{3}$ = ·000058
3·616008

Log tan of 88° 51′ 10″ = 11·698417

To find accurately an arc of not more than 2° from its log sine.

To the given log sin, add 5·314425, and one-third of the decimal portion of the secant of the nearest arc to that whose log sin is given, the sum, rejecting 10 from the characteristic, will be the logarithm of the number of seconds in the arc.

Example.

Required the arc whose log sin is 8·314719.

Given log sin . . . . . = 8·314719
Constant log . . . . . = 5·314425
Log sec of nearest arc = ·000093 ÷ 3 = ·000031

Arc required 1° 10′ 58″ = 4258″ = 3·629175

To find accurately an arc of not more than 2° from its log tan.

To the given log tan add 5·314425, and from the sum subtract two-thirds of the decimal portion of the log sec of the arc whose log tan is nearest to that given, and the remainder, rejecting 10 from the characteristic, will be the log of the number of seconds in the arc.

Example.

Required the arc whose log tan is 8·231461.

Given log tan . . . . . = 8·231461
Constant log . . . . . = 5·314425
13·545886
Log sec of nearest arc = ·000063 × $\frac{2}{3}$ = ·000042
Required arc = 58′ 34″ = 3514″ = 13·545844

To find accurately an arc greater than 88° *from its log tan.*

Add to the given log tan two-thirds of the decimal portion of the log cosec of the nearest arc to that whose log tan is given, and subtract the sum from 15·314425, the remainder is the log of the number of seconds that the arc is *less than* 90°.

Example.

Required the arc whose log tan is 11·695900.

Constant log		= 15·314425
Given log tan	= 11·695900	
Log cosec of nearest arc ·000088 × ⅔ =	·000059	
		11·695959
Required arc = 1° 9′ 14″ = 4154″ =		3·618466

ON THE ROOTS AND POWERS OF TRIGONOMETRICAL QUANTITIES.

In extracting the root, or raising the power of any trigonometrical quantity by means of its logarithm, it will always be found most convenient to reduce the assumed radius to unity, by subtracting 10 from the characteristic of the logarithm, observing where the characteristic thus becomes negative the rules given at pages 47 and 48. An example of this alteration of the characteristic is given at page 143.

TRIGONOMETRICAL QUANTITIES OF ANGLES GREATER THAN 90°.

Although Table IV. only purports to give the sines, tangents, &c., of angles less than, or equal to 90°, any of these functions of angles *greater* than 90° may readily be obtained, since any function of an angle greater than 90° is equal to the same function of the *supplement* of that angle; due regard being had to changing the sign, if requisite, according to the table given at page 136. Thus the log sin of 141° 15′ is 9·796521, the same as the sin of its supplement 38° 45′; the log sec of 95° 43′ is — 11·001701, or the sec of 84° 17′ with its sign changed; the log cosec of the same is 10·002165, or the cosec of 84° 17′, the sign remaining the same; and the log tan of 173° 4′ is — 9 084947, or the tan of 6° 56′, with its sign altered.

SECT. III. *General Propositions.*

I. The chord of any arc is a mean proportional between the versed sine of that arc and the diameter of the circle.

II. As radius is to the cosine of any arc, so is twice the sine of that arc to the sine of double the arc.

III. The secant of any arc is equal to the sum of its tangent, and the tangent of half its complement.

IV. The sum of the tangent and secant of any arc, is equal to the tangent of an arc exceeding that by half its complement. Or, the

sum of the tangent and secant of an arc is equal to the tangent of 45° *plus* half the arc.

V. The chord of 60° is equal to the radius of the circle; the versed sine and cosine of 60° are each equal to half the radius, and the secant of 60° is equal to double the radius.

VI. The tangent of 45° is equal to the radius.

VII. The square of the sine of half any arc or angle is equal to a rectangle under half the radius and the versed sine of the whole arc; and the square of its cosine is equal to a rectangle under half the radius and the versed sine of the supplement of the whole arc or angle.

VIII. The rectangle under the radius and the sine of the sum or the difference of two arcs is equal to the sum or the difference of the rectangles under their alternate sines and cosines.

IX. The rectangle under the radius and the cosine of the sum or the difference of two arcs, is equal to the difference or the sum of the rectangles under their respective cosines and sines.

X. As the difference or sum of the square of the radius and the rectangle under the tangents of two arcs, is to the square of the radius; so is the sum or difference of their tangents, to the tangent of the sum or difference of the arcs.

XI. As the sum of the sines of two unequal arcs, is to their difference; so is the tangent of half the sum of those two arcs to the tangent of half their difference.

XII. Of any three equidifferent arcs, it will be as radius is to the cosine of their common difference, so is the sine of the mean arc, to half the sum of the sines of the extremes; and, as radius is to the sine of the common difference, so is the cosine of the mean arc to half the difference of the sines of the two extremes.

(A.) If the sine of the mean of three equidifferent arcs (radius being unity) be multiplied into twice the cosine of the common difference, and the sine of either extreme be deducted from the product, the remainder will be the sine of the other extreme.

(B.) The sine of any arc above 60°, is equal to the sine of another arc as much below 60°, together with the sine of its excess above 60°.

Cor. From this latter proposition, the sines below 60° being known, those of arcs above 60° are determinable by addition only.

XIII. In any right-angled triangle, the hypothenuse is to one of the legs, as the radius is to the sine of the angle opposite to that leg; and one of the legs is to the other as the radius is to the tangent of the angle opposite to the latter.

XIV. In any plane triangle, as one of the sides is to another, so is the sine of the angle opposite to the former to the sine of the angle opposite to the latter.

XV. In any plane triangle it will be, as the sum of the sides about the vertical angle is to their difference, so is the tangent of half the sum of the angles at the base, to the tangent of half their difference.

XVI. In any plane triangle it will be, as the cosine of the difference of the angles at the base, is to the cosine of half their sum, so is the sum of the sides about the vertical angles to the third side. Also, as the sine of half the difference of the angles at the base, is to the sine of half their sum, so is the difference of the sides about the vertical angle to the third side, or base *.

XVII. In any plane triangle it will be, as the base is to the sum of the two other sides, so is the difference of those sides to the difference of the segments of the base made by a perpendicular let fall from the vertical angle.

XVIII. In any plane triangle it will be, as twice the rectangle under any two sides, is to the difference of the sum of the squares of those two sides and the square of the base, so is the radius to the cosine of the angle contained by the two sides.

Cor. When unity is assumed as radius, then if A C, A B, B C, are the sides of a triangle and C the angle opposite the side A B, this Prop. gives $\cos C = \frac{A C^2 + B C^2 - A B^2}{2 C B . C A}$: and similar expressions for the other angles.

XIX. As the sum of the tangents of any two unequal angles is to their difference, so is the sine of the sum of those angles to the sine of their difference.

XX. As the sine of the difference of any two unequal angles is to the difference of their sines, so is the sum of those sines to the sine of the sum of the angles.

These and other propositions are the foundation of various formulæ, for which the reader who wishes to pursue the inquiry may consult the best treatises on Trigonometry.

SECT. IV. *Solution of the Cases of Plane Triangles.*

Although the three sides and three angles of a plane triangle, when combined three and three, constitute twenty varieties, yet they furnish only *three* distinct cases in which separate rules are required.

CASE I.

When a side and an angle are two of the given parts.

The solution may be effected by Prop. XIV. of the preceding section, wherein it is affirmed that the sides of plane triangles are respectively proportional to the sines of their opposite angles.

In practice, if a *side* be required, begin the proportion with a sine, and say,

As the sine of the given angle,
Is to its opposite side;
So is the sine of either of the other angles,
To its opposite side.

* These propositions were first given by *Thacker* in his *Mathematical Miscellany*, published in 1743; their practical utility has been recently shown by *Professor Wallace*, in the *Edinburgh Philosophical Transactions*.

If an *angle* be required, begin the proportion with a side, and say,

As one of the given sides,
Is to the sine of its opposite angle;
So is the other given side,
To the sine of its opposite angle.

The third angle becomes known by taking the sum of the two former from 180°.

Note.—It is usually best to work the proportions in trigonometry by means of the logarithms, taking the logarithm of the *first* term from the sum of the logarithms of the *second* and *third*, to obtain the logarithm of the *fourth* term. Or, adding the *arithmetical complement* of the logarithm of the first term to the logarithms of the other two, to obtain that of the fourth.

CASE II.

When two sides and the included angle are given.

The solution may be effected by means of Props. XV. and XVI. of the preceding section.

Thus: take the given angle from 180°, the remainder will be the sum of the other two angles.

Then say,
As the sum of the given sides,
Is to their difference;
So is the tangent of half the sum of the remaining angles,
To the tangent of half their difference.

Then, secondly say,
As the cosine of half the said difference,
Is to the cosine of half the sum of the angles;
So is the sum of the given sides,
To the third, or required side.

Or,
As the sine of half the diff. of the angles,
Is to the sine of half their sum;
So is the difference of the given sides,
To the third side.

Example.—In the triangle A B C (Fig. 9.) are given A C = 450, B C = 540, and the included angle C = 80°; to find the third side, and the two remaining angles. Then,

Log (B C − A C	= 90) =	1·954243
Log tan ($\frac{1}{2}$ A + B	= 50°) =	10·076187
		12·030430
Log (B C + A C	= 990) =	2·995685
Log tan ($\frac{1}{2}$ A − B	= 6° 11″) =	9·034795

Then, Log cos $(\frac{1}{2}\overline{A + B} = 50°) =$ 9·808068
Log $(BC + AC = 990) =$ 2·995635
12·803703
Log cos $(\frac{1}{2}\overline{A - B} = 6° 11') =$ 9·997466
Log $(AB \quad \ldots = 640{\cdot}08) =$ 2·806237

Also, $\frac{1}{2}(A + B) + \frac{1}{2}(A - B) = 56° 11' = A$; and $\frac{1}{2}(A + B) - \frac{1}{2}(A - B) = 43° 49' = B$.

Here, much time will be saved in the work by taking log cos $\frac{1}{2}(A + B)$ from the tables, at the same time with log tan $\frac{1}{2}(A + B)$; and log cos $\frac{1}{2}(A - B)$ as soon as log tan $\frac{1}{2}(A - B)$ is found. Observe, also, that the log of $BC + AC$ is the same in the second operation as in the first. Thus the tables need only be opened in *five* places for both operations.

Another solution to Case II.

Supposing C to be the given angle, and CA, CB, the given sides; then the third side may be found by this theorem, viz.:—

$$AB = \sqrt{(AC^2 + BC^2 - 2AC \cdot CB \cdot \cos C)}.$$

Thus, taking $AC = 450$, $BC = 540$, $C = 80°$, its cos ·1736482

$$AB = \sqrt{(450^2 + 540^2 - 2 \times 450 \times 540 \times {\cdot}1736482)}$$
$$= \sqrt{\{90^2 (5^2 + 6^2 - 2 \times 5 \times 6 \times {\cdot}1736482)\}}$$
$$= 90 \sqrt{50{\cdot}58118} = 90 \times 7{\cdot}112 = 640{\cdot}08, \text{ as before.}$$

CASE III.

When the three sides of a plane triangle are given, to find the angles.

1st *Method.*—Assume the longest of the three sides as base, then say, conformably with Prop. XVI., last section,

As the base,
Is to the sum of the two other sides;
So is the difference of those sides,
To the difference of the segments of the base.

Half the base added to the said difference gives the greater segment, and made less by it gives the less; and thus, by means of the perpendicular from the vertical angle, divides the original triangle into two, each of which falls under the first case.

2nd *Method.*—Find any one of the angles by means of Prop. XVIII. of the preceding section; and the remaining angles either by a repetition of the same rule, or by the relation of the sides to the sines of their opposite angles, viz.:—

$$\cos C = \frac{AC^2 + BC^2 - AB^2}{2AC \cdot BC}; \quad \cos B = \frac{AB^2 + BC^2 - AC^2}{2AB \cdot BC}$$

and $\cos A = \dfrac{BA^2 + AC^2 - BC^2}{2AB \cdot AC}$.

RIGHT-ANGLED PLANE TRIANGLES.

Right-angled triangles may (as well as others), be solved by means of the rule to the respective case under which any specified example falls; and it will then be found, since a right angle is always one of the data, that the rule usually becomes simplified in its application.

When two of the sides are given, the third may be found by means of the property in *Geom.*, Prop. XVI. Sect. III.

$$\text{Hypoth.} = \sqrt{(\text{base}^2 + \text{perp.}^2)}$$

$$\text{Base} = \sqrt{(\text{hyp.}^2 - \text{perp.}^2)} = \sqrt{(\text{hyp.} + \text{perp.}) \,.\, (\text{hyp.} - \text{perp.})}$$

$$\text{Perp.} = \sqrt{(\text{hyp.}^2 - \text{base}^2)} = \sqrt{(\text{hyp.} + \text{base}) \,.\, (\text{hyp.} - \text{base})}.$$

There is another method for right-angled triangles, known by the phrase *making any side radius;* which is this.

"*To find a side.* Call any one of the sides radius, and write upon it the word *radius;* observe whether the other sides become sines, tangents, or secants, and write those words upon them accordingly. Call the word written upon each side the *name* of each side: then say,

As the *name* of the given side,
Is to the given side;
So is the *name* of the required side,
To the required side."

"*To find an angle.* Call either of the *given* sides radius, and write upon it the word *radius;* observe whether the other sides become sines, tangents, or secants, and write those words on them accordingly. Call the word written upon each side the *name* of that side. Then say,

As the side made radius,
Is to radius;
So is the other given side,
To the *name* of that side,
which determines the opposite angle."

When the numbers which measure the sides of the triangle are either under 12, or resolvable into factors which are each less than 12, the solution may be obtained, conformably with this rule, easier without logarithms than with them. For,

Let A B C (Fig. 103) be a right-angled triangle, in which A B, the base, is assumed to be radius; B C is the tangent of A, and A C its secant, to that radius; or, dividing each of these by the base, we shall have the tangent and secant of A, respectively, radius being unity. Tracing in like manner the consequences of assuming B C (as fig. 104), and A C (as fig. 105), each for radius, we shall readily obtain these expressions.

1. $\frac{\text{perp.}}{\text{base}}$ = tan angle at base. (Fig. 103.)

2. $\frac{\text{hyp.}}{\text{base}}$ = sec angle at base. (Fig. 103.)

3. $\frac{\text{base}}{\text{perp.}}$ = tan angle at vertex. (Fig. 105.)

4. $\frac{\text{hyp.}}{\text{perp.}}$ = sec angle at vertex. (Fig. 105.)

5. $\frac{\text{perp.}}{\text{hyp.}}$ = sin angle at base. (Fig. 104.)

6. $\frac{\text{base}}{\text{hyp.}}$ = sin angle at vertex. (Fig. 104.)

SECT. V. *On the application of Trigonometry to measuring Heights and Distances.*

Trigonometry receives its principal practical application in the operations of surveying, and measuring heights and distances; as, however, the methods of its application (depending on the peculiar circumstances of each case) are exceedingly various, we cannot lay down any general rules, but must content ourselves with giving a selection of such examples as are most likely to occur; and the principles developed in which, will be sufficient to guide any person in its further application to other cases.

The instruments employed to measure angles are quadrants, sextants, theodolites, &c., the use of either of which may be sooner learnt from an examination of the instruments themselves than from any description independently of them. For military men and for civil engineers, a good pocket sextant, and an accurate micrometer (such as Cavallo's) attached to a telescope, are highly useful. For measuring small distances, as bases, 50 feet and 100 feet chains, and a portable box of graduated tape, will be necessary.

For the purposes of surveying, it is usual to employ a chain 66 feet in length, subdivided into 100 links, each 7·92 inches; the reason for using a chain of this length is, that ten of such square chains are equal to an acre, and therefore the acreage of the several divisions of an estate are found with much greater facility when measured in chains and links, than when the measurements are taken in feet.

Example 1.

In order to find the distance between two trees, A and B (Fig. 9), which could not be directly measured because of a pool which occupied much of the intermediate space, I measured the distance of each of them from a third object, C, viz., A C = 588, B C = 672, and then at the point C took the angle A C B between the two trees = 55° 40′. Required their distance.

This is an example to Case II. of plane triangles, in which two sides, and the included angle, are given. The work, therefore, may exercise the student: the answer is 593·8.

Example 2.

Wanting to know the distance between two inaccessible objects, C and D (Fig. 106), which lay in a direct line from the bottom of a tower on whose top I stood, I took the angles of *depression* of the two objects, viz., of the most remote $25\frac{1}{2}°$, of the nearest 57°. What is the distance between them, the height of the tower A B being 120 feet?

H A D = 25° 30′, hence B A D = B A H − H A D = 64° 30′.

H A C = 57° 0′, hence B A C = B A H − H A C = 33° 0′.

Hence the following calculation, by means of the *natural* tangents. For, if A B be regarded as radius, B D and B C will be the tangents of the respective angles B A D, B A C, and C D the difference of those tangents. It is, therefore, equal to the product of the difference of the natural tangents of those angles into the height A B.

Thus, nat. tan 64° 30′ =	2·0965436
nat. tan 33° =	0·6494076
difference	1·4471360
multiplied by height,	120
gives distance C D =	173·6563200

* * The natural sines, tangents, &c., are easily obtained from Table IV., by subtracting 10 from the characteristics, and then finding the natural number answering to the logarithms with their characteristics so altered.

Example 3.

Standing at a measurable distance A B (Fig. 103), on a horizontal plane, from the bottom of a tower, I took the angle of elevation of the top (C); it is required from thence to determine the height of the tower.

In this case there would be given A B and the angle A, to find B C = A B × tan A.

By logarithms, when the numbers are large, it will be log B C = log A B + log tan A.

Note.—If angle A = 11° 19′ then B C = $\frac{1}{5}$ A B very nearly.

„ A = 16 42 „ B C = $\frac{3}{10}$ A B „

„ A = 21 48 „ B C = $\frac{4}{7}$ A B „

„ A = 26 34 „ B C = $\frac{1}{2}$ A B „

„ A = 30 58 „ B C = $\frac{2}{3}$ A B „

„ A = 35 0 „ B C = $\frac{7}{10}$ A B „

„ A = 38 40 „ B C = $\frac{4}{5}$ A B „

„ A = 45 0 „ B C = A B, *exactly*.

To save the time of computation, therefore, the observer may set the instrument to one of these angles, and advance or recede, till it

accords with the angle of elevation of the object; its height above the horizontal level of the observer's eye will at once be known, by taking the appropriate fraction of the distance A B.

Example 4.

Wanting to know the height of a church steeple, to the bottom of which I could not measure on account of a high wall between me and the church, I fixed upon two stations at the distance of 93 feet from each other, on a horizontal line from the bottom of the steeple, and at each of them took the angle of elevation of the top of the steeple, that is, at the nearest station 55° 54′, at the other 33° 20′. Required the height of the steeple.

Recurring to figure 106, we have given the distance C D, and the angles of elevation at C and D. The quickest operation is by means of the natural tangents, and the theorem $A B = \frac{C D}{\cot D - \cot C}$.

Thus $\cot D = \cot 33° 20′ = 1{\cdot}5204261$
$C = \cot 55\ 54 = {\cdot}6770509$

Their difference $= {\cdot}8433752$

Hence, $A B = \frac{93}{{\cdot}8433752} = 110{\cdot}27$ feet.

Example 5.

Wishing to know the height of an obelisk standing at the top of a regularly sloping hill, I first measured from its bottom a distance of 36 feet, and there found the angle formed by the inclined plane and a line from the centre of the instrument to the top of the obelisk 41°; but after measuring on downward in the same sloping direction 54 feet farther, I found the angle formed in like manner to be only 23° 45′. What was the height of the obelisk, and what the angle made by the sloping ground with the horizon?

The figure being constructed (see fig. 107), there are given in the triangle A C B, all the angles and the side A B, to find B C. It will be obtained by this proportion, as sin C (= 17° 15′ = B − A) : A B (= 54) : : sin A (= 23° 45′) : B C = 73·3392. Then, in the triangle D B C are known B C as above, B D = 36, C B D = 41°; to find the other angles, and the side C D. Thus, first, as C B + B D : C B − B D : : tan ½ (D + C) = ½ (139°) : tan ½ (D − C) = 42° 24½′. Hence 69° 30′ + 42° 24½′ = 112° 54½′ = C D B, and 69° 30′ − 42° 24½′ = 27° 5½′ = B C D. Then, sin B C D : B D : : sin C B D : C D = 51·86, height of the obelisk.

The angle of inclination D A E = H D A = C D B − 90° = 22° 54½′.

Remark.—If the line B D cannot be measured, then the angle D A E of the sloping ground must be taken, as well as the angles C A B and C B D. In that case D A E + 90° will be equal to C D B:

ıfter C B is found from the triangle A C B, C D may be the triangle C B D, by means of the relation between the . the sines of their opposite angles.

Example 6.

; on a horizontal plane, and wanting to ascertain the height of standing on the top of an inaccessible hill, I took the angle ation of the top of the hill 40°, and of the top of the tower en measuring in a direct line 180 feet farther from the hill, I ı the same vertical plane the angle of elevation of the top of wer 33° 45′. Required from hence the height of the tower.

e figure being constructed (see fig. 108), there are given, A B 80, C A B = 33° 45′, A C B = C B E − C A E = 17° 15′, C B D .1°, B D C = 180° − (90° − D B E) = 130°. And C D may be ıd from the expression C D . rad^4 = A B . sin A . sin C B D . cosec ;B . sec D B E.

Or, using logarithms,

log A B	= 180°	=	2·255273
+ log sin A	= 33° 45′	=	9·744739
+ log sin C B D	= 11° 0′	=	9·280599
+ log cosec A C B	= 17° 15′	=	10·527914
+ log sec D B E	= 40° 0′	=	10·115746
			41·924271
	− log rad^4	=	40·000000
log C D	= 83·9983	=	1 924271

Example 7.

In order to determine the distance between two inaccessible objects A and B (Fig. 108), on a horizontal plane, we measured a convenient base, C D, of 536 yards, and at the extremities C and D took the following angles, viz., D C B = 40° 16′, B C A = 57° 40′, C D A = 42° 22′, A D B = 71° 7′. Required the distance A B.

First, in the triangle C D A are given all the angles, and the side C D to find A D. So, again, in the triangle C D B, are given all the angles, and C D to find D B. Lastly, in the triangle D A B are given the two sides A D, B D, and the included angle A D B, to find A B = 939·52 yards.

Remark.—In like manner the distances taken two and two, between any number of remote objects posited around a convenient station line, may be ascertained.

Example 8.

Suppose that in carrying on an extensive survey, the distance between two spires A and B (Fig. 109) has been found equal to 6594 yards, and that C and D are two eminences conveniently situated for extending the triangles, but not admitting of the determination

of their distance by actual admeasurement: to ascertain it, therefore, we took at C and D the following angles, viz.:—

$$\begin{cases} \text{A C B} = 85^\circ\ 46' \\ \text{B C D} = 23^\circ\ 56' \end{cases} \qquad \begin{cases} \text{A D C} = 31^\circ\ 48' \\ \text{A D B} = 68^\circ\ 2' \end{cases}$$

Required C D from these data.

In order to solve this problem, construct a similar quadrilateral A cdb, assuming cd equal to 1, 10, or any other convenient number: compute A b from the given angles, according to the method of the preceding example. Then, since the quadrilaterals A cdb, A C D B, are similar, it will be, as $Ab : cd ::$ A B : C D; from which C D is found to be equal to 4694 yards.

Example 9.

Given the angles of elevation of any distant object, taken at three places in a horizontal right line, which does not pass through the point directly below the object; and the respective distances between the stations; to find the height of the object, and its distance from either station.

Let A E C (Fig. 110) be the horizontal plane, F E the perpendicular height of the object above that plane, A, B, C, the three places of observation, F A E, F B E, F C E, the angles of elevation, and A B, B C, the given distances. Then, since the triangles A E F, B E F, C E F, are all right angled at E, the distances A E, B E, C E, will manifestly be as the cotangents of the angles of elevation at A, B, and C.

Put A B = D, B C = d, E F = x, and then express algebraically the theorem given in Geom. Sect. III. Prop. XXV., which in this case becomes,

$$\text{A E}^2 \cdot \text{B C} + \text{C E}^2 \cdot \text{A B} = \text{B E}^2 \cdot \text{A C} + \text{A C} \cdot \text{A B} \cdot \text{B C}.$$

The resulting equation is

$$d x^2 (\cot \text{A})^2 + \text{D} x^2 (\cot \text{C})^2 = (\text{D} + d) x^2 (\cot \text{B})^2 + (\text{D} + d)\,\text{D} d.$$

From which is readily found

$$x = \sqrt{\frac{(\text{D} + d)\,\text{D} d}{d(\cot \text{A})^2 + \text{D}(\cot \text{C})^2 - (\text{D} + d)(\cot \text{B})^2}}.$$

Thus E F becoming known, the distances A E, B E, C E, are found, by multiplying the cotangents of A, B, and C, respectively, by E F.

Remark.—When D = d, or D + d = 2 D = 2 d, that is, when the point B is midway between A and C, the algebraic expression becomes,

$$x = d \div \sqrt{\tfrac{1}{2}(\cot \text{A})^2 + \tfrac{1}{2}(\cot \text{C})^2 - (\cot \text{B})^2},$$

which is tolerably well suited for logarithmic computation. The rule may, in that case, be thus expressed.

Double the log cotangents of the angles of elevation of the extreme stations, find the natural numbers answering thereto, and take half their sum; from which subtract the natural number answering

to twice the log cotangent of the middle angle of elevation: then half the log of this remainder subtracted from the log of the measured distance between the first and second, or the second and third stations, will be the log of the height of the object.

The distance from either station will be found as above.

Note.—The case explained in this example, is one that is highly useful, and of frequent occurrence. An analogous one is when the angles of elevation of a remote object are taken from the three angles of a triangle on a horizontal plane, the sides of that triangle being known, or measurable: but the above admits of a simpler computation, and may usually be employed.

Example 10.

From a convenient station P (Fig. 111), where could be seen three objects, A, B, and C, whose distances from each other were known (viz. A B = 800, A C = 600, B C = 400 yards), I took the horizontal angles A P C = 33° 45′, B P C = 22° 30′. It is hence required to determine the respective distances of my station from each object.

Here it will be necessary, as preparatory to the computation, to describe the manner of

Construction.—Draw the given triangle A B C from any convenient scale. From the point A draw a line A D to make with A B an angle equal to 22° 30′, and from B a line B D to make an angle D B A = 33° 45′. Let a circle be described to pass through their intersection D, and through the points A and B. Through C and D draw a right line to meet the circle again in P: so shall P be the point required. For, drawing P A, P B, the angle A P D is evidently = A B D, since it stands on the same arc A D: and for a like reason B P D = B A D. So that P is the point where the angles have the assigned value.

The result of a careful construction of this kind, upon a good-sized scale, will give the values of P A, P C, P B, true to within the 200th part of each.

Manner of Computation.—In the triangle A B C, where the sides are known, find the angles. In the triangle A B D, where all the angles are known, and the sides A B, find one of the other sides A D. Take B A D from B A C, the remainder, D A C, is the angle included between the two known sides, A D, A C; from which the angles A D C and A C D may be found, by Case II., p. 149. The angle C A P = 180° − (A P C + A C D). Also, B C P = B C A − A C D; and P B C = A B C + P B A = A B C + sup. A D C. Hence, the three required distances are found by these proportions. As sin A P C : A C :: sin P A C : P C :: sin P C A : P A; and lastly, as sin B P C : B C :: sin B P C : B P. The results of the computation are, P A = 709·33, P C = 1042·66, P B = 934 yards.

*** The computation of problems of this kind, however, may be a little shortened by means of an analytical investigation. Those who wish to pursue this department of trigonometry may consult the treatises by *Bonnycastle*, *Gregory*, and *Woodhouse*.

Note.—If C had been nearer to P than A B, the general principles of construction and computation would be the same; and the modification in the process very obvious.

DETERMINATION OF HEIGHTS AND DISTANCES BY APPROXIMATE MECHANICAL METHODS.

1. *For Heights.*

1. *By shadows*, when the sun shines.—Set up vertically a staff of known length, and measure the length of its shadow upon a horizontal or other plane; measure also the length of the shadow of the object whose height is required. Then it will be, as the length of the shadow of the staff, is to the length of the staff itself; so is the length of the shadow of the object, to the object's height.

2. *By two rods or staves set up vertically.*—Let two staves, one, say, of 6 feet, the other of 4 feet long, be placed upon horizontal circular or square feet, on which each may stand steadily. Let A B (Fig. 112) be the object, as a tower or steeple, whose altitude is required, and A C the horizontal plane passing through its base. Let C D and E F, the two rods, be placed with their bases in one and the same line C A, passing through A the foot of the object; and let them be moved nearer to, or farther from, each other, until the summit B of the object is seen, in the same line as D and F, the tops of the rods. Then by the principle of similar triangles, it will be, as D H (= C E) : F H :: D G (= C A) : B G; to which add A G = C D, for the whole height A B.

3. *By reflection.*—Place a vessel of water upon the ground, and recede from it, until you see the top of the object reflected from the smooth surface of the liquid. Then, since by a principle in optics, the angles of incidence and reflection are equal, it will be as your distance measured horizontally from the point at which the reflection is made, is to the height of your eye above the reflecting surface; so is the horizontal distance of the foot of the object from the vessel to its altitude above the said surface *.

4. *By means of a portable barometer and thermometer.*—Observe the altitude B, of the mercurial column, in inches, tenths, and hundredths, at the *bottom* of the hill, or other object whose altitude is required; observe, also, the altitude, *b*, of the mercurial column at the *top* of the object; observe the temperatures on Fahrenheit's thermometer, at the times of the two barometrical observations, and take the mean between them.

* *Leonard Digges*, in his curious work, the *Pantometria*, published in 1571, first proposed a method for the determination of altitudes by means of a geometrical square and plummet, which has been described by various later authors, as Ozanam, Donn, Hutton, &c. But, as it does not seem preferable to the methods above given, I have not repeated it here.

$55000 \times \frac{B - b}{B + b}$ = height of the hill, in feet, for the tempera-

of 55° on Fahrenheit. Add $\frac{1}{210}$ of this result for every degree ch the mean temperature exceeds 55°; subtract as much for every ree below 55°.

This will be a good approximation when the height of the hill is s than 2000 feet; and it is easily remembered, because 55°, the sumed temperature, agree with 55, the significant figures in the co-ficient; while the significant figures in the denominator of the cor-ecting fraction are two *fours*.

5. *By an extension of the principle of page* 153.—Set the sextant, r other instrument, to the angle 45°, and find the point C (Fig. 103) n the horizontal plane, where the object A B has that elevation: then set the instrument to 26° 34′, and recede from C, in direction B C D, till the object has that elevation.

The distance C D *between the two stations will be* = A B.

So, again, if C = 40°, D = 24° 31½′, C D will be = A B.
or, „ if C = 35°, D = 22° 23′, C D „ = A B.
or, „ if C = 30°, D = 20° 6′, C D „ = A B.
or, „ if C = 20°, D = 14° 56′, C D „ = A B.
or, generally, if cot D − cot C = rad, C D „ = A B.

6. *For deviation from level.*—Let E represent the elevation of the tangent line to the earth above the true level, in feet and parts of a foot, D the distance in miles: then $E = \frac{2}{3} D^2$.

This gives 8 inches for a distance of one mile; and is a near approximation when the distance does not exceed 2 or 3 miles.

2. *For Distances.*

1. *By means of a rhombus set off upon a horizontal plane.*—Suppose O (Fig. 113) the object and O B the required distance. With a line or measuring tape, whose length is equal to the side of the intended rhombus, say 50 or 100 feet, lay down one side B A in the direction B O towards the object, and B C another side in any convenient direction (for whether B be a right angle, or not, is of no consequence); and put up rods or arrows at A and C. Then fasten two ends of two such lines at A and C, and extend them until the two other ends just meet together at D; let them lie thus stretched upon the ground, and they will form the two other sides of the rhombus A D, C D. Fix a mark or arrow at R, directly between C and O, upon the line A D; and measure R D, R A upon the tape. Then it will be as R D : D C :: C B : B O, the required distance.

Otherwise. To find the length of the inaccessible line Q R.

At some convenient point B (Fig. 114) lay down the rhombus BADC, so that two of its sides, B A, B C, are directed to the extremities of the line Q R. Mark the intersections, O and P, of A R,

C Q, with the sides of the rhombus (as in the former method): then the triangle O D P will be similar to the triangle R B Q; and the inaccessible distance R Q will be found $= \frac{O P \times B A^2}{O D \times D P}$.*

Thus, if B A = B C, &c. = 100 ft., O D = 9 ft. 5 ins., D P = 11 ft. 10 ins., O P = 13 ft. 7 ins., then Q R $= \frac{10000 \times 13\frac{7}{12}}{9\frac{5}{12} \times 11\frac{10}{12}} = 1219$ feet.

2. *By means of a micrometer attached to a telescope.*—Portable instruments for the purpose of measuring extremely small angles, have been invented by Martin, Cavallo, Dollond, Brewster, and others. In employing them for the determination of distances, all that is necessary in practice is to measure the angle subtended by an object of known dimensions, placed either vertically or horizontally, at the remoter extremity of the line whose length we wish to ascertain. Thus, if there be a house, or other erection, built with bricks, of the usual size; then four courses in *height* are equal to *a foot*, and four in *length* equal to *a yard:* and distances measured by means of these will be tolerably accurate, if care be taken while observing the angle subtended by the horizontal object, to stand directly in front of it. A man, a carriage-wheel, a window, a door, &c., at the remoter extremity of the distance we wish to ascertain, may serve for an approximation. But in all cases where it is possible, let a foot, a yard, or a six-feet measure, be placed vertically, at one end of the line to be measured, while the observer with his micrometer stands at the other. Then, if h be the height of the object,

either $\frac{1}{2} h \times \cot \frac{1}{2}$ angle subtended
or $h \times \cot$ angle

will give the distance, according as the eye of the observer is horizontally opposite to the *middle*, or to *one extremity* of the object whose angle is taken.

When a table of natural tangents is not at hand, a *very* near approximation for all angles less than *half a degree*, and a tolerably near one up to angles of *a degree*, will be furnished by the following rules.

If the distant object whose angle is taken be 1 foot in length, then

3437·73 ÷ the angle in minutes } will give the distance in
or 206264· ÷ the angle in seconds } feet.

If the remote object be 3, 6, 9, &c., feet in length, multiply the former result by 3, 6, 9, &c., respectively.

Ex. 1. What is the distance of a man 6 feet high, when he subtends an angle of 30 seconds?

* For P D : D A :: A B : B R $= \frac{A B^2}{P D}$; and O D : O P :: B R : R Q $= \frac{A B^2 . O P}{O D . D P}$.

206264 × 6 ÷ 30 = 206264 ÷ 5 = 41252·8 feet = 13750·9 yards, the distance required.

Ex. 2. In order to ascertain the length of a street, I put up a foot measure at one end of it, and standing at the other found that measure to subtend an angle of 2 minutes: required the length of the street.

$$3437{\cdot}73 \div 2 = 1718{\cdot}86 \text{ feet} = 572{\cdot}95 \text{ yards.}$$

3. *By means of the velocity of sound.*—Let a gun be fired at the remoter extremity of the required distance, and observe, by means of a chronometer that measures tenths of seconds, the interval that elapses between the flash and the report: then estimate the distance for one second by the following rule, and multiply that distance by the observed interval of time; the product will give the whole distance required.

At the temperature of freezing, 32°, the velocity of sound is 1100 feet per second; for lower temperatures deduct, or for higher temperatures add, half a foot per second for every degree of difference from 32° on Fahr. thermometer; the result will show the velocity of sound, very nearly, at all such temperatures.

Thus, at the temperature of 50°, the velocity of sound is,

$$1100 \times \tfrac{1}{2}(50 - 32) = 1109 \text{ feet.}$$

At temperature 60°, it is $1100 + \tfrac{1}{2}(60 - 32) = 1114$ feet.

For a more accurate method of determining the velocity of sound, see page 281.

CHAP. VI.

CONIC SECTIONS.

SECT. I. *Definitions.*

1. *Conic Sections* are the figures made by a plane cutting a cone.

2. According to the different positions of the cutting plane there arise five different figures or sections, viz., a *triangle*, a *circle*, an *ellipse*, an *hyperbola*, and a *parabola:* of which the three last are peculiarly called *Conic Sections.*

3. If the cutting plane pass through the vertex of the cone, and any part of the base, the section will be a *triangle*, as A B. (Fig. 115.)

4. If the plane cut the cone parallel to the base, or make no angle with it, the section will be a *circle*, as A B. (Fig. 116.)

5. The section is an *ellipse* when the cone is cut obliquely through both sides, or when the plane is inclined to the base in a less angle than the side of the cone is, as A B C D. (Fig. 120.)

6. The section is a *parabola* when the cone is cut by a plane parallel to the side, or when the cutting plane and the side of the cone make equal angles with the base, as $P_1 A P_2$. (Fig. 138.)

7. The section is an *hyperbola* when the cutting plane makes a greater angle with the base than the side of the cone makes, as $P_1 A P_2$. (Fig. 130.)

Note.—In all the above definitions the cone is supposed to be a *right cone.*

8. If all the sides of the cone be continued through the vertex, forming an opposite equal cone, and the plane be also continued to cut the opposite cone, this latter section is called the *opposite hyperbola* to the former, as $S_1 B S_2$. (Fig. 130.)

9. And if there be two other cones, with their axes in the same plane and their sides touching the sides of the former cones, then will the same plane cut all the cones and form four hyperbolas, as $P_1 A P_2$, $R_1 C R_2$, $S_1 B S_2$, $Q_1 D Q_2$ (fig. 118), each opposite pair of which are similar; these hyperbolas are said to be *conjugate* to each other, and the figure thus formed is called the *figure of the conjugate hyperbolas*, as fig. 118.

Note.—In the following definitions, the letters refer to fig. 117 for the ellipse; fig. 118 for the hyperbola; and fig. 119 for the parabola.

10. The *vertices* of any section are the points where the cutting plane meets the opposite sides of the cone, as A and B.

Cor. Hence the ellipse and the opposite hyperbolas have each two vertices; but the parabola only one; unless we consider the other as at an infinite distance.

11. The *major axis*, or *transverse diameter* of a conic section, is the line or distance A B between the vertices.

Cor. Hence the axis of a parabola is infinite in length, A B being only a part of it.

12. The *centre* O is the middle of the axis.

Cor. Hence the centre of a parabola is infinitely distant from the vertex. And of an ellipse, the axis and centre lie within the curve: but of an hyperbola, without.

13. The *minor*, or *conjugate axis*, is the line C D, drawn through the centre perpendicular to the transverse axis, and bounded each way by the curve.

Cor. Hence the parabola has not any conjugate axis, unless we suppose it at an infinite distance, and infinite in length.

14. A *tangent* to a curve at any point, is a line as $A_1 T_1$, which touches the curve in that point; but being produced either way does not cut it.

15. A *diameter* is any right line, as $A_1 B_1$, drawn through the centre, and terminated on each side by the curve; and the extremities of the diameter, or its intersections with the curve, as A_1 and B_1, are its *vertices*.

Cor. Hence all the diameters of a parabola are parallel to the axis, and infinite in length. Hence, also, every diameter of the ellipse and hyperbola has two vertices; but of the parabola, only one; unless we consider the other as at an infinite distance.

16. The *conjugate* to any diameter is the line drawn through the centre, and parallel to the tangent of the curve at the vertex of the diameter. So, $C_1 D_1$, parallel to the tangent at A_1, is the conjugate to $A_1 B_1$.

17. An *ordinate* to any diameter is a line parallel to its conjugate, or to the tangent at its vertex, and terminated by the diameter and curve. So, K I and $K_2 A_1$ are ordinates to the axis A B; and $K_1 I$ and $E_1 I_2$ ordinates to the diameter $A_1 B_1$.

Cor. Hence the ordinates of the axes are perpendicular to it; but of other diameters, the ordinates are oblique to them.

18. An *abscissa* is a part of any diameter, contained between its vertex and an ordinate to it; as A K or B K, and $A_1 E_1$ or $B_1 E_1$.

Cor. Hence, in the ellipse and hyperbola, every ordinate has two abscissæ; but in the parabola only one; the other vertex of the diameter being infinitely distant.

19. The vertex from which the abcissæ are measured is called the *origin* of those abcissæ, and any abcissa and its ordinate are called *co-ordinates*, as A K and K I.

20. The *parameter* of any diameter is that double ordinate which is a third proportional to the transverse and conjugate axes in the

ellipse or hyperbola, and to any abscissa and its ordinate in the parabola. The parameter of the transverse axis is called the *principal parameter*, or the *latus rectum;* thus, if $A_1 B_1 : C_1 D_1 :: C_1 D_1 : I_2 I_4$, then $I_2 I_4$ is the *parameter* of $A_1 B_1$; and if $AB : CD :: CD : I_1 I_3$, then $I_1 I_3$ is the *latus rectum*.

21. The point where the parameter cuts the transverse axis is called the *focus*, as E, F; and the distance of the focus from the nearest vertex of the same axis is called the *focal distance*, as AE, BF.

Cor. Hence, the ellipse and hyperbola have each two foci, but the parabola only one. The foci (or burning points) were so called because all rays are united or reflected into one of them, which proceed from the other focus, and are reflected from the curve.

22. The point E is also frequently called the *pole;* and a line drawn from any point in the curve to the pole, is called the *radius vector*, as EI. And the angle BEI, contained between the radius vector and the transverse axis, is called the *traced angle*.

23. The *directrix* is a right line drawn perpendicular to the transverse axis of a conic section, through an assignable point in the prolongation of that axis; such that lines drawn from any points in the curve parallel to the axis to meet the directrix, shall be to lines drawn from the same points to the focus, in a constant ratio for the same curve.

Thus, if $AE : AT :: EI_2 : I_2 y :: EI_3 : I_3 x :: EI : IX$, then XY is the directrix.

In the *ellipse* AE is *less* than AT.
In the *parabola* AE is *equal* to AT.
In the *hyperbola* AE is *greater* than AT.

24. The *subtangent* to any point in a curve, is that portion of the transverse axis which is contained between the tangent and ordinate to the same point, as $K_2 T_1$.

25. A *normal* at any point is a line drawn from that point perpendicular to the tangent, to meet the transverse axis, as $A_1 L$.

26. *The Radius of Curvature* of a conic section or other curve, is the radius of that circle which is precisely of the same curvature as the curve itself, at any assigned point, or the radius of the circle which *fits* the curve and coincides with it, at a small distance on each side of the point of contact. The circle itself is called the *osculatory* circle, or the *equicurve* circle; and if the curve be of incessantly varying curvature, each point has a distinct equicurve circle, the radius of which is perpendicular to the tangent at the point of contact.

27. An *asymptote* is a right line towards which a certain curve line approaches continually nearer and nearer, yet so as never to meet, except both be produced indefinitely. The hyperbola has two asymptotes, as UV, WZ.

SECT. II. *Properties of the Ellipse.*

1. Let the frustum of a right cone A G B H (fig. 120), be cut by a plane forming a tangent to each end, then will the section (A C B D) thus produced be an *ellipse*. Let $d = GB$, the diameter of the lesser end; $D = AH$ the diameter of the greater; $h = AG$, or BH, the slant height; $t = AB$, the transverse diameter of the ellipse; $c = CD$, the conjugate diameter; and $f = EF$, the distance between the two foci; then,

$$t^2 = h^2 + Dd \ldots\ldots (1.)$$
$$c^2 = Dd \ldots\ldots (2.)$$
$$f = h \ldots\ldots (3.)$$

2. Let $y = IK$ (fig. 117), any ordinate; $x = AK$, its abscissa measured from A; $z = OK$, its abscissa measured from the centre, O; $v = IE$, the radius vector from the focus E; $\beta =$ the traced angle IEF; and $q = \sqrt{\frac{1}{4}t^2 - \frac{1}{4}c^2}$; then,

$$y = \frac{c}{t}\sqrt{(tx - x^2)} \ldots\ldots (4.)$$

which is the equation to the ellipse when the abscissæ are measured from the vertex;

$$y = \frac{c}{t}\sqrt{(\tfrac{1}{4}t^2 - z^2)} \ldots\ldots (5.)$$

which is the equation when the abscissæ are measured from the centre; and

$$v = \frac{\frac{1}{4}c^2}{\frac{1}{2}t - q \cdot \cos\beta} \ldots\ldots (6.)$$

which is called the *polar equation*, and finds its principal use in the investigations of astronomy.

3. For the relation between the abscissæ and ordinates in any ellipse, we have,

As the square of the transverse axis,
: the square of the conjugate;
:: the rectangle of the abscissæ,
: the square of their ordinate.

Or, $t^2 : c^2 :: x(t - x) : y^2 \ldots\ldots (7.)$

The same proportion obtains between the ordinate and abscissæ to any diameter; or putting x_1 for A_1K_1 (fig. 117); x_2 for B_1K_1, and y_1 for IK_1; then

$$t_1^2 : c_1^2 :: x_1 x_2 : y_1^2 \ldots\ldots (8.)$$

The sum of the squares of any pair of conjugate diameters is equal to the sum of the squares of the two axes; or putting t_1 for A_1B_1 (fig. 117), and c_1 for C_1D_1; then

$$t^2 + c^2 = t_1^2 + c_1^2 \ldots\ldots (9.)$$

All the parallelograms that can be circumscribed about an ellipse are equal to the rectangle of the two axes; or $= ct$..... (10.)

The sum of the two radius vectors, drawn from the two foci, to any point in the curve, are equal to the transverse axis; or putting $v = EI$ (fig. 117), the radius vector from E, and $V = FI$, the same from F; we have

$$V + v = t \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (11.)$$

Cor. The distance of either focus from either extremity of the conjugate diameter is equal to half the transverse axis; or,

$$CE = CF = \frac{t}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (12.)$$

The square of the distance between the two foci equals the difference of the squares of the two axes; or,

$$f^2 = t^2 - c^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (13.)$$

4. If TM (fig. 121) be the tangent to an ellipse at any point I, and let T be the point in which the tangent meets the transverse axis produced; also from the two foci draw FM, EN, perpendicular to the tangent, and produce EN and FI to meet each other in P; then,

$$OK : AO :: AO : OT; \text{ or, } OT = \frac{t^2}{4x} \quad \ldots\ldots\ldots \quad (14.)$$

Cor. If there be any number of ellipses described on the same transverse axis (fig. 122), and any ordinate be drawn, cutting all the ellipses in the points I_1, I_2, I_3, I_4, &c., the tangents to the several ellipses at those points, will all meet the transverse axis produced in one common point, T ..... (15.)

The angles made by the two radius vectors and the tangent are equal; or,

$$\angle TIE = \angle LIF \ldots\ldots\ldots\ldots\ldots \quad (16.)$$

The points M and N (fig. 121) fall in the circumference of a circle whose diameter is the transverse axis, AB ..... (17.)

$$\text{Also, } EN \cdot FM = CO^2 = \left(\frac{c}{2}\right)^2 \quad \ldots\ldots \quad (18.)$$

$$\text{And, } EN^2 = \left(\frac{c}{2}\right)^2 \cdot \frac{v}{V} \quad \ldots\ldots\ldots\ldots\ldots \quad (19.)$$

If m be in the middle of KI_1, then, Am produced will meet the two tangents TQ, BQ, in their point of intersection, Q ... (20.)

If the ordinate II_1 passes through the focus, then the point T of intersection of the tangent and the transverse axis produced, will be a point in the directrix ..... (21.)

In any ellipse the parameter $= \frac{c^2}{t}$ ..... (22.)

5. Let R = the radius of curvature at any point in an ellipse, and V, v, the distances of the same point from the two foci; then

$$R = 4\,\frac{(Vv)^{\frac{3}{2}}}{tc} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (23.)$$

It is a *maximum* at the extremity of the conjugate axis, when

$$R_1 = \frac{t^2}{2c} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (24.)$$

And a *minimum* at the extremity of the transverse axis, when

$$R_2 = \frac{c^2}{2t} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (25.)$$

The area of any ellipse is a mean proportional between the areas of the circles described on its two axes ..... (26.)

PROBLEMS RELATING TO THE ELLIPSE.

PROB. I. *To find the two foci* E, F, *of an ellipse, of which the transverse axis* A B, *and conjugate* C D, *are given.*

From C or D as a centre, with a radius equal to the semi-transverse axis A O or B O, describe an arc cutting the transverse axis in E, F, which are the two foci required. (Fig. 123.)

PROB. II. *To construct an ellipse whose two axes are given.*

Find the distance E F, from Prob. I., and let a fine thread, E I F E, in length = E F + A B, be put round two pins fixed at the points E F; then, if a pencil be put within the cord, and carried round, care being taken that the cord is always tight, the pencil will describe an ellipse, A I C B D A. (Fig. 124.)

PROB. III. *The transverse and conjugate axis of an ellipse being given, to describe the same with a trammel.*

Let the distance between the pencil A and the first pin B, be equal to half the conjugate axis, and the distance between A and the second pin C, be equal to half the transverse; then, the pins being put into the grooves of the trammel, and the pencil A being moved, will describe the ellipse. (Fig. 125.)

Otherwise.—Let there be provided three rulers, of which the two E P, F H, are of the same length as the transverse axis A B, and the third H P, equal in length to E F, the focal distance. Then, connecting these rulers so as to move freely about E F H and I, their intersection I will always be in the curve of the ellipse: so that, if there be slits running along the two rulers, and the apparatus turned freely about the foci, a pencil put through the slits at their point of intersection will describe the curve. (Fig. 126.)

PROB. IV. *To find the two axes of any given ellipse.*

Draw any two parallel lines across the ellipse, as M L, F K: bisect them in the points I and D, through which draw the right line

N I E P, and bisect it in O. From O as a centre, with any adequate radius, describe an arc of a circle cutting the ellipse in the points G, H. Join G, H, and parallel to the line G, H, draw through O the minor axis C D; perpendicular to which through O draw A B, which will be the major axis. (Fig. 127.)

PROB. V. *From any given point out of an ellipse to draw a tangent to it.*

Let T be the given point, through it and the centre C draw the diameter A B; and parallel to it any line H I terminated by the curve. Bisect H I in O; and C O produced will be the conjugate to A B. Draw any line T S = T B, and make T R = T C. Draw R A, and parallel to it, S P cutting A B in P. Through P, draw P M parallel to C D, and join T M, which will be the tangent required. (Fig. 128.)

PROB. VI. *To find the length of an elliptic arc* $I_1 C I_2$.

Produce the height C E to O, the centre of the ellipse. Join I_1 and O, and from O as a centre with radius C O equal the semi-conjugate, describe an arc cutting $I_1 O$ in a. Bisect $I_1 a$ in b, and from O as a centre with radius O b describe the arc bc, cutting C O produced in c; then the circular arc bc is equal in length to half the elliptic arc $I_1 C I_2$. (Fig. 129.)

SECT. III. *Properties of the Hyperbola.*

1. Let two opposite right cones (Fig. 130) be cut by a plane making a less angle with their axes than the sides of the cones make, then the sections thus produced will be two *opposite hyperbolas*. Let d = A H, the lesser diameter of the cone at the vertex of one hyperbola; D = B G, the greater diameter of the cone at the vertex of the opposite hyperbola; h = A G, or B H, the slant height; t = A B, the transverse axis; c = C D, the conjugate axis; and f = E F, the distance between the two foci; then we have

$$t^2 = h^2 - D d \quad . \quad . \quad . \quad (1.)$$
$$c^2 = D d \quad . \quad . \quad . \quad . \quad (2.)$$
$$f = h \quad . \quad . \quad . \quad . \quad (3.)$$

which will be seen to be identical with the corresponding equations for the ellipse, with the exception of the first, which, however, only differs in the sign of the last term. This at once indicates a general analogy between the properties of the two curves; and if we employ the same letters to represent the corresponding lines in figures 117, 121, 122, and figures 118, 131, 132, the formulæ already given for the ellipse will apply to the hyperbola, only making the following alterations.

(4.) Becomes $y = \frac{c}{t}\sqrt{(tx + x^2)}$.

(5.) „ $y = \frac{c}{t}\sqrt{(x^2 - \frac{1}{4}t^2)}$.

(6.) „ $v = \frac{\frac{1}{4}c^2}{\frac{1}{2}t + q \cdot \cos\beta}$.

(7.) and (8.) remain unaltered.

(9.) Becomes $t_1^2 - c_1^2 = t^2 - c^2$.

(10.) All the parallelograms that can be *inscribed* between four conjugate hyperbolas, are equal to the rectangle of the two axes; or, $= c\,t$.

(11.) Becomes $V - v = t$.

(12.) „ O E or O F = A C or C B.

(13.) „ $f^2 = t^2 + c^2$.

(14.) to (25.), both inclusive, remain unaltered.

2. Besides these, however, there are several curious properties which relate to the *asymptotes* of the hyperbola. Let S_1, s_3, S_2, (fig. 133) be an hyperbola; O V and O Z its asymptotes; s_1 K s_5, a double ordinate to the transverse axis; O z_1, O z_2, O z_3, O z_4, O z_5, abscissæ taken on the asymptote O Z; and $z_1\,s_1$, $z_2\,s_2$, $z_3\,s_3$, $z_4\,s_4$, $z_5\,s_5$, ordinates to the same, drawn parallel to the other asymptote O V:

Then the parallelograms O v_5 s_1 z_1, O v_4 s_2 z_2, O v_3, s_3 z_3, &c. are all equal; or

$$O z_1 \cdot z_1 s_1 = O z_2 \cdot z_2 s_2 = O z_3 \cdot z_3 s_3 = O z_4 \cdot z_4 s_4 \quad (26.)$$

$$\text{Also, } m s_1 \cdot s_1 s_5 = n s_5 \cdot s_1 s_5 = O s_3^2 = \frac{t^2}{4} \quad . \quad . \quad (27.)$$

And the triangle O T I_1 (Fig. 131) = the triangle O B Q. (28.)

Also, if the abscissæ O z_1, O z_2, O z_3, &c. of any hyperbola, be taken on one of the asymptotes in an increasing geometrical progression, the ordinates $z_1\,s_1$, $z_2\,s_2$, $z_3\,s_3$, &c. parallel to the other asymptote are in decreasing geometrical progression, having the same ratio. . . . . (29.)

And, when the distances O z_2, O z_3, &c. are in geometrical progression, the asymptotic spaces $z_1\,s_1\,s_2\,z_2$, $z_1\,s_1\,s_3\,z_3$, &c. will be in arithmetical progression, and will, therefore, be analogous to the logarithms of the former . . . . . (30.)

The *radix* of the *system* of logarithms will depend upon the value of the angle made by the two asymptotes. In *Napier's* logarithms V O Z is a right angle; while in the common logarithms it is 25° 44′ 25½″ *.

* See Sect. 6, page 93.

3. In the case of four conjugate hyperbolas, the transverse and conjugate axes of one pair become the conjugate and transverse of the other pair, and therefore they are said to be mutually conjugate to each other.

PROBLEMS RELATING TO HYPERBOLAS.

PROB. I. *To describe an Hyperbola of which the transverse axis and two foci are given.*

Let one end of a long ruler, E O (fig. 134) be fastened at one of the foci, E, so as to turn freely about that point as a centre. Then take a thread F I_1 O, of such a length, that when one end is fixed at the other focus F, and the other end to the ruler at O, the doubled part will just reach to B, one end of the transverse axis. Then if the ruler E O be turned about the fixed point E, at the same time keeping the thread O I_1 F always tight, and its part I_1 O close to the side of the ruler, by means of the pencil I_1; the curve line B S_1 described by the motion of the pencil I, is one part of an hyperbola. And if the ruler be turned, and move on the other side of the fixed point F, the other part B S_2 of the same hyperbola may be described after the same manner. But if the end of the ruler be fixed in F, and that of the thread in E, the opposite hyperbola P_1 A P_2 may be described.

Otherwise; also by continued motion. Let E and F (Fig. 135) be the two foci, and A and B the two vertices of the hyperbola. Take three rulers, C D, D I, I F, and fix them so that E D = C F = A B, and D C = E F; the rulers D I and G I being of an indefinite length beyond I, and having slits in them for a pin to move in; and the rulers having holes in them at E and F, to fasten them to the foci E and F by means of pins, and at the points D and C they are to be joined by the ruler D C. Then, if a pencil be put in the slits, at the common intersection of the rulers D I and F I, and moved along, causing the two rulers F I, I D, to turn about the foci E and F, that pencil will describe the portion A P_1 of an hyperbola.

PROB. II. *To describe an hyperbola of which the two axes are given, by finding points in the curve.*

Let A B be the transverse axis, and C D the conjugate. First from O as a centre, with radius equal A C or C B, describe arcs cutting the transverse axis produced in E and F, which will be the two foci. Then assume any number of points, a_1, a_2, a_3, &c. beyond F, and from E as a centre, with radii equal to B a_1, B a_2, B a_3, &c., describe arcs b_1, b_2, b_3, &c.; also from F as a centre, with radii equal to A a_1, A a_2, A a_3, &c., describe arcs cutting the former arcs in the points c_1, c_2, c_3, &c., which will all be points in the hyperbola required. (Fig. 136.)

PROB. III. *To describe an hyperbola, of which the transverse axis* A B, *and any abscissa* A G, *and its double ordinate* H I, *are given.*

Through A draw $a\,b$ parallel to H I; from H and I draw H a and I b parallel to A G; then divide aH and bI, each into any number of equal parts, 1, 2, 3, &c., and through the points of division draw lines radiating to A as a centre; also divide K H and K I into the same number of equal parts, and through the points of division draw lines radiating to B, then will the several points c_1, c_2, c_3, &c, in which these lines intersect the former lines, be so many points in the hyperbola required. (Fig. 137.)

SECT. IV. *Properties of the Parabola.*

1. If the right cone K H I (fig. 138) be cut by a plane parallel to the side K H, then will the section P_1 A P_2, thus produced, be a *parabola*. Let the cone also be cut by a plane, M A, perpendicular to the plane of the parabola, and by another plane, G A, parallel to the base of the cone, and both passing through the point A; bisect the distance M G (between their two vertices) in L, and through L, take a plane L T, parallel to the base of the cone, and let E be the focus of the parabola. Let b = B P_2, half the base of the parabola; d = G A, the diameter of the cone at the vertex of the parabola; f = A E, the focal distance; and h = K A, the slant height of the cone above the same; then

The line T X, in which the plane L T meets the plane of the parabola, is its directrix . . . . . (1.)

$$\text{T A} = f = \frac{d^2}{4h} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

$$f = \frac{b^2}{4d} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

$$2\,(\text{M G}) = 4f = \text{the parameter} \quad . \quad . \quad (4.)$$

2. Let p = I_1 I_3 (fig. 119), the parameter of a parabola; x = A K, any abscissa; y = I_1 K, the corresponding ordinate; v = E I, the radius vector; and β = the traced angle I E B, E being the focus; then

$$y^2 = px \quad . \quad . \quad . \quad . \quad (5.)$$

$$v = \frac{2f}{1 \mp \cos\beta} \quad . \quad . \quad (6.)$$

the equations to the parabola: in the latter of which, or the polar equation, the sign + obtains when K is between A and E, and — when K is below E.

$$\text{Rad. of curvature at I} = \frac{(4x + p)^{\frac{3}{2}}}{2\sqrt{p}} \quad . \quad . \quad (7.)$$

At the vertex, A, x vanishes, and we have rad. of curv. at vertex $= \frac{1}{2}p$ . . . . (8.)

3. In the same figure, where XY is the directrix, the following properties obtain; viz.

$$AE = AT,\ EI_2 = I_2 y,\ EI_3 = I_3 x,\ EI = IX,\ \&c. \quad (9.)$$

$$\text{As } AK : AK_3 :: KI^2 : K_3P_1{}^2;\ \text{or } \frac{AK}{AK_3} = \frac{KI^2}{K_3P_1{}^2} \quad (10.)$$

where AK and AK_3 are any abscissæ, and KI, K_3P_1 their corresponding ordinates. Also

$$EI = AK + AE,\ EP_1 = AK_3 + AE \quad . \quad (11.)$$

$$AE = \frac{EI_3}{2} = \frac{I_1I_3}{4} \quad . \quad . \quad . \quad . \quad . \quad (12.)$$

I_1I_3 being the parameter $= p$.

$$\left.\begin{array}{l}\text{As } p : K_3P_1 + K_2I :: K_3P_1 - K_2I : AK_3 - AK_2 \\ \text{or, as } p : P_2a :: aP_1 : K_2K_3 \quad . \quad . \quad . \quad . \quad .\end{array}\right\} \quad (13.)$$

$$\text{And } p = \frac{IK_2{}^2}{AK_2} = \frac{K_3P_1{}^2}{AK_3} \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.)$$

4. Again, let A_1T_1 be the tangent to a parabola at any point A_1, and let B_1A_1Y be drawn through A_1 parallel to the axis AB; let A_1L be perpendicular to A_1T_1; then is K_2T_1 the subtangent, A_1L the normal, K_2L the subnormal; and the following properties obtain; viz.

$$\text{angle } EA_1T_1 = \text{angle } ET_1A_1 = \text{angle } T_1A_1Y \quad . \quad . \quad (15.)$$

$$\text{angle } LA_1B_1 = \text{angle } LA_1E \quad . \quad . \quad . \quad . \quad . \quad (16.)$$

$$EA_1 = ET_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.)$$

$$AK_2 = AT_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.)$$

$$\text{subtangent } K_2T_1 = 2AK_2 \quad . \quad . \quad . \quad . \quad . \quad (19.)$$

$$\text{subnormal } K_2L = 2AE = \frac{p}{2} \text{ a constant quantity} \quad . \quad . \quad (20.)$$

5. In figure 139 also, where CQ is a tangent to the parabola at the point C, and IK, OM, QL, &c., parallel to the axis AD.

$$\text{Then } IE : EK :: CK : KL \quad . \quad . \quad . \quad (21.)$$

and a similar property obtains, whether CL be perpendicular or oblique to TD.

The external parts of the parallels IE, TA, ON, QL, &c., are always proportional to the squares of the intercepted parts of the tangent; that is,

$$\left.\begin{array}{l}\text{the external parts } IE, TA, ON, QL, \\ \text{are proportional to } CI^2, CT^2, CO^2, CQ^2, \\ \text{or to the squares } CK^2, CD^2, CM^2, CL^2,\end{array}\right\} \quad . \quad . \quad (22.)$$

And as this property is common to every position of the tangent, if the lines I E, T A, O N, &c., be appended to the points I, T, O, &c., of the tangent, and movable about them, so as always to hang vertically, and of such lengths that their extremities E, A, N, &c., be in the curve of a parabola in any one position of the tangent; then making the tangent revolve about the point C, the extremities E, A, N, &c., will always form the curve of some parabola, in every position of the tangent.

The same properties, too, that have been shown of the axis, and its abscisses and ordinates, &c., are true of those of any other diameter.

PROBLEMS RELATING TO THE PARABOLA.

PROB. I. *To construct a Parabola, of which the base and height are given.*

Construct an isosceles triangle A B D (fig. 140) whose base A B shall be the same as that of the proposed parabola, and its altitude C D *twice* the altitude C V of the parabola. Divide each side A D, D B, into 10, 12, 16, or 20, equal parts [16 is a good number, because it can be obtained by continual bisections], and suppose them numbered 1, 2, 3, &c. from A to D, and 1, 2, 3, &c. from D to B. Then draw right lines 1, 1; 2, 2; 3, 3; 4, 4; &c., and their mutual intersection will beautifully approximate to the curve of the parabola A V B.

PROB. II. *To describe a Parabola by finding points in the curve, an abscissa* A B, *and its double ordinate* C D, *being given.*

Through A (fig. 141) draw $a\,b$ parallel to C D; and through C and D draw a C, b D parallel to A B; then divide a C and b D into any convenient number of parts, 1, 2, 3, &c., and through these divisions draw lines radiating to A; also divide B C and B D into the same number of parts, and through the divisions draw lines perpendicular to C D, then the points c_1, c_2, c_3, &c., in which these lines intersect the former, will be in the parabola.

Let the ruler, or directrix B C (fig. 142) be laid upon a plane with the square G D O, in such a manner that one of its sides D G lies along the edge of that ruler; and if the thread F M O, equal in length to D O, (the other side of the square,) have one end fixed in the extremity of the ruler at O, and the other end in the focus F: then slide the side of the square D G along the ruler B C, and at the same time keep the thread continually tight by means of the pencil M, with its part M O close to the side of the square D O; so shall the curve A M X, which the pencil describes by this motion, be one part of a parabola.

And if the square be turned over, and moved on the other side of the fixed point F, the other part of the same parabola A M Z will be described.

PROB. III. *Any right line being given in a parabola, to find the corresponding diameter; also, the axis, parameter, and focus.*

Draw H I (fig. 143) parallel to the given line D E. Bisect D E, and H I, in ● and G, through which draw A O G for the diameter. Draw H R perpendicular to A G and bisect it in B; and draw V B parallel to A G for the axis. Make V B : H B :: H B : parameter of the axis. Then ¼ the parameter set from V to F gives the focus.

PROB. IV. *To draw a tangent to a Parabola.*

If the point of contact C (fig. 144) be given, draw the ordinate, C B, and produce the axis until A T = A B: then join T C, which will be the tangent.

Or if the point be given in the axis produced: take A B = A T, and draw the ordinate B C, which will give C the point of contact; to which draw the line T C as before.

If D be any other point, neither in the curve nor in the axis produced, through which the tangent is to pass: draw D E G perpendicular to the axis, and take D H a mean proportional between D E and D G, and draw H C parallel to the axis; so shall C be the point of contact, through which and the given point D the tangent D C T is to be drawn.

When the tangent is to make a given angle with the ordinate at the point of contact: take the abscis A I equal to half the parameter, or to double the focal distance, and draw the ordinate I E: also draw A H to make with A I the angle A H I equal to the given angle; then draw H C parallel to the axis, and it will cut the curve in C the point of contact, where a line drawn to make the given angle with C B will be the tangent required.

CHAP. VII.

PROPERTIES OF CURVES.

SECT. I. *Definitions.*

1. A ***curve line***, as already defined, (Def. 5, page 98,) is a line which continually changes its direction.

2. A ***plane*** curve, is one the several points of which are all situate in the same plane; and a curve of ***double curvature*** is one whose several points are not all in the same plane, and which, therefore, curves in two directions.

3. Curve lines have been further divided into *Algebraical* or *Geometrical*, and *Transcendental* or *Mechanical*.

4. An ***algebraical*** curve, is one in which the relation which the abscissa bears to the ordinate, taken for any possible point in that curve, may always be expressed by a constant algebraical formula.

5. A ***transcendental*** curve, is one in which no such constant relation between its abscissæ and ordinates exists, and which can, therefore, only be described by mechanical means.

6. The algebraical formula which expresses this relation is called the ***equation*** of the curve; and curves are classed into orders, according to the number of dimensions contained in such equations. Thus any curve line which might be represented by the equation

$$0 = a + bx + cy + dx^2 + exy + fy^2,$$

would be called a line of the *second* order, because the abscissa (x) and ordinate (y) are involved only to the *second* power; and a curve which might be expressed by

$$0 = a + bx + cy + dx^2 + exy + fy^2 + gx^3 + hx^2y + hy^2x + ly^3$$

would be called a line of the *third* order, because the same quantities are involved to the *third* power. The letters a, b, c, &c., are merely the coefficients depending upon the particular curve expressed by the equation.

7. An ***asymptote*** to a curve is a straight line, to which the curve line continually approaches, without ever meeting it; as the asymptotes U V and W Z, to the hyperbola, fig. 118.

8. If a curve cuts itself by passing *twice* through the same point, as A, fig. 145, this point is called a *punctum duplex:* if *three* times, as B, it is called a *punctum triplex*, &c.

9. The oval contained between A and C, fig. 145, is called a *nodus:* when the distance between A and C becomes indefinitely small, so that the nodus vanishes, the point A is called a *punctum conjugatum.*

10. A point in which two branches of a curve terminate, is called a *cuspis.*

11. The *involute* of any curve, is another curve traced by the end of a string in being unwound from the first curve, which is called the *evolute.* Thus, let A B C (fig. 146) be a curve having a thread laying in contact with every part of the same; then, if the thread be unwound from A, the curve line A D E, which its end will describe, will be the *involute* to the curve A B C, which latter will be the *evolute* of the curve A D E.

The principal lines connected with curves, such as *tangent, ordinate, abscissæ,* &c., having been already defined in the first section of Chap. VI., it is unnecessary to recapitulate them here.

12. The equations of the conic sections have been already given, for the ellipse at page 165, the hyperbola at page 169, and the parabola at page 171: the equation for the circle is

$$y = \sqrt{dx - x^2},$$

d being the diameter; the whole of these four curves are therefore lines of the *second* order, the quantities x and y, only being raised to that power. We shall now pass on to describe the principal properties of those curves which are most frequently employed for useful purposes.

SECT. II. *The Conchoid.*

Conchoid, or *Conchiles,* (from the Latin *concha,* a shell,) is the name given to a curve by its inventor, *Nicomedes,* about 200 years before the Christian era.

The conchoid is thus constructed: A P and B D (fig. 147) being two lines intersecting at right angles: from P draw a number of other lines P F D E, &c., on which make always D E = D F = A B or B C; so shall the curve line drawn through all the points E, E, E, be the first conchoid, or that of Nicomedes; and the curve drawn through all the other points, F, F, F, is called the second conchoid; though, in reality, they are both but parts of the same curve, having the same pole P, and four infinite legs, to which the line D B D (called the directrix) is a common asymptote.

The inventor, Nicomedes, contrived an instrument for describing his conchoid by a mechanical motion, in the following manner: in the ruler D D (fig. 148) is a channel or groove cut, so that a smooth nail firmly fixed in the movable ruler C A, in the point D, may slide freely within it: into the ruler A P is fixed another nail at P, for the movable ruler A P to slide upon. If, therefore, the ruler A P be so moved as that the nail D passes along the groove D D, the style, or point in A, will describe the first conchoid.

Conchoids of all possible varieties may also be constructed with great facility by Mr. Jopling's apparatus for curves, now well known.

1. Let AB = BC = DE = DF = a, PB = b, BG = EH = x, and GE = BH = y: then the equation to the first conchoid will be

$$x^2(b+x)^2 + x^2y^2 = a^2(b+x)^2,$$

or, $x^4 + 2bx^3 + b^2x^2 + x^2y^2 = a^2b^2 + 2a^2bx + a^2x^2$;

and, changing only the sign of x, as being negative in the other curve, the equation to the second conchoid will be

$$x^2(b-x)^2 + x^2y^2 = a^2(b-x)^2,$$

or, $x^4 - 2bx^3 + b^2x^2 + x^2y^2 = a^2b^2 - 2a^2bx + a^2x^2$.

2. Of the conchoids expressed by these two equations (or rather the same equation with different signs), there are three species; first, when a is *less* than b, the conchoid will be of the form shown in fig. 148, the inferior branch having a *punctum conjugatum* at A; secondly, when a is *equal* to b, the conchoid will assume the form shown in fig. 149, the punctum conjugatum giving place to a *cuspis* at B; and thirdly, when a is *greater* than b, the cuspis becomes a *nodus*, the conchoid taking the form shown in fig. 150.

Newton approved of the use of the conchoid for trisecting angles, or finding two mean proportionals, or for constructing other solid problems. But the principal modern use of this curve, and of the apparatus by which it is constructed, is to sketch the contour of the section that shall represent the diminution of columns in architecture.

SECT. III. *The Cissoid or Cyssoid.*

The *cissoid* is a curve invented by an ancient Greek geometer and engineer named *Diocles*, for the purpose of finding two continued mean proportionals between two given lines. This curve admits of an easy mechanical construction; and is described very beautifully by means of Mr. Jopling's apparatus.

At the extremity B of the diameter AB (fig. 151), of a given circle AOBo, erect the indefinite perpendicular eBE, and from the other extremity A draw any number of right lines, AC, AD, AE, &c., cutting the circle in the points R, O, M, &c.; then, if CL be taken = AR, DO = AO, EN = AM, &c., the curve passing through the points A, L, O, N, &c. will be the *cissoid*.

1. Here the circle AOBo is called the *generating circle;* and AB is the *axis* of the curves ALON, &c., Alon, &c., which form a *cuspis* at A, and, passing through the middle points O, o, of the two semicircles, tend continually towards the *directrix* eBE, which is their common asymptote.

2. Letting fall perpendiculars LP, RQ, from any corresponding points L, R: then is AP = BQ, and AL = CR.

3. Also, $AP : PB :: PL^2 : AP^2$. So that, if the diameter AB of the circle $= a$, the abscissa $AP = x$, and the ordinate $PL = y$; then

$$x : a - x :: y^2 : x^2, \text{ or } x^3 = (a - x)y^2,$$

which is the equation to the curve.

4. The arc AM of the circle = arc BR, and arc Am = Br.

5. The whole infinitely long cissoidal space, contained between the asymptote eBE and the curves NOLA, &c., and $ALon$, &c., is equal to three times the area of the generating circle $AOBo$.

SECT. IV. *The Cycloid, and Epicycloid.*

The *cycloid*, or *trochoid*, is an elegant mechanical curve first noticed by *Descartes*, and an account of it was published by *Mersenne*, in 1615. It is, in fact, the curve described by a nail in the rim of a carriage-wheel while it makes one revolution on a flat horizontal plane.

1. Thus, if a circle EPF (fig. 152), keeping always in the same plane, be made to roll along the right line AB, until a fixed point P, in its circumference, which at first touched the line at A, touches it again at B after a complete revolution; the curve APVPB described by the motion of the point P is called a *cycloid*.

2. The circle EPF is called the *generating circle;* and the right line AB, on which it revolves, is called the *base* of the cycloid.

Also, the right line, or diameter CV, of the circle, which bisects the base AB at right angles, is the *axis* of the cycloid; and the point V, where it meets the curve, is the *vertex* of the cycloid.

3 If P (fig. 153) be a point in the fixed diameter AF produced, and the circle AEF be made to roll along the line AB as before, so that the point A, which first touched it at one extremity, shall touch it again at B, the curve PVP, described by the point P, is called a *curtate cycloid*.

4. And, if the point P (fig. 154) be anywhere in the unproduced diameter AP, and the circle AEF be made to roll along AB from A to B; the curve PVP, is, in that case, called the *inflected* or *prolate cycloid*.

The following are the chief properties of the common cycloid.

5. The circular arc VE (fig. 155) is equal to the line EG between the circle and cycloid, parallel to AB.

6. The semi-circumference VEC is equal to the semi-base CB.

7. The arc VG equals twice the corresponding chord VE.

8. The semi-cycloidal arc VGB equals twice the diameter VC.

9. The tangent TG is parallel to the chord VE.

10. The radius of curvature at V equals 2 CV.

11. The area of the cycloid AVBCA is triple that of the circle CEV; and consequently that circle and the spaces VECBG, VE′CAG′, are equal to one another.

12. A body falls through any arc LK (fig. 156) of a cycloid reversed, in the same time whether that arc be great or small; that is, from any point L, to the lowest point K, which is the vertex reversed: and that time is to the time of falling perpendicularly through the axis MK, as the semi-circumference of a circle is to its diameter, or as 3·141593 : 2. And hence it follows that if a pendulum be made to vibrate in the arc LKN of a cycloid, all the vibrations will be performed in the same time.

13. The *evolute* of a cycloid is another equal cycloid, so that if two equal semi-cycloids OP, OQ (fig. 156), be joined at O, so that OM is equal to MK the diameter of the generating circle, and the string of a pendulum hung up at O, having its length equal to OK or the curve OP; then, by plying the string round the curve OP, to which it is equal, if the ball be let go, it will describe, and vibrate in the other cycloid PKQ; where OP equals QK and OQ equals PK.

14. The cycloid is the curve of swiftest descent: or a heavy body will fall from one given point to another, by the way of the arc of a cycloid passing through those two points, in a less time than by any other route. Hence, this curve is at once interesting to men of science and to practical mechanics.

15. If the generating circle, instead of rolling along a straight line is made to roll upon the circumference of another circle, the curve described by any point in its circumference is called an *epicycloid*.

16. This curve derives its importance in practical mechanics, from its being the curve most frequently adopted for the teeth of wheels.

17. If the generating circle revolves upon the *convex* circumference of the quiescent circle, as in fig. 157, it is then called an *exterior epicycloid* ABCD. But if it revolves on the *concave* circumference, as in fig. 158, the curve is called an *interior epicycloid*.

18. The area of an epicycloid, either interior or exterior, may be found from the following proportion: as $AB : 3\overline{AB} + BC :: abc : abD$ :: the area of the generating circle : the area of the *epicycloid* DBE*cb*D.

19. If the diameter of the generating circle is equal to half that of the quiescent circle, the epicycloid becomes a straight line; which circumstance has been taken advantage of for converting a continued circular motion into an alternating rectilinear motion, in the manner explained in the article on "Select Mechanical Expedients," in the Appendix.

SECT. V. *The Quadratrix.*

The *quadratrix* is a species of curve by means of which the quadrature of the circle and other curves is determined mechanically. For the quadrature of the circle, curves of this class were invented by *Dinostrates* and *Tschirnhausen*, and for that of the hyperbola by Mr. *Perks*. We shall simply describe in this place the quadratrix of

Dinostrates; in order to show its use in the division of an arc or angle.

To construct this quadratrix, divide the quadrantal arc A B (fig. 159) into any number of equal parts, A *b*, *bc*, *cd*, *d*B; and the radius A C into the same number of equal parts, A *f*, *fg*, *gh*, *h*C. Draw radii C *b*, C *c*, &c., to the points of division upon the arc; and let lines *fl*, *gm*, &c., drawn perpendicularly to A C from the several points of division upon it, meet the radii in *l*, *m*, *n*, &c., respectively. The curve A *l m n* D that passes through the points of intersection *l*, *m*, &c. is the quadratrix of Dinostrates.

The figure A C D *n m l* A thus constructed may be cut out from a thin plate of brass, horn, or pasteboard, and employed in the division of a circular arc.

Thus, suppose the arc I L (fig. 160) or the angle I K L is to be divided into five equal parts. Apply the side A B of the quadratrix upon I K, the point B corresponding with the angle K. Draw a line along the curve A S, cutting K L in F. Remove the instrument, and from F let fall the perpendicular F E upon I K. Divide E I into five equal parts, by Prob. VII. page 113, and through the points of division draw C M, D N, &c. parallel to E F. Then, through their intersections M, N, O, P, draw the lines K M, K N, K O, K P, radiating from K, and they will divide the angle I K L into five equal parts, as required.

Note 1.—If, instead of dividing the arc into equal parts, it were proposed to divide it into a certain number of parts having given ratios to each other; it would only be necessary to divide E I into parts having the given ratio, and proceed in other respects as above.

Note 2.—If the arc or angle to be divided exceed 90 degrees, bisect it, divide that bisected arc or angle into the proposed number of parts, and take *two* of them for *one* of the required divisions of the whole arc.

SECT. VI. *The Catenary.*

The *catenary* is a mechanical curve, being that which is assumed by a chain or cord of uniform substance and texture, when it is hung upon two points or pins of suspension (whether those points be in a horizontal plane or not), and left to adjust itself in equilibrio in a vertical plane.

The catenarian curve was for a long while regarded as the proper form for the chains of a suspension bridge, but a slight consideration of the above definition will show its inapplicability. For, since the chains of a suspension bridge ought never to be of uniform substance, and have also in addition to their own weight to support that of the roadway, which latter is very differently distributed to the former; the form of the curve which the chains will assume is materially modified, and approaches more nearly to that of a parabola*.

* See page 207.

Let A B (fig. 161) be the points of suspension of such a cord, $A a C b B$ the cord itself when hanging at rest in a vertical position. Then the two equal and symmetrical portions $A a C$, $C b B$, both exposed to the force of gravity upon every particle, balance each other precisely at C. And, if one half, as $C b B$, were taken away, the other half, $A a C$, would immediately adjust itself in the vertical position under the point A were it not prevented. Suppose it to be prevented by a force acting horizontally at C, and equal to the weight of a portion of the cord or chain equal in length to C M; then is C M the measure of the *tension* at the vertex of the curve; it is also regarded as the *parameter* of the catenary. Whether the portion $A a C$ hang from A, or a shorter portion, as $a C$, hang from a, the tension at C is evidently the same: for, in the latter case, the resistance of the pin at a, accomplishes the same as the tension of the line at a when the whole $A a C$ hangs from A*.

Let the line C M which measures the tension at the vertex $= p$, let Cd (or the *deflexion* as it is termed) $= x$, ad (or the semi-span) $= y$, $Ca = z$, $CD = h$, $AB = d$, $CaA = CbB = \frac{1}{2}l$. Then,

$$1.\ y = p\left(\text{hyp. log}\ \frac{p + x + \sqrt{2px + x^2}}{p}\right)$$

$$= p\left(\text{hyp. log}\ \frac{p + x + z}{p}\right) = p\left(\text{hyp. log}\ \frac{z + p}{z - p}\right)$$

$$= p\,\mathrm{M} \,.\, \log \tan\left(45° + \tfrac{1}{2}\mathrm{S}\right);$$

where $\mathrm{M} = 2{\cdot}30255851$, the number by which the common logarithms must be multiplied to obtain the hyperbolic logarithms†; and S = the angle which the tangent to the curve at the point a makes with the horizon.

2. If the angle S of suspension made between the tangent to the curve at A or B, and the horizon be 45°; then $d : l :: 1 : 1{\cdot}1346$.

3. When $l = 2d$, then $h = {\cdot}7966\,d$, and $\mathrm{S} = 77° 3'$.

4. When the angle S of suspension is $56° 28'$, then p, x, y, and z, are as 1, 0·81, 1·1995, and 1·5089 respectively. In this case t, the tension at the point of suspension, is a *minimum* with respect to y.

5. Generally, $\frac{d}{l} = -\tan \mathrm{S}\,\mathrm{M} \,.\, \log \tan\left(\frac{1}{2}\mathrm{S}\right)$.

Or, $\log \frac{d}{l} = \log \tan \mathrm{S} + \log\{\log \cot\left(\frac{1}{2}\mathrm{S}\right) - 10\} + {\cdot}3622157 - 10$.

This last formula serving to compute an approximative result.

* This may easily be determined experimentally, by letting the cord hang very freely over a pulley at C, and lengthening or shortening the portion there suspended, until it keeps $A a C$, in its due position; then is the portion so hanging beyond the pulley equal in length to C M.

† See page 93.

6. The distance of the centre of gravity of the whole curve $2s$, from the vertex $= \frac{1}{2}\left(x + \frac{p\,y}{z} - p\right)$.

7. The radius of curvature $\frac{(p+x)^2}{p} = \frac{t^2}{p}$: this at the vertex is rad. curv. $= p$.

8. When S and p are given; then

$$s = p \,.\, \tan S$$
$$t = p \,.\, \sec S$$
$$x = p \,.\, \sec (S - 1) = \frac{p \,.\, \text{versin } S}{\cos S}$$
$$y = p\,\mathrm{M} \,.\, \log \tan (45° + \tfrac{1}{2} S).$$

9. When S and z, or $\frac{1}{2}l$, are given: then

$$p = z \,.\, \cot S$$
$$t = z \,.\, \text{cosec } S$$
$$x = z \,.\, \text{cosec } S \,.\, \text{versin } S$$
$$y = \mathrm{M}\, z \,.\, \cot S \,.\, \log \tan (45° + \tfrac{1}{2} S).$$

10. When S and y are given: then

$$p = \frac{y}{\mathrm{M} \,.\, \log \tan (45° + \frac{1}{2} S)}$$
$$t = \frac{y}{\mathrm{M} \,.\, \cos S \,.\, \log \tan (45° + \frac{1}{2} S)}$$
$$z = \frac{y \,.\, \tan S}{\mathrm{M} \,.\, \log \tan (45° + \frac{1}{2} S)}$$
$$x = \frac{y \,.\, \text{versin } S}{\mathrm{M} \,.\, \cos S \,.\, \log \tan (45° + \frac{1}{2} S)}$$

11. When x and y are given; then

$$\frac{\log \tan (45° + \frac{1}{2} S)}{\sec S \,.\, \text{versin } S} = \frac{y}{\mathrm{M}\, x};$$

from which S may be found by an approximative process; also

$$p = \frac{x}{\sec S - 1} \ldots\ t = \frac{x}{\text{versin } S} \ldots\ z = \frac{x \,.\, \sin S}{\text{versin } S}$$

in all these cases t is determined in *length of chain or cord* of wh the catenary is actually constructed.

12. *To draw the catenary mechanically.* — If the distance A (fig. 161) between the points of suspension, and the depth DC of lowermost point, be given, hang one extremity of a fine unifo chain or cord at one of the points A, and (letting the chain or c adjust itself as a festoon in a vertical plane) lengthen or shorte as it is held near the other end, over a pin at B, until, when at r

it just reaches the point C: so shall the cord form the catenary; and a pencil passed along the cord, from A by a, C, b, to B, will mark the curve upon a vertical board brought into contact with it.

13. All catenaries that make equal angles with their ordinates at their points of suspension are similar, and have x to y a constant ratio: and of any two which do not make equal angles, but have x to y in different ratios, a portion may be cut from one curve similar to the other. Thus, let A C B and A′ C′ B′ (fig. 162), be the two curves, of which A′ C′ B′ is the flatter. Suppose them placed upon one axis D C C′, and the tangent T′ S′, to the lower curve, at B′, the point of suspension, to be drawn. Then, parallel to T′ S′ draw another line T S to touch the other curve in b. Through b draw $b a$ parallel to B′ A′. So shall the portion a C b of the upper catenary be similar to the lower catenary A′ C′ B′.

14. With reference to the practical uses of the catenary, we may now blend the geometrical and the mechanical consideration of its properties. Taking any portion C b (fig. 163) of the catenary, from the lowest point C; its weight may be regarded as supported by tensions acting in the tangential directions C N, b N. The strains at C and b may be conceived as acting at the point of intersection N; above which, therefore, in the vertical direction N G, the weight of the portion C b may be conceived to act at its centre of gravity, G.

Hence, strain at C : weight of C b :: sin G N b : sin b N C :: cos b N R : sin b N R :: radius : tan b N R :: radius : tan d b N.

Hence, the horizontal tension at C being constant, the weight, and consequently the length of any portion c b of the uniform chain must be proportional to the tangent of the inclination of the catenary to the horizon at the extremity b of the said portion. *This may be regarded as the characteristic property of the catenary.*

15. In like manner, the horizontal strain at C : oblique strain at b :: sin b N G : sin C N G :: cos b N R : radius :: radius : sec b N R.

Therefore, the strain exerted tangentially, at any point b, is proportional to the *secant* of the inclination at that point.

Also, from § 14 and § 15, tangential strain at b : weight of B b C :: sec d b N : tan d b N.

These properties evidently accord with the preceding equations.

16. Let, then, C O, in the axis produced downwards, be equal to the parameter, or the measure of the horizontal strain at C; and upon O as a centre with radius C O describe a circle. A tangent dt drawn to this circle from d, will be parallel to the tangent b N h of the curve at the point b to which d b is the ordinate. That tangent dt (to the circle) will also be equal in length to the corresponding portion b C of the curve: while the tension at b will be expressed by a length of the chain equal to the secant O d. So again, if D T be a tangent to the circle drawn from D, it will be equal in length to B b C, and parallel to the tangent to the catenary at B; while the secant O D will measure the oblique tension at B; evidently, exceeding the constant horizontal tension or strain at C, by the abscissa C D.

17. When the parameter of the catenary, or the line which measures the tension at the lowest point, is equal to the deflexion DC; if each of these be supposed equal to 1, then AB = 2·6339, the length of chain ACB = 3·4641; the strain at the points of suspension A and B will each be 2, that at the lowest point being 1; and the chain at A and B will make an angle of 60° with the horizon.

18. If the strain at C be equal to the weight of the chain, and each denoted by 1: then AB = ·96242, DC = ·1180340, the tension at A or B = 1·118, the angle of suspension at those points nearly 26° 34′; the width of the curve is 8·1536 times, and the length 8·4719 times, the deflexion DC.

19. If the strain or tension at the lowest point be double the weight of the chain: then if the parameter be 1, AB will be ·49493, CD = ·03078, the strain at A or B 1·03078, the angle of suspension about 14° 2′, the width or span 16·0816 times, and the length of chain 16·2462 times the deflexion.

The magnitudes of the lines, angles, and strains in many other cases, may be seen in the table below. The whole theory may be verified experimentally, by means of spring steelyards applied to a chain of given length and weight, placed in various positions, according to the method suggested at page 244, when treating of the mechanical powers.

20. Taking AB = d, CD = h, length ACB = l, strain at C or parameter = p, then, in all cases where the deflexion is small compared with the length of the chain, *Professor Leslie* shows*, that

$$p = \frac{d^2}{8h} + \tfrac{1}{6}h \ldots\ \text{strain at A or B} = \frac{d^2}{8h} + \tfrac{7}{6}h$$

$$\text{or } p = \frac{l^2}{8h} - \tfrac{1}{2}h \ldots\ \text{strain at A or B} = \frac{l^2}{8h} + \tfrac{1}{2}h$$

$$l = d + \frac{8h^2}{3d}.$$

In this case, too, the strains at C and A or B are nearly in the inverse ratio of the deflexion†.

21. The following table is abridged from a very extensive one given by *Mr. Ware*, in his "Tracts on Vaults and Bridges." Two examples will serve to illustrate its use.

Ex. 1. Suppose that the span of a proposed suspension bridge is to be 560 feet, and the deflexion in the middle 25⅞ feet; what will be the length of the chain, the angle of suspension at the extremities, and the ratio of the horizontal pressure at the lowest point, and the oblique pressures at the points of suspension, with the entire weight of the chain?

* Elements of Natural Philosophy, p. 63.

† For a very complete investigation of the proper forms of catenaries for suspension bridges, with remarks on the Menai Bridge, and on the failure of the suspension bridge at Broughton, see Mr. Eaton Hodgkinson's paper in the Memoirs of the Manchester Society, vol. v., New Series.

Table of Relations of Catenarian Curves, the Parameter being denoted by 1.

Angle of suspension.	Deflexion, or D C.	Semi-span, or D B.	Length of chain, or C *b* D.	Tension at the point of suspension.	Semi-span divided by the deflexion, or $\frac{DB}{DC}$.
1° 0′	·00015	·01745	·01745	1·0001	114·586
2 0	·00061	·03491	·03492	1·0006	57·279
3 0	·00137	·05238	·05241	1·0014	38·171
4 0	·00244	·06987	·06993	1·0024	28·613
5 0	·00382	·08738	·08749	1·0038	22·874
6 0	·00551	·10491	·10510	1·0055	19·046
7 0	·00751	·12248	·12278	1·0075	16·309
8 0	·00983	·14008	·14054	1·0098	14·254
9 0	·01247	·15773	·15838	1·0125	12·654
10 0	·01543	·17542	·17633	1·0154	11·372
11 0	·01872	·19318	·19438	1·0187	10·320
12 0	·02234	·21099	·21256	1·0223	9·444
13 0	·02630	·22887	·23087	1·0263	8·701
14 0	·03061	·24681	·24933	1·0306	8·062
15 0	·03528	·26484	·26795	1·0353	7·508
16 0	·04030	·28296	·28675	1·0403	7·021
17 0	·04569	·30116	·30573	1·0457	6·591
18 0	·05146	·31946	·32492	1·0515	6·208
19 0	·05762	·33786	·34433	1·0576	5·863
20 0	·06418	·35637	·36397	1·0642	5·553
21 0	·07114	·37502	·38386	1·0711	5·271
22 0	·07853	·39376	·40403	1·0786	5·014
23 0	·08636	·41267	·42447	1·0864	4·778
24 0	·09484	·43169	·44523	1·0946	4·562
25 0	·10338	·45087	·46631	1·1034	4·361
26 0	·11260	·47021	·48773	1·1126	4·176
28 0	·13257	·50940	·53171	1·1326	3·843
30 0	·15470	·54930	·57735	1·1547	3·551
32 4	·18004	·59120	·62649	1·1800	3·284
34 16	·21003	·63710	·68130	1·2100	3·034
36 52	·24995	·69320	·74991	1·2499	2·773
39 11	·29011	·74430	·81510	1·2901	2·567
41 44	·34004	·80290	·89201	1·3400	2·362
44 0	·39016	·85660	·96569	1·3902	2·196
46 1	·43999	·90660	1·03610	1·4400	2·060
48 11	·49981	·96230	1·11780	1·4998	1·925
50 8	·56005	1·01420	1·19740	1·5800	1·811
52 9	·62978	1·07060	1·28690	1·6297	1·699
54 13	·71021	1·13040	1·38740	1·7102	1·592
56 28	·81021	1·19950	1·50890	1·8102	1·481
58 3	·88972	1·25100	1·60340	1·8897	1·416
60 0	1·00000	1·31690	1·73210	2·0000	1·317
64 6	1·28940	1·47020	2·05940	2·2894	1·140
67 28	1·60950	1·61350	2·41020	2·6095	1·002
67 32	1·61680	1·61640	2·41820	2·6168	0·9998

Here D B ÷ D C = 280 ÷ 25·875 = 10·82, a number which is to be found in the table.

Opposite to that number, we find 11° for the angle of suspension, D B = ·19318, C B = ·19438, tension at A or B = 1·0187, the constant tension at the vertex being 1. (Fig. 161).

Consequently, ·19318 : ·19438 :: 560 : 563·48 length of the chain.

Also, horizon, pressure at C being taken as 1·0000
the oblique pressure at A or B will = 1·0187
and the entire weight of chain will = ·39876

Ex. 2. Suppose that while the span remains 560, the deflexion is increased to 51.

Here D B ÷ D C = 280 ÷ 51 = 5·49. This number is not to be found exactly in the table. The nearest is 5·553 in the last column, agreeing with 20°, the angle of suspension.

Now, 5·55 — 5·49 = ·06, and 5·55 — 5·27 = ·28, the former difference being nearly one-fifth of the latter. Hence, adding to each number, in the line agreeing with 20°, one-fifth of the difference between that and the corresponding number in the next line, we shall have

Angle of suspension = 20° 12′, D C = ·06556, D B = ·36010, C D = ·36797, tension at A = 1·10656.

Hence ·36010 : ·36797 :: 560 : 572·24, length of chain.

Also, horizontal pressure at C being taken as 1·0000
the oblique pressure at A or B will = 1·10656
and the entire weight of chain will = ·73594

Comparing this with the former case, it will be seen that the tensions at C and A, in reference to the weight of the chain, are diminished nearly in the inverse ratio of the two values of D C; thus confirming the remark in art. 20.

In making use of this table, the remark at page 180 must be borne in mind, since the results obtained from it, will be only approximations more or less correct, as the weight of the chain exceeds that of the roadway and load. Where greater accuracy is required, the formula given at page 208 must be employed in its stead.

PART II.

MIXED MATHEMATICS.

CHAP. I.

MECHANICS IN GENERAL.

1. *Mechanics* is the science of equilibrium and of motion.

2. Every cause which tends to move a body, or to stop it when in motion, or to change the direction of its motion, is called a *force* or *power*.

3. The *direction* of a force is that straight line in which the point to which the force is applied tends to move by virtue of that force.

4. When the forces that act upon a body, destroy or annihilate each other's operation, so that the body remains quiescent, they are said to be in *equilibrium*, and are then called *pressures*.

5. The *direction* of a pressure is the straight line in which that pressure tends to prevent the motion of the point to which it is applied.

6. The effect of any force or pressure is found to be the same, at whatever point in the line of its direction it be applied.

7. Two forces or pressures are said to be *equal*, when, being applied in *opposite* directions to the same point, no motion ensues. If, however, both forces are supposed to act in the *same* direction, the single force or pressure which would be required in the opposite direction to keep the point in equilibrium, is said to be *double* either of the former forces. And one of the former forces being taken as the unit, the latter force would be represented by two of such units.

8. When two forces can be expressed in terms of the same unit, they are said to be *commensurable*.

9. It is usual to represent forces and pressures by lines, the *direction* of the line coinciding with the actual *direction* of the force, and (a line of a certain length being taken as the representative of a unit of force,) the *length* of the line expressing the *amount* or *magnitude* of the given force or pressure; and a force thus denoted is said to be *represented both in magnitude and direction*.

10. When several forces or pressures act in different directions upon the same point, it is possible to find the direction and magnitude of another force or pressure which would replace all the others; that

is, if the others were removed, and this one force applied in their stead, the effect produced would be the same. Such a force is called the *resultant* of the others, which are named the *components*, and the operation which we have supposed is called the *composition* of forces, and its converse, or finding the directions and magnitudes of any number of forces which would produce the same effect as any one given force, is called the *resolution* of forces.

11. The *moment* of a force or pressure about any point, is the perpendicular distance from that point to the line of direction in which the force acts, multiplied by the number of units expressing that force.

12. *Vis inertia*, or power of inactivity, is defined by Newton to be a power implanted in all matter, by which it resists any change attempted to be made in its state, that is, by which it requires force to alter its state, either of rest or motion.

13. *Vis viva*, or living force, a term used by Leibnitz to denote the force or power of a body in motion; or the force which would be required to bring it to a state of rest.

14. Mechanics is usually divided into five branches; viz. —

I. *Statics*, which relates to the *equilibrium* of pressures applied to solid bodies, and of the weight and pressure of bodies when at rest.

II. *Dynamics*, which relates to the motion produced in solid bodies by the application of force.

III. *Hydrostatics*, which relates to the equilibrium and pressure of *non-elastic* fluids, and of the weight, pressure, and stability of bodies immersed in them.

IV. *Hydrodynamics*, which relates to the motion of *non-elastic* fluids by the application of force.

V. *Pneumatics*, which relates to the various circumstances attending the equilibrium or motion of *elastic* fluids.

CHAP. II.

STATICS.

SECT. I. *Statical Equilibrium.*

1. IF any *two* pressures applied to a point keep it in equilibrium, they must be equal to each other, and must act in the same straight line and in contrary directions.

2. If any *three* pressures applied to a point keep it in equilibrium, they must all act in the same plane; and any one of those pressures is represented in magnitude and direction by the diagonal of a *parallelogram*, whose sides represent the other two pressures in magnitude and direction.

3. If any *four* pressures whose directions are not in the same plane, applied to a point, keep it in equilibrium, any one of those pressures is represented in magnitude and direction by the diagonal of a *parallelopipedon*, whose contiguous edges represent the other three pressures in magnitude and direction.

4. If any number of pressures whose directions are all in the same plane, applied to a point, keep it in equilibrium, those pressures will be represented in magnitude and direction by a *polygon* whose sides are made parallel to the direction of those pressures, and proportional to their magnitude.

5. If any number of pressures in the same plane be in equilibrium, and *any* point be taken in that plane from which their moments are measured, then the sum of the moments of those pressures which tend to turn the plane in one direction about that point, will be equal to the sum of the moments of those which tend to turn it in the opposite direction.

6. If any number of pressures acting in the same plane, and which are in equilibrium about a given point, be moved parallel to the directions in which they act, until they all intersect in any given point, they will still be in equilibrium about that point.

7. The resultant of two parallel pressures, if acting in the *same* direction, is equal to their *sum;* but, if acting in *contrary* directions, is equal to their *difference*, the direction of their resultant being in the direction of the greater pressure. And generally the resultant of any number of parallel pressures is equal to the several pressures added together with their proper signs.

From the foregoing propositions we may deduce the following formulæ:—

8. Let P_1 and P_2 represent any two pressures, and β the angle formed by their two lines of direction; let R be their resultant; γ the angle which its line of direction makes with that of P_1, and δ that which it makes with P_2; then

$$R = \sqrt{P_1^2 + P_2^2 + 2 P_1 P_2 \cos \beta} \quad . \quad . \quad . \quad \text{(I.)}$$

$$\tan \gamma = \frac{P_2 \sin \beta}{P_1 + P_2 \cos \beta} \quad . \quad . \quad . \quad . \quad . \quad \text{(II.)}$$

$$P_1 : P_2 :: \sin \delta : \sin \gamma \quad . \quad . \quad . \quad . \quad . \quad \text{(III.)}$$

or, three forces being in equilibrium, any two of them are to each other inversely as the sines of their inclinations to the third.

9. To determine the resultant of any number of pressures in the same plane:—Let P_1, P_2, P_3, &c. represent the pressures in magnitude, and α_1, α_2, α_3, the inclinations which their lines of direction make with some line given in position; let R be their resultant, and ι its inclination to the same line; then

$$R \cos \iota = P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + \ldots + P_n \cos \alpha_n \quad \text{(IV.)}$$

$$R \sin \iota = P_1 \sin \alpha_1 + P_2 \sin \alpha_2 + \ldots + P_n \sin \alpha_n \quad \text{(V.)}$$

in which the several terms are taken positive or negative according to the direction in which they act. From these we obtain

$$R = \{ \sqrt{(P_1 \sin \alpha_1 + P_2 \sin \alpha_2 + \ldots + P_n \sin \alpha_n)^2 + (P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + \ldots + P_n \cos \alpha_n)^2} \} \quad \text{(VI.)}$$

$$\tan \iota = \frac{P_1 \sin \alpha_1 + P_2 \sin \alpha_2 + \ldots + P_n \sin \alpha_n}{P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + \ldots + P_n \cos \alpha_n} \quad \text{(VII.)}$$

10. Let P_1, P_2, P_3, &c., be any number of parallel pressures, (being + or − according to their direction,) and D_1, D_2, D_3, &c., be their perpendicular distance from any given parallel plane: let R be their resultant, and Δ its perpendicular distance from the same plane: then

$$R = P_1 + P_2 + \ldots + P_n \quad . \quad . \quad . \quad . \quad \text{(VIII.)}$$

$$\Delta = \frac{P_1 D_1 + P_2 D_2 + \ldots + P_n D_n}{P_1 + P_2 + \ldots + P_n} \quad \text{(IX.)}$$

As these formulæ and propositions are of universal application in the constructions of civil engineers, architects, and mechanicians, we shall give a few simple examples, to render their use clear to those who are not familiar with mathematical formulæ.

Ex. 1. Suppose that a weight B is attached by a stirrup to the foot of a king-post A B, which is attached to two rafters A C, A D, in the respective positions shown in fig. 164. Then if A E be set off upon

A B, equal in numerical value to the vertical strain upon A B, and the parallelogram A F E G be completed, A F measured upon the same scale will show the strain upon the rafter A C, and A G the strain upon A D.

Ex. 2. Let it be proposed to compare the strains upon the tie-beams A D, and the struts A C, when they sustain equal weights B, in the two different positions indicated by figs. 165 and 166. Let A E in one figure be equal to the corresponding vertical line A E in the other, and in each represent the numerical value of the weight B, that hangs from A. Through E in both figures, draw lines parallel to D A, A C, respectively, and let them meet A C, and D A produced in F and G: then A F E G in each figure is the *parallelogram of forces* by which the several strains are to be measured. A G represents the tension upon the tie-beam A D, and A F the strain upon the strut A C. Both these lines are evidently shorter in the lower figure than they are in the upper, A E being of the same length in both: therefore the first figure exhibits the least advantageous position of the beams. It is evident also, that while C A tends upwards and D A downwards, the greater the angle D A C, the less is its supplement C A G, and the less the sides F A, A G, of the parallelogram.

Ex. 3. Let it be required to determine geometrically, and by computation, the magnitude and direction of the four pressures P_1, P_2, P_3 and P_4, (fig. 167,) all applied to the point A, and acting in the same plane; let $P_1 = 12$, $P_2 = 9$, $P_3 = 16$, $P_4 = 15$, and the angles which their directions make with the line B A C, (given in position,) be 77°, 37°, 9° and 48°, respectively.

1. *Geometrically.*—From the point A (fig. 168) lay off the angle C A D equal 77°, and upon the line A D set off 12, from any scale of equal parts; then through D draw the line D E, equal to 9 parts by the same scale, and making an angle of 37° with B A C; again, through E draw the line E F equal to 16, and inclined 9° to B A C; and lastly, through F draw E G, equal to 15, and inclined 48° to B A C; then the straight line A G, connecting the first and last points, and completing the polygon A D E F G A, will be the resultant of the four given pressures; and being measured by the same scale, will be found equal to 36·72 parts nearly, and to be inclined 13° to B A C.

2. By *Computation.*—As the pressures P_1, P_2, and P_3, are not on the same side of the line B A C (fig. 167) as P_4, the signs of their *sines* must be taken differently; but as they are all on the same side of the line H A I, (perpendicular to B A C,) the signs of their *cosines* must be taken alike; then if R is their resultant, and ι its inclination to B A C, we obtain from formula (VI.)—

$$R = \sqrt{\{(12 \times \cdot9744 + 9 \times \cdot6018 + 16 \times \cdot1564 - 15 \times \cdot7431)^2 + (12 \times \cdot225 + 9 \times \cdot7986 + 16 \times \cdot9877 + 15 \times \cdot6691)^2\}} = 36\cdot7138;$$

and from formula (VII.)—

$$\tan \iota = \frac{12 \times \cdot9744 + 9 \times \cdot6018 + 16 \times \cdot1564 - 15 \times \cdot7431}{12 \times \cdot225 + 9 \times \cdot7986 + 16 \times \cdot9877 + 13 \times \cdot6691} = \cdot23105$$

$$\therefore \iota = 13°.$$

In which the values are the same as were found geometrically.

SECT. II. *Centre of Gravity.*

1. *Gravity* is the force in virtue of which bodies left to themselves fall to the earth in directions perpendicular to its surface.

2. We may distinguish between the effect of *gravity* and that of *weight*, by observing that the former is the power of transmitting to every particle of matter a certain velocity which is absolutely independent of the number of material particles; while the latter is the effort which must be exercised to prevent a given mass from obeying the law of gravity. *Weight*, therefore, *depends upon the mass;* but *gravity* has no dependence at all upon it.

3. The *centre of gravity* of any body, or system of bodies, is that point about which the body or system, acted upon only by the force of gravity, will balance itself in all positions: or it is a point which, when supported, the body or system will be supported, however it may be situated in other respects.

The centre of gravity of a body is not always within the body itself; thus the centre of gravity of a ring is not in the substance of the ring, but in the axis of its circumscribing cylinder; and the centre of gravity of a hollow staff, or of a bone, is not in the matter of which it is constituted, but somewhere in its imaginary axis; every body, however, has a centre of gravity, and so has every system of bodies.

4. Varying the position of the body will not cause any change in the relative position of the centre of gravity; since any such mutation will be nothing more than changing the directions of the forces, without their ceasing to be parallel; and although the amount of the forces may not continue the same, in consequence of the body being supposed at different distances from the earth, still the forces upon all the moleculæ vary proportionally, and the position of their centre remains unchanged.

5. When a heavy body is suspended by any other point than its centre of gravity, it will not rest unless that centre is in the same vertical line with the point of suspension; for in all other positions the force which is intended to ensure the equilibrium will not be directly opposite to the resultant of the parallel forces of gravity upon the several particles of the body, and of course the equilibrium will not be obtained. (See Art. 9, on *Pendulums*, page 222.)

6. If a heavy body be sustained by two or more forces, their directions must meet, either at the centre of gravity of that body, or in the vertical line which passes through it.

7. When a body stands upon a plane, if a vertical line passing through the centre of gravity fall within the base on which the body stands, it will not fall over; but if that vertical line passes without the base, the body will fall, unless it be prevented by external means. When the vertical line falls upon the extremity of the base, the body *may* stand, but its state (which is called *unstable equilibrium*) may be disturbed by a very trifling force; and the nearer this line passes to any edge of the base the more easily may the body be thrown

over; the nearer it falls to the middle of the base, the more firmly the body stands.

Upon this principle it is that *leaning towers* have been built at Pisa, and various other places; the vertical line of direction from the centre of gravity falling within the base. And, from the same principle it may be seen that a waggon loaded with heavy materials, as B, may stand with perfect safety, on the side of a convex road, the vertical line from the centre of gravity falling between the wheels; while a waggon A with a high load, as of hay, or of wool-packs, shall fall over, because the vertical line of direction falls without the wheels.

8. Owing to the great distance of the earth's centre from its surface, the directions of the force of gravity of the several moleculæ composing a body may be considered parallel without any appreciable error, and therefore all that has been said in the foregoing section on the subject of parallel pressures will apply to the force of gravity; and the formula for finding the position of the resultant of any number of parallel pressures, may be applied to determine the position of the centre of gravity.

Thus, if B_1, B_2, B_3, &c., denote the weights of the particles of any body, and D_1, D_2, D_3, &c., the perpendicular distances of their respective centres of gravity from any given plane; then, the distance (Δ) of their common centre of gravity from the same plane is found by formulæ (IX.) to be

$$\Delta = \frac{B_1 D_1 + B_2 D_2 + \ldots\ldots + B_n D_n}{B_1 + B_2 + \ldots\ldots + B_n}$$

Therefore, if by means of this formula we ascertain the distance of the centre of gravity from any *three* planes given in position, we shall have determined its exact situation in space.

9. The common centre of gravity of two bodies divides the right line which joins their respective centres of gravity, in the inverse ratio of their weights. The centre of gravity of any number of bodies may be found, by finding the common centre of gravity of any two of the bodies, and then considering this as the centre of gravity of a body equal to the combined weight of the two, we may find the common centre of gravity of this imaginary body and a third; and thus proceed, *ad libitum*.

10. If the particles or bodies of any system be moving uniformly and rectilineally, with any velocities and directions whatever, the centre of gravity is either at rest, or moves uniformly in a right line.

Hence, if a rotatory motion be given to a body and it be then left to move freely, the axis of rotation will pass through the centre of

gravity: for that centre, either remaining at rest or moving uniformly forward in a right line, has no rotation.

Here too it may be remarked, that *a force applied at the centre of gravity of a body, cannot produce a rotatory motion.*

11. To find the centre of gravity mechanically, it is only requisite to dispose the body successively, in two positions of equilibrium, by the aid of two forces in vertical directions, applied in succession to two different points of the body; the point of intersection of these two directions will show the centre.

This may be exemplified by particularising a few methods. If the body have plane sides, as a piece of board, hang it up by any point (A fig. 169), then a plumb-line suspended from the same point will pass through the centre of gravity; therefore mark that line (A B) upon it: and after suspending the body by another point (C), apply the plummet to find another such line (C D); then will their intersection show the centre of gravity.

Or thus: hang the body from a tack successively by two strings attached to different parts of it, and each time mark upon it the line of a plummet attached to the same tack; then will the intersection of the two lines be the centre of gravity.

Another method: Lay the body on the edge of a triangular prism, or such like, moving it to and fro till the parts on both sides are in equilibrio, and mark a line upon it close by the edge of the prism: balance it again in another position, and mark the fresh line by the edge of the prism; the vertical line passing through the intersection of these lines, will likewise pass through the centre of gravity. The same thing may be effected by laying the body on a table, till it is just ready to fall off, and then marking a line upon it by the edge of the table: this done in two positions of the body will in like manner point out the position of the centre of gravity.

When it is proposed to find the centre of gravity of the arch of a bridge, or any other structure, let it be laid down accurately to scale upon pasteboard; and the figure being carefully cut out, its centre of gravity may be ascertained by the preceding process.

12. The centre of gravity of a right line, or of a parallelogram, prism, or cylinder, is in its middle point; as is also that of a circle, or of its circumference, or of a sphere, or of a regular polygon; the centre of gravity of an ellipse, a parabola, a cone, a conoid, a spheroid, &c., is situated in its axis. And the same of all symmetrical figures.

13. The centre of gravity of a triangle is the point of intersection of lines drawn from the three angles to the middles of the sides respectively opposite: it divides each of those lines into two portions in the ratio of 2 to 1.

14. *In a Trapesium.* Divide the figure into two triangles by the diagonal A C (fig. 170) and find the centres of gravity E and F of these triangles; join E F, and find the common centre G of these two by this proportion, A B C : A D C :: F G : E G, or A B C D : A D C :: E F : E G. Or, divide the figure into two triangles by a diagonal

BD; then find their centres of gravity; and the line which joins them will intersect EF in G, the centre of gravity of the trapezium.

15. In like manner, for any other plane figure, whatever be the number of sides, divide it into several triangles, and find the centre of gravity of each; then connect two centres together, and find their common centre as above; then connect this and the centre of a third, and find the common centre of these; and so on, always connecting the last found common centre to another centre, till the whole are included in this process; so shall the last common centre be that which is required.

16. The centre of gravity of a circular arc is distant from the centre of the arc a fourth proportional to the arc, the radius, and the chord of the arc.

17. In a circular sector, the distance of the centre of gravity from the centre of the circle is $\frac{2\,c\,r}{3\,a}$; where a denotes the arc, c its chord, and r the radius.

18. The centres of gravity of the *surfaces* of a cylinder, of a cone, and of a conic frustum, are respectively at the same distances from the origin as are the centres of gravity, of the parallelogram, triangle, and trapezoid, which are vertical sections of the respective solids.

19. In the segment of a sphere, or spheroid, A being the whole axis, and h the height of the segment, the distance of the centre of gravity from the vertex is equal $\frac{4\,A - 3\,h}{6\,A - 4\,h}$; and when the height becomes half A, or the segment becomes a hemi-sphere, the distance from the vertex is equal to $\frac{3}{8}\,h$.

20. The centre of gravity of the *surface* of a spheric segment is at the middle of its versed sine or height.

21. The centre of gravity of the *convex surface* of a spherical zone, is in the middle of that portion of the axis of the sphere which is intercepted by the two bases of the zone.

22. In a cone, as well as any other pyramid, the distance from the vertex is $\frac{3}{4}$ of the axis.

23. In a conic frustum, the distance on the axis from the centre of the lesser end, is $\frac{1}{4}\,h.\ \frac{3\,R^2 + 2\,R\,r + r^2}{R^2 + R\,r + r^2}$: where h equals the height, and R r the radii of the greater and lesser ends.

24. The same theorem will serve for the frustum of any regular pyramid, taking R and r for the sides of the two ends.

25. In the paraboloid, the distance from the vertex is $\frac{2}{3}$ of the axis.

26. In the frustum of the paraboloid, the distance on the axis from the centre of the lesser end, is $\frac{1}{3}\,h.\ \frac{2\,R^2 + r^2}{R^2 + R^2}$: where h equals the height, and R r the radii of the greater and lesser ends.

27. Every figure, whether superficial or solid, generated by the motion of a line or surface, is equal to the product of the generating magnitude multiplied by the path of its centre of gravity.

As an example, let A B C (fig. 171) be a right angled triangle, the revolution of which on the leg A B will produce a cone; let A B = 9, B C = 6, and D be the centre of gravity; then by § 13, A D = $\frac{2}{3}$ A F, therefore D E = $\frac{2}{3}$ F B = 2 equal the radius of the circular path described by the centre of gravity, the circumference of which will therefore be 4 × 3·1416 = 12·5664, which multiplied by (3 × 9) = 27, the area of the triangle, will be 339·2928, the solid content of the cone; and by the rule given at page 130 for finding the solidity of a cone, we obtain 12^2 × ·7854 × 3 = 339·2928, the same result.

SECT. III. *General application of the principles of Statics to the equilibrium of Structures.*

Every structure is exposed to the operation of a system of forces; so that the examination of its stability involves the application of the general conditions of equilibrium.

Now, no part of a structure can be dislocated, except it be either by a progressive, or a rotatory motion. For either the part is displaced, *without changing its form*, in which case it is (as a system of invariable form) incapable of receiving any instantaneous motion, which is not either progressive or rotatory; or else it happens to be displaced, *changing at the same time its form*, which, considering the cohesion of its parts, cannot take place without its breaking in its weakest section; in which case a progressive motion is generated if the force acts *perpendicularly* to the section, and a rotatory motion, if it acts *obliquely*.

We shall here consider the most useful cases; indicating by the word *stress*, that force which tends to give motion to the structure, and by *resistance*, that which tends to hinder it.

EQUILIBRIUM OF PIERS, OR ABUTMENTS.

1. If we suppose figure 172 to be the vertical section of a pier, we may reason upon that section instead of the pier itself, if it be of uniform structure.

Let G be the place of the centre of gravity of the section A B C D, S Z the direction in which the stress acts, meeting X I, the vertical line through the centre of gravity, in I. Then, supposing the stress to be resolved into two forces, one (P_1) vertical, the other (P_2) horizontal; the pier (regarding it as one body) can only give way either by a progressive motion from B towards A, or by a rotatory motion about A.

2. The progressive motion is resisted by friction. If W denote the weight of the pier, P_1 the stress estimated vertically, and P_2 its horizontal effort, then the pressure on the base = W + P_1, and

putting f for the coefficient of friction, its friction $= f(W + P_1)$, which is the amount of the resistance to progressive motion. So that to ensure stability in this respect we must have

$$f(W + P_1) > P_2 \ldots\ldots\ldots\ldots\ldots\ldots \text{(I.)}$$

3. To ensure stability in regard to rotation, taking the moments of the forces about the point A, we must have

$$W \cdot \overline{AX} + P_1 \cdot \overline{AE} > P_2 \cdot \overline{AQ} \ldots\ldots \text{(II.)}$$

Or, supposing the stress not to be resolved, its moment about the point A must be less than that of the weight of the pier about the same point; that is, putting S for the whole stress,

$$W \cdot \overline{AX} > S \cdot \overline{AZ} \ldots\ldots\ldots\ldots\ldots \text{(III.)}$$

Or, by a graphical process, suppose the two forces W and S to be applied at I, and complete the parallelogram, having sides which represent these forces. Then must the diagonal representing the resultant of those forces produced meet the base on the side of A, towards B, to ensure stability.

4. If, as is very frequently the case, the vertical section of the pier is a rectangle, putting h for the height of the pier, b for its base, w for the weight of a cube unit of the material of which it is formed, and R a horizontal stress applied at the summit of the pier; then the pier will be in a state of *unstable* equilibrium, as far as regards its progressive motion, when

$$bfhw = R \ldots\ldots\ldots\ldots\ldots\ldots \text{(IV.)}$$

and as regards its being overthrown when

$$\frac{b^2 w}{2} = R \ldots\ldots\ldots\ldots\ldots\ldots \text{(V.)}$$

But in order that the structure may really be secure, these equations must become

$$bfhw > R, \text{ and } \frac{b^2 w}{2} > R,$$

and the more the first member exceeds the second the greater will be the stability of the pier.

Example 1. Suppose a rectangular wall 39·4 feet high, and of a material weighing 125 lbs. per cube foot, is to sustain a horizontal strain of 9900 lbs. avoirdupois on each foot in length, applied at its summit: what must be its thickness that there may be an equilibrium, taking $f = 1{\cdot}75$.

By transposing formula (IV.) above, we obtain

$$b = \frac{R}{fhw},$$

and substituting the several values given above, we have

$$b = \frac{9900}{{\cdot}75 \times 39{\cdot}4 \times 125} = 2{\cdot}68 \text{ feet};$$

therefore the thickness of the pier at its base must be more than 2·68 feet, otherwise it will be moved horizontally by the stress R. But we must also inquire, what thickness is requisite to prevent the pier being overthrown; and this we shall ascertain from formula (V.), which by transposition becomes

$$b = \sqrt{\frac{2\,R}{w}},$$

$$\text{and } b = \sqrt{\frac{2 \times 9900}{125}} = 12{\cdot}58 \text{ feet.}$$

Therefore, we see, that the pier must be more than 12·58 feet upon its base, otherwise it will be overthrown by the horizontal strain at R.

Here, as the thickness required to prevent overturning is much the greatest, the computation in reference to the other kind of equilibrium may usually be avoided.

Example 2. An embankment, or dam, A B C D (fig. 173), formed of clay between two rubble walls, is exposed to the pressure of 30 feet of water; it is required to ascertain whether the dam will be overthrown, and also whether it will slip upon its bed. Its dimensions are, D C = 4 feet, B C which is vertical = 32 feet, and A B = 12 feet; its weight may be taken at 135 lbs. per cubic foot, and the coefficient of friction or $f = {\cdot}5$.

First, by the method laid down in § 14, page 194, we ascertain the distance of the centre of gravity of the wall from its interior face B C, to be 4·333; therefore the distance (A X) of its line of direction from A will be 7·667 feet. The weight of the wall (W) will be $\frac{12+4}{2} \times 32 \times 135 = 34560$ lbs.; the pressure of the water (P_2 or S) will be $\frac{30}{2} \times 62{\cdot}5 = 937{\cdot}5$, and the centre of pressure of the water being at two-thirds of its depth (as explained at page 250), the distance (A Z) of its line of direction above the point A will be 10 feet.

Now, in order to ascertain its stability to resist progressive motion, we must substitute these values in formula (I.), when we obtain

$$\cdot 5 \times 34560 > 937{\cdot}5, \text{ or}$$
$$17280 > 937{\cdot}5;$$

therefore there would be no fear of the embankment slipping upon its base. And from formula (III.) we have,

$$34560 \times 7{\cdot}667 > 937{\cdot}5 \times 10, \text{ or}$$
$$264972 > 9375;$$

therefore the wall cannot be overthrown.

PRESSURE OF EARTH AGAINST WALLS.

1. Let D A C B (fig. 174) be the vertical section of a wall behind which is posited a bank or terrace of earth, of which a prism whose

section is represented by C B H would detach itself and fall down, were it not prevented by the wall. Then B H is denominated the *line of rupture* or the *natural slope*, or *natural declivity*. In sandy or loose earth, the angle C H B seldom exceeds 30°; in stronger earth it becomes 37°; and in some favourable cases more than 45°.

2. Now, the prism whose vertical section is C B H, has a tendency to descend along the inclined plane B H by reason of the force of gravity; but it is retained in its place, not only by the force opposed to it by the wall, but also by its cohesive attachment to the face B H, and by its friction upon the same surface.

If we resolve the weight of this prism into two forces, one acting perpendicular to the plane B H, and the other parallel to the same, the latter minus the force required to overcome either the cohesion or the friction of the surface B H, will be the strain acting in a direction passing through the centre of gravity of the prism C B H and parallel to B H, upon the back of the wall, and tending to overthrow it; and the amount of this strain, and its direction, being ascertained, the formulæ already given may be employed to determine the stability of the wall,

$$\text{I H} = \text{L H} + \text{cohesion} + \text{friction}.$$

3. It is evident, therefore, that as the angle at which the earth will stand is one of the elements in the calculation, the solution to this inquiry must be, in a great measure, experimental. It has been found, however, theoretically, by M. Prony*, and confirmed experimentally, that the angle formed with the vertical (C B H) by the prism of earth that exerts the greatest horizontal stress against a wall, is *half* the angle which the natural slope of the earth makes with the vertical: and this curious result greatly simplifies the whole inquiry.

Putting h for the height of the wall, β for the angle C B F, or half C B H, the natural slope of the ground, w_1 for the weight of a cubit unit of the ground, and f for the coefficient of friction we have

$$\frac{h^2 w_1 \cdot \tan \beta}{2} =$$

the weight of the prism C B F; and resolving this into two forces, one (P_3) perpendicular, the other (P_4) parallel to B H, we have for the former

$$\frac{h^2 w_1 \cdot \tan \beta \cdot \sin \beta}{2} = P_3,$$

and for the latter

$$\frac{h^2 w_1 \cdot \sin \beta}{2} = P_4.$$

* See a demonstration at p. 369, vol. ii. tenth edition of Dr. Hutton's Course of Mathematics.

Therefore the friction along the surface B H, will be

$$\frac{f h^2 w_1 . \tan\beta . \sin\beta}{2}, \quad \ldots\ldots\ldots\ldots \text{(VI.)}$$

which subtracted from P_4, gives

$$(1 - f . \tan\beta) \frac{h^2 w_1 . \sin\beta}{2} = S \ldots\ldots\ldots \text{(VII.)}$$

equal the strain acting through the centre of gravity of the prism upon the back of the wall; now, as the centre of gravity of a triangle is situate at a third of its height, and the direction of the strain S is parallel to the sloping base B H, it will meet the wall at one-third of its height, and b being put for the breadth of the base of the wall, we shall have

$$S\left(\frac{h}{3} . \sin\beta - b . \cos\beta\right) \ldots\ldots\ldots \text{(VIII.)}$$

equal the moment of the strain S about the point A, tending to overthrow the wall.

Example. Suppose a wall is to be built of brickwork (weighing 117 lbs. per cubic foot), to support a terrace 39 feet in height, the earth composing which weighs 105 lbs. per cubic foot, its natural slope being 53° from the vertical, and the coefficient of friction ·45; it is required to find the breadth which must be given to the wall at its base in order to ensure its not being overthrown.

By substituting these values in formula (VIII.), we obtain

$$S = (1 - \cdot 45 \times \cdot 4986) \frac{39^2 \times 105 \times \cdot 4462}{2} = 27636;$$

the moment of which from formula (VIII.) is

$$27636\left(\frac{39}{3} \times \cdot 4462 - b \times \cdot 8949\right) = 160322 - 24731 . b.$$

Therefore by formula (III.), we have

$$\frac{39\, b^2 \times 117}{2} > 160322 - 24731\, b, \text{ or}$$

$$b > 4{\cdot}58;$$

that is, the base of the wall must be more than 4·58 feet, in order that it shall not be overthrown by the pressure of the earth.

4. Of the experimental results, the best which we have seen are those of M. Mayniel, from which the following are selected; in all of which the upper surface of the earth and of the wall which supports it are supposed to be both in one horizontal plane.

1st. Both theory and experiment indicate that the resultant of the thrust of a bank, behind a vertical wall, is at a distance B K from the bottom of the wall equal to one-third of its height.

2dly. That the friction is *half* the pressure, in vegetable earths, and *four-tenths* in sand; or that $f = \cdot 5$ in one case and ·4 in the other.

ly. The cohesion which vegetable earths acquire, when cut in and well laid, course by course, diminishes their thrust by full *hirds;* or in this case $f = \cdot 667$.

ıe following table contains the value of the angle C B F, (being the angle formed by the natural slope and the vertical,) for al different kinds of earth, the authorities being given in the last nn.

Nature of Earth.	Angle C B F = β		Authority.
	°	′	
Rubble	22	30	Mayniel.
Loose shingle perfectly dry . . .	25	30	Pasley.
Earth the most dense and compact . .	17	30	Barlow.
Common earth slightly damp . . .	18	0	Rondelet.
Idem pulverized and dry . .	21	35	Id.
Vegetable earth	31	43	Mayniel.
Idem mixed with large gravel .	31	43	Id.
Idem mixed with small gravel .	35	52	Id.
Sand	34	6	Id.
Fine dry sand	25	30	Barlow.
Idem	27	45	Rondelet.
Idem (a single experiment) .	34	30	Gadroy.

THICKNESS OF WALLS, BOTH FACES VERTICAL.

The following table exhibits the thickness which ought to be ı to a parallel wall of various materials, and supporting the ure of different kinds of earth.

Nature of Earth.	Thickness of the wall or DC, its height being 1.		
	Wall of brick, 109 lbs. per cub. ft.	Wall of unhewn stone, 135 lbs. per cub. ft.	Wall of hewn freestone.
table earth, carefully laid course by course	·16	·15	·13
, well rammed	·17	·16	·14
h mixed with large gravel . . .	·19	·17	·16
. . . . .	·33	·30	·26

For walls with an interior slope, or a slope towards the bank, ıe base of the slope be $\frac{1}{n}$ of the height, and let S and s be the fic gravities of the wall and of the earth; then

$$DC = h\sqrt{\frac{1}{3n^2} + m\frac{s}{S}} - \frac{h}{n};$$

e $m = \cdot 0424$, for vegetable or clayey earth, mixed with large

gravel; ·0464, if the earth be mixed with small gravel; ·1528, for sand; and ·166, for semi-fluid earths.

Example. Suppose the height of a wall to be 20 feet, and $\frac{1}{20}$ of the height for the base of the *talus* or slope; suppose, also, the specific gravities of the wall and of the bank to be 2600, and 1400, and the earth semi-fluid; what, then, must be the thickness of the wall at the crown?

Here the theorem will become,

$$DC = 20\sqrt{\frac{1}{1200} + \cdot 166 \cdot \tfrac{14}{26} - \tfrac{9}{20}}$$

$$= 20\sqrt{\cdot 0008333 + \cdot 0894} - 1 = (20 \times \cdot 3) - 1$$

$$= 6 - 1 = 5 \text{ feet:}$$

while the thickness of the wall at *bottom* will be 6 feet.

EQUILIBRIUM OF POLYGONS.

1. Let there be any number of lines, bars, or beams, AB, BC, CD, DE, &c. (fig. 175), all in the same vertical plane, connected together and freely moveable about the joints or angles, A, B, C, D, E, &c., and kept in equilibrio by weights laid on the angles: it is required to assign the proportion of those weights; as also the force or push in the direction of the said lines; and the horizontal thrust at every angle.

Through any point, as D, draw a vertical line a D H g; to which, from any point, as C, draw lines in the direction of, or parallel to, the given lines or beams, viz., C a parallel to AB, C b parallel to BC, C e to DE, C f to EF, C g to FG, &c.; also CH parallel to the horizon, or perpendicular to the vertical line a d g, in which also all these parallels terminate.

Then will all these lines be exactly proportional to the forces acting or exerted in the directions to which they are parallel, whether vertical, horizontal, or oblique. That is, the oblique forces or thrusts in the direction of the bars ..... AB, BC, CD, DE, EF, FG, are proportional to their parallels ... C a, C b, C d, C e, C f, C g; and the vertical weights on the angles B, C, D, E, F, &c. are as the parts of the vertical ..... a b, b D, D e, e f, f g, and the weight of the whole frame ..... ABCDEFG, is proportional to the sum of all the verticals, or to ..... a g; also the horizontal thrust at every angle, is everywhere the same constant quantity, and is expressed by the constant horizontal line CH.

Corol. 1. It is worthy of remark that the lengths of the bars AB, BC, &c. do not affect or alter the proportions of any of these loads or thrusts; since all the lines C a, C b, a b, &c., remain the same, whatever be the lengths of AB, BC, &c. The positions of the bars, and the weights on the angles depending mutually on each other, as well as the horizontal and oblique thrusts. Thus, if there be given

the position of D C, and the weights or loads laid on the angles D, C, B; set these on the vertical, D H, D b, b a, then C b, C a, give the directions or positions of C B, B A, as well as the quantity or proportion of C H the constant horizontal thrust.

Corol. 2. If C H be made radius; then it is evident that H a is the tangent, and C a the secant of the elevation of C a or A B above the horizon; also H b is the tangent, and C b the secant of the elevation of C b or C B; also H D and C D the tangent and secant of the elevation of C D; also H e and C e the tangent and secant of the elevation of C e or D E; also H f and C f the tangent and secant of the elevation of E F; and so on; also the parts of the vertical ab, bD, ef, fg, denoting the weights laid on the several angles, are the differences of the said tangents of elevations. Hence then in general,

1st. The oblique thrusts, in the directions of the bars, are to one another, directly in proportion to the secants of their angles with the horizon; or, which is the same thing, reciprocally proportional to the cosines of the same angles, or reciprocally proportional to the sines of the vertical angles, a, b, D, e, f, g, &c., made by the vertical line with the several directions of the bars; because (formula K . 3, page 137) the secants of any angles are always reciprocally in proportion to their cosines.

2. The weight or load laid on each angle is directly proportional to the difference between the tangents of the elevations above the horizon, of the two lines which form the angle.

3. The horizontal thrust at every angle is the same constant quantity, and has the same proportion to the weight on the top of the uppermost bar, as radius has to the tangent of the elevation of that bar. Or, as the whole vertical ag, is to the line C H, so is the weight of the whole assemblage of bars, to the horizontal thrust.

4. It may hence be deduced also, that the weight or pressure laid on any angle, is directly proportional to the continual product of the sine of that angle and of the secants of the elevations of the bars or lines which form it.

Scholium. This proposition is very fruitful in its practical consequences, and contains the whole theory of centerings, and indeed of arches, which may be deduced from the premises by supposing the constituting bars to become very short, like arch stones, so as to form the curve of an arch. It appears too, that the horizontal thrust, which is constant or uniformly the same throughout, is a proper measuring unit, by means of which to estimate the other thrusts and pressures, as they are all determinable from it and the given positions; and the value of it, as appears above, may be easily computed from the uppermost or vertical part alone, or from the whole assemblage together, or from any part of the whole, counted from the top downwards.

In the most important cases, a model of the structure may be made, and the relations of the pressures at any angle, whether horizontal, vertical, or in the directions of the beams, may be determined by a spring steel-yard applied successively in the several directions.

2. If the whole figure in the preceding problem be inverted, or turned round the horizontal line A G (fig. 176) as an axis, till it be completely reversed, or in the same vertical plane below the first position, each angle D, *d*, &c., being in the same plumb line; and if weights *i*, *k*, *l*, *m*, *n*, which are respectively equal to the weights laid on the angles B, C, D, E, F, of the first figure, be now suspended by threads from the corresponding angles *h*, *c*, *d*, *d*, *f*, of the lower figure; those weights keep this figure in exact equilibrio, the same as the former, and all the tensions or forces in the latter case, whether vertical, horizontal, or oblique, will be exactly equal to the corresponding forces of weight, pressure, or thrust in the like directions of the first figure.

This, again, is a proposition most fertile in its application, especially to the practical mechanic, saving the labour of tedious calculations, but making the results of experiment equally accurate. It may thus be applied to the practical determination of arches for bridges, with any proposed roadway; and to that of the position of the rafters in a curb or mansard roof.

3. Thus, suppose it were required to make such a roof, with a given width A E (fig. 177), and of four proposed rafters A B, B C, C D, D E. Here, take four pieces that are equal or in the same given proportions as those proposed, and connect them closely together at the joints A, B, C, D, E, by pins or strings, so as to be freely moveable about them; then suspend the whole from two pins, A, E, fixed in the same horizontal line, and the several pieces will arrange themselves in such a form, A *b c d* E, that all its parts will come to rest in equilibrio. Then, by inverting the figure, it will exhibit the form of the framing of a curb roof A B C D E, which will also be in equilibrio, the thrusts of the pieces now balancing each other, in the same manner as was done by the mutual tensions of the hanging festoon A *b c d* E.

4. If the mansard be constituted of four equal rafters; then, if angle C A E = m, angle C A B = x; it is demonstrable that $2 \sin 2x = \sin 2m$. So that if the span A E, and height M C, be given, it will be easy to compute the lengths A B, B C, &c.

Example. Suppose A E = 24 feet, M C 12.

$$\text{Then, } \frac{\text{M C}}{\text{M A}} = 1 = \tan 45^\circ \text{ angle C A M} = m.$$

$$\therefore \sin 2m = \sin 90^\circ = 1, \text{ and } \sin 2x = \tfrac{1}{2}$$

$$\therefore 2x = 30^\circ, \text{ and } x = 15^\circ = \text{C A B}.$$

$$\text{Hence M A B} = 45^\circ + 15^\circ = 60^\circ$$

$$\text{and M B A} = \tfrac{1}{2}(180^\circ - 2 \times 15^\circ) = 90^\circ - 15^\circ = 75^\circ$$

$$\text{also A M B} = 180^\circ - (75^\circ + 60^\circ) = 45^\circ$$

and lastly, $\sin 75^\circ : \sin 45 :: \text{A M} = 12 : \text{A B} = 8{\cdot}7846$ feet.

Note.—In this example, since A M = M C, as well as A B = B C, it is evident that M B bisects the right angle A M C; yet it seemed preferable to trace the steps of a general solution.

STABILITY OF ARCHES.

1. If the effect of the force of gravity upon the ponderating matter of an arch and pier, be considered apart from the operation of the cements which unite the stones, &c., the investigation is difficult to practical men, and it furnishes results that require much skill and care in their application. But, in an arch whose component parts are united with a very powerful cement, those parts do not give way in vertical columns, but by the separation of the entire mass (including arches and piers), into three, or, at most, into four parts; and in this case the conditions of equilibrium are easily expressed and applied.

Let fF, f'F', (fig. 178) be the joints of rupture, or places at which the arch would most naturally separate, whether it yield in two pieces or in one. Let G be the centre of gravity of the semi-arch fF K k, and G′ that of the pier A B Ff. Let F I be drawn parallel to the horizon, and G H be demitted perpendicularly upon it; also let G′ D be a perpendicular passing through G′, and F E drawn from F parallel to it. Then,

2. The first case is when the arch fF F′f' tends to fall vertically in one piece, removing the sections fF, f'F′; if W be the weight of the semi-arch fF K k, and P that of the pier up to the joint fF, the equilibrium will be determined by these two equations:—viz.,

$$\varphi \cdot P = W\left(\frac{CI}{FI} - \varphi\right) \quad \text{(I.)}$$

$$P \cdot \frac{AD}{FE} = W\left(\frac{CI}{FI} - \frac{AE}{FE}\right) \quad \text{(II.)}$$

where φ is the measure of the friction, or the tangent of the angle of repose of the material, and the first equation is that of the equilibrium of the horizontal thrusts, while the second indicates the equilibrium of rotation about the exterior angle A of the pier.

3. In the second case, when each of the two semi-arches F k, kF′, tend to turn about the vertex k of the arch, removing the points F, F′, the equilibrium of horizontal translation, and of rotation, will be respectively determined by the following equations:—viz.,

$$\varphi \cdot P = W\left(\frac{FH}{kI} - \varphi\right) \quad \text{(III.)}$$

$$P \cdot \frac{AD}{FE} = W\left(\frac{FH}{kI} - \frac{AE}{FE}\right) \quad \text{(IV.)}$$

4. Hence it will be easy to examine the stability of any cemented arch, upon the hypothesis of these two propositions. Assume different points, such as F in the arch, for which let the numerical values of the equations (I.) and (II.), or (III.) and (IV.) be computed. To ensure stability, the first members of the respective equations must exceed the second; and those parts will be weakest where the excess is least.

5. The following table extracted from Prof. Moseley's "Mechanical Principles of Engineering," page 151, contains the value of the angle φ for the materials most usually employed in the construction of arches.

Nature of Materials.	Coefficient of friction.	Limiting angle $= \varphi$.	
		°	′
Soft calcareous stone, well dressed, upon the same	·74	36	30
Hard calcareous stone, ditto	·75	36	52
Common brick, ditto	·67	33	50
Oak, endwise, ditto	·63	32	13
Wrought iron, ditto	·49	26	7
Hard calcareous stone, well dressed, upon hard calcareous stone	·70	35	0
Soft, ditto	·75	36	52
Common brick, ditto	·67	33	50
Oak, endwise, ditto	·64	32	37
Wrought iron, ditto	·42	22	47
Soft calcareous stone upon soft calcareous stone, with fresh mortar of fine sand	·74	36	30
EXPERIMENTS BY DIFFERENT OBSERVERS.			
Smooth freestone upon smooth freestone, dry. (Rennie.)	·71	35	23
Ditto, with fresh mortar. (Rennie.)	·66	33	26
Hard polished calcareous stone upon hard polished calcareous stone	·58	30	7
Calcareous stone upon ditto, both surfaces being made rough with the chisel. (Bouchardi.)	·78	37	58
Well dressed granite upon rough granite. (Rennie.)	·66	33	26
Ditto, with fresh mortar, ditto. (Rennie.)	·49	26	7
Box of wood upon pavement. (Regnier.)	·58	30	7
Ditto, upon beaten earth. (Herbert.)	·33	18	16
Libage stone upon a bed of dry clay	·51	27	2
Ditto, the clay being damp and soft	·34	18	47
Ditto, the clay being equally damp, but covered with thick sand. (Grève.)	·40	21	48

If the section be drawn on smooth drawing pasteboard, upon a good sized scale, the places of the centres of gravity may be found experimentally, as well as the relative weights of the semi-arch and piers, and the measures of the several lines from the scale employed in the construction.

If the dimensions of the arch were given, and the thickness of the pier required; the same equations would serve; and different thicknesses of the pier might be assumed, until the first members of the equations come out largest.

The same rules are applicable to domes, simply taking the ungulas* instead of the profiles.

* The *ungulas*, as mentioned above, are the solids generated by the revolution of the sections f F k K and f F A B about the vertical axis k C.

EQUILIBRIUM OF SUSPENSION BRIDGES.

1. The several parts of a suspension bridge should always be proportioned in such a manner, that the tensile strain per square inch of section would be uniform throughout the whole length of the chains; and as the tension varies with the inclination of the chain, and to fulfil the above condition, the area of the chain must vary as the tension, it is quite evident that the chains of a suspension bridge ought not under any circumstances to be made of a uniform sectional area.

If the chain was of a uniform section, and had only its own weight to carry, the form which it would assume would be a catenary; if on the other hand, the chain is supposed devoid of weight, and the load suspended from it to be uniformly distributed horizontally, (as in the case of the roadway of a bridge,) then the form which the chain would assume would be the common parabola. Neither, however, of these supposed cases ever occur in practice, there being always three loads very differently distributed, viz., the weight of the chain itself, that of the roadway suspended from it, and that of the vertical rods by which the same is suspended; consequently, the form which the chain when in equilibrium would assume is neither a catenary, nor a common parabola, but is between the two; and it approaches nearer to one or the other, as the weight of the chain or that of the roadway predominates. Taking these three loads into account, and varying the section of the chain so that its tensile strain is uniform throughout, the determination of the true form of the curve which it would assume when all its parts were in equilibrium, becomes a very complicated problem; it has, however, been very ably solved by Professor Moseley*, and the following formula which he has deduced, contains all that is required for determining the form of the curve and the dimensions of the chains.

If μ_1 = the weight of a bar of the material of the chain 1 square inch in section and 1 foot long, μ_2 = the weight of a foot in length of the roadway, and, supposing the vertical suspending rods to be diffused over the whole space between the chain and the roadway, forming a uniform plate of such a thickness that its weight is precisely the same as that of the actual suspending rods, let μ_3 = the weight of a square foot of this plate; also, let K = the sectional area in square inches of the chain at any point P (fig. 179), x and y being the ordinate and abscissa at the same point; let a = the semi-span, H = the deflexion, b = the length of the shortest suspending rod, c = the tension upon the lowest point of the chain, and $\alpha = \frac{m\mu_1}{\tau}$, where τ represents the tenacity of the material of the chain per square inch, and m = the number of times that τ exceeds the actual tensile strain upon the chains; then

* "The Mechanical Principles of Engineering and Architecture," by the Rev. Henry Moseley, page 547.

$$x^2 = \left(\frac{2c}{\mu_3 b + \alpha c + \mu_2}\right)(y - b) \ldots\ldots\ldots\ldots \text{(I.)}$$

$$c = \left(\frac{\mu_2 + \mu_3 b}{2H - 2b - \alpha s^2}\right) s^2 \ldots\ldots\ldots\ldots \text{(II.)}$$

$$K = \frac{\alpha c}{\mu_1}\left\{2(y - b)\left(\frac{\mu_3 b + \mu_2}{c} + \alpha\right) + 1\right\}^{\frac{1}{2}} \ldots \text{(III.)}$$

2. As, however, the use of these formulæ for determining the requisite number of points in a bridge would be attended with much labour, it is only in cases where great accuracy is requisite that they need be employed; for most purposes it will be sufficiently correct to assume that the form of the curve is a common parabola, in which case the formulæ for determining the several elements of the bridge become much more simple, and easy of application.

Let t = the *coefficient of tension* at the lowest point of the chain, or the quantity by which half the weight of the bridge must be multiplied to give the actual strain upon the chains, t_2 = the coefficient at the point of suspension, t_3 = the mean coefficient of tension for the whole chain, s = the semi-span, d = the deflexion, λ = the length of half the chain, K_1 = the sectional area of the chains at the lowest point, K_2 = the same at the point of suspension, K_3 = the mean sectional area of the whole chain, W = the weight of half the whole bridge, including the chain, γ = the angle which the tangent to the curve at the point of suspension makes with the horizon, and b_1 = the length of the suspending rod at any point P. The letters, μ_1, μ_2, m, τ, b, x, y, and c, represent the same quantities as before.

Then, the curve being a parabola, the length of the suspending rod at any point P may easily be found by means of the following property of the parabola;

$$y : d :: x^2 : s,$$

therefore, supposing the roadway to be horizontal, we have

$$b_1 = b + \frac{d x^2}{s} \ldots\ldots\ldots\ldots \text{(IV.)}$$

And for the length of the chain, we have from the formula at page 129, for the length of parabolic arcs,

$$\lambda = \sqrt{s^2 + \frac{4}{3} d^2} \ldots\ldots\ldots\ldots \text{(V.)}$$

Also, from the method given at page 174, for drawing a tangent to any point in a parabola, we can easily deduce the value of γ, as follows:—if D E (fig. 180) be made equal to C D, the line drawn from A to E will be the tangent to the parabola at the point A, and therefore the angle C A E = γ, is determined from the proportion

$$s : \text{rad} :: 2d : \tan\gamma;$$

$$\therefore \tan\gamma = \frac{2d}{s} \quad \text{..... (VI.)}$$

And since the chain A D B with the roadway suspended from it is nothing more than an equilibriated polygon, such as is described at page 202, having an infinite number of sides, all the relations there mentioned as existing between the several strains, may be applied to the present inquiry; therefore, if C E $= 2d$ be made to represent the whole weight of half the bridge $=$ W, then A C $= s$ will represent the tension at D, and A E the tension at A; therefore,

$$t_1 = \frac{s}{2d} \quad \text{..... (VII.)}$$

$$t_2 = \frac{s \cdot \sec\gamma}{2d} \quad \text{..... (VIII.)}$$

$$\text{and } t_3 = (1 + \sec\gamma)\frac{s}{4d} \quad \text{..... (IX.)}$$

Then, for the mean sectional area of the chain, we have

$$K_3 = \frac{\mu_2 s}{\frac{\tau}{m t_3} - \lambda\mu_1} \quad \text{..... (X.)}$$

and for the weight of the roadway and chain for half the bridge,

$$W = K_3 \lambda\mu_1 + \mu_2 s \quad \text{..... (XI.)}$$

Then since $t_1 W = c$, the tension at the lowest part of the chain, we have for the sectional area of the chain at the same place,

$$K_1 = \frac{m t_1 W}{\tau} \quad \text{..... (XII.)}$$

and $t_2 W$ being the tension at the point of suspension, the sectional area of the chain at that point will be

$$K_2 = \frac{m t_2 W}{\tau} \quad \text{..... (XIII.)}$$

3. In order to render the practical application of the foregoing formulæ quite clear, we subjoin the following example. Let it be required to determine all the elements of a suspension bridge, the span of which is to be 360 feet, and the deflexion 30 feet; let the weight of the roadway for every foot in length $=$ 4500 lbs., the tenacity of a square inch of the chain $=$ 67200 lbs., the weight of a bar 1 inch square and 1 foot long $=$ 3·4 lbs., and the number of times that the cohesive strength of the iron is to exceed the load $= 6$.

Then substituting these values for the letters representing the several quantities, we obtain from formula (V.)

$\lambda = \sqrt{180 + \frac{4 \times 30^2}{3}} = 183{\cdot}33$ feet, equal the length o
the chain.

From formula (VI.)

$\tan \gamma = \frac{2 \times 30}{180} = \frac{1}{3} = {\cdot}3333$, which is the tangent* of 1
the angle made by the chains with the horizon at their po
suspension.

From formula (VII.)

$t_1 = \frac{180}{2 \times 30} = 3$, the coefficient of tension at the lowest p
the chain; from formula (VIII.)

$t_2 = \frac{180 \times 1{\cdot}0541\dagger}{2 \times 30} = 3{\cdot}1623$ equal the coefficient of ten
the point of suspension; and from formula (IX.)

$t_3 = (1 + 1{\cdot}0541)\frac{180}{4 \times 30} = 3{\cdot}0812$ for the mean coeffici
tension for the whole length of the chain.

From formula (X.)

$$K_3 = \frac{4500 \times 180}{\frac{67200}{6 \times 3{\cdot}0812} - 183{\cdot}33 \times 3{\cdot}4} = 269 \text{ square inches f}$$

mean sectional area of the chain.

From formula (XI.)

$W = 269 \times 183{\cdot}33 \times 3{\cdot}4 + 4500 \times 180 = 977{,}647$ lbs. t
total weight of half the bridge; and finally, from formula (XI
obtain

$K_1 = \frac{6 \times 3 \times 977{,}674}{67{,}200} = 262$ square inches for the section
of the chain in the centre; and from formula (XIII.)

$K_2 = \frac{6 \times 3{\cdot}1623 \times 977{,}674}{67{,}200} = 276$ square inches for the se
area of the chain at the point of suspension.

* In order to find the angle of which ·3333 is the tangent, take th
·3333 = $\bar{1}{\cdot}522835$, and adding 10 to the characteristic (for the reason
which see page 153), we have 9·522835, which we find from Table IV
log tan of 18° 26′, as above.

† The secant of 18° 26′ is obtained by a reverse process to that expl
the foregoing note; thus, from Table IV. we find the log sec of 18° 2
10·022875, then subtracting 10 we have 0·022875, the number answ
which in Table III. is 1·0541, the secant of 18° 26′ radius being 1.

CHAP. III.

DYNAMICS.

SECT. I. *General Definitions.*

1. THE *mass* of a body is the quantity of matter of which it is composed; and is proportional to its weight, or to the *force* which must be applied to the body to prevent its gravitating to the earth, and which, being greater or less as the mass is greater or less, we regard as a measure of the mass itself.

2. *Density* is a word by which we indicate the comparative closeness or otherwise of the particles of bodies, and is synonymous with the term *specific gravity*. Those bodies which have the greatest number of particles, or the greatest quantity of matter, in a given magnitude, we call *most dense;* those which have the least quantity of matter, *least dense*. Thus *lead* is more dense than freestone; *freestone* more dense than oak; and *oak* more dense than cork.

3. The *velocity* with which a body in motion moves, is measured by the space over which it passes in any given time; the unit usually assumed being one *second*.

4. If the body passes over an equal space in each successive unit of time, the body is said to move *uniformly*, or to have a *uniform velocity*, and the measure of such velocity is the space actually passed over by the body in each second.

5. If, however, the body passes over a *greater* space in each successive second than it did in the preceding, then it is said to move with an *accelerated velocity;* when the differences between the spaces moved over in any two successive seconds is the same, at whatever period of the body's motion they be taken, or in other words, when the successive spaces form an arithmetical progression, the body is said to move with a *uniformly accelerated* velocity; but when the spaces passed over in successive seconds increase according to any other law, the body is then said to have its velocity *variably accelerated*.

6. If, on the other hand, the body passes over a *smaller* space in each successive second than it did in the preceding, then it is said to move with a *retarded* velocity; which, if the successive spaces form a decreasing arithmetical series, is said to be *uniformly retarded;* if otherwise, it is said to be *variably retarded*.

7. The *velocity* of a body whose motion is variable is expressed at any moment, by the space which it *would pass over in a second*, if its velocity at the moment spoken of were to continue *uniform for that period.*

8. *Mechanical effect* is measured by the product of the *mass* or *weight* of the body into the space over which it has been moved; no regard being had to the time occupied. The unit of mechanical effect employed in the subsequent pages, is a weight of 1 pound raised through a space of 1 foot, and is designated by the letter U.

9. The *momentum* of a body in motion means the mechanical effect which such a body will produce in a *moment* (or second) of time, and varies as the weight of the body multiplied by its *velocity.*

10. The *vis viva* of a body in motion is the whole mechanical effect which it will produce in *being brought to a state of rest*, no regard being had to the time in which the effect is produced, and it varies as the weight of the body multiplied by the *square of its velocity.*

SECT. II. *On the general Laws of Uniform and Variable Motion.*

1. As a proper understanding of the actual difference between the *momentum* and the *vis viva* of a body in motion, is essential to a correct application of the principles of dynamics, we shall take some pains to set this difference in as clear a light (for the student) as possible. And it is of the more importance to do so, as a diversity of opinion upon this subject has existed amongst some of the most eminent mathematicians, and much time and talent has been expended by them in supporting errors which have arisen entirely in a misconception of terms, and in excluding from their conclusion the consideration of *time* which they included in their premises.

2. From carefully conducted and often repeated experiments, the following results with regard to bodies in motion have been obtained:—

I. If a body of a certain weight, and moving with a given velocity, meet another body of double that weight, and moving with half the velocity, the two bodies will destroy each other's motion, and *both* will be brought to a *state of rest.*

II. A body of a certain weight and moving with a given velocity, being subject to a uniformly retarding force, (*i.e.* a uniform force acting constantly in a contrary direction to the body's motion,) will move over a certain space in being brought to rest, and will occupy a certain time in doing so; then another body of the same weight, but moving with *half* the velocity of the former, being subject to the same uniformly retarding force, will move over *one quarter* of the

space moved over by the former, in being brought to a state of rest, and will occupy in doing so *half* the time. And another body of the same weight, but moving with *one-third* of the velocity of the first, will move over *one-ninth* of the space, and occupy *one-third* the time of the first, in being brought to a state of rest.

Now the diversity of opinion to which we have alluded above, has arisen from the (at first sight) apparently contradictory nature of these two results: one party has drawn a conclusion from the first experiment that the *force of a body in motion* is directly *as its velocity;* and the other party has drawn a conclusion from the second experiment that the *force of a body in motion* is directly *as the square of its velocity*. These errors have arisen from the term "force of a body in motion" being used without any fixed and definite meaning being attached to the same.

3. The proper measure of the *whole* force (i. e. *vis viva*) of a body in motion, is the mechanical effect, or (as Professor Moseley very significantly terms it) the *work*, which it is capable of performing in being brought to a state of rest. That evidently being the force *due to its motion* which is required to *destroy the same*, and which is directly as the *square of its velocity*, as found by the second experiment.

4. If, however, we only consider the mechanical effect (or the work) which a body in motion is capable of performing *in a given time*, (i. e. its *momentum*,) we shall find from the second experiment, that although the body having twice the velocity *ultimately* produced four times the effect, in doing so it occupied just twice the time; and again, although the body moving with three times the velocity *ultimately* produced nine times the effect, it occupied in doing so three times the time; and, therefore, the mechanical effect produced *in a given time* by the bodies was directly *as the velocities*. And this conclusion, drawn from the second experiment, is in accordance with that which must be drawn from the first experiment, in which it is evident that the effect produced by both bodies must be in the same (although indefinitely small) time.

5. The force of gravity being constantly the same, both in amount and direction, and being practically uniform* in its action for such small distances from the earth's surface as come under consideration in ordinary dynamical investigations, has been universally adopted as the unit of measure for all other forces.

6. The actual amount of the force of gravity is measured by the velocity which a body falling in vacuo, in the latitude of London, will

* The force of gravity varies as the square of the distance from the earth's centre, and therefore becomes less as we ascend above the surface of the earth, in the proportion of 1 to ·9994, a difference too small to require notice in any question of terrestrial mechanics.

acquire in one second of time, and which, by carefully conducted experiments, has been ascertained to be 386·289 inches, or about $32\frac{1}{6}$ feet per second, and this latter quantity is usually represented by the symbol g.

MOTION UNIFORMLY ACCELERATED.

1. If we now put W to denote the quantity of matter, or weight of a body in motion, v its velocity in feet per second, M its momentum, and V its *vis viva*, both expressed in units of mechanical effect, or pounds raised through a space of 1 foot; we have

$$M = W v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (I.)$$

$$V = W \frac{v^2}{2g} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (II.)$$

2. And further, if s be put to represent the space passed over in the time t, by a body subject to the uniform force f; then we have the following relations between all these quantities:—

$$V = sW = \frac{tv}{2} W = \frac{ft^2}{2} W = \frac{v^2}{2f} W \quad . \quad . \quad (III.)$$

$$W = \frac{V}{s} = \frac{2V}{tv} = \frac{2V}{ft^2} = \frac{2fV}{v^2} \quad . \quad . \quad (IV.)$$

$$s = \frac{tv}{2} = \frac{ft^2}{2} = \frac{v^2}{2f} \quad . \quad . \quad . \quad . \quad . \quad . \quad (V.)$$

$$v = ft = \frac{2s}{t} = \sqrt{2fs} \quad . \quad . \quad . \quad . \quad (VI.)$$

$$t = \frac{v}{f} = \frac{2s}{v} = \sqrt{\frac{2s}{f}} \quad . \quad . \quad . \quad . \quad (VII.)$$

$$f = \frac{v}{t} = \frac{2s}{t^2} = \frac{v^2}{2s} \quad . \quad . \quad . \quad . \quad . \quad (VIII.)$$

MOTION OF BODIES UNDER THE ACTION OF GRAVITY.

1. When the uniform force is that of gravity, $f = g$; and g being proportional to the weight of the body (W), the foregoing relations are simplified, and then become as follows:—

$$s = \frac{tv}{2} = \frac{gt^2}{2} = \frac{v^2}{2g} \quad . \quad . \quad . \quad . \quad . \quad . \quad (IX.)$$

$$v = gt = \frac{2s}{t} = \sqrt{2gs} \quad . \quad . \quad . \quad . \quad . \quad . \quad (X.)$$

$$t = \frac{v}{g} = \frac{2s}{v} = \sqrt{\frac{2s}{g}} \quad \ldots\ldots \text{(XI.)}$$

$$g = \frac{v}{t} = \frac{2s}{t^2} = \frac{v^2}{2s} \quad \ldots\ldots \text{(XII.)}$$

2. Any two of these quantities being given, the other two may be immediately ascertained from the above equations; the following table shows their actual numerical values for the first four seconds of the motion of a heavy body falling freely by the action of gravity:—

The times in seconds being	1″,	2″,	3″,	4″, &c.
The velocities in feet will be	$32\frac{1}{6}$,	$64\frac{1}{3}$,	$96\frac{1}{2}$,	$128\frac{2}{3}$, &c.
The spaces in the whole times	$16\frac{1}{12}$,	$64\frac{1}{3}$,	$144\frac{3}{4}$,	$257\frac{1}{3}$, &c.
And the space for each second	$16\frac{1}{12}$,	$48\frac{1}{4}$,	$80\frac{5}{12}$,	$112\frac{7}{12}$, &c.

of which spaces the common difference is $32\frac{1}{6}$ feet, equal g, the measure of the force of gravity.

3. If, instead of a heavy body being allowed to fall freely, it be *propelled* vertically upwards or downwards with a given velocity, v, then

$$s = tv \mp \frac{gt^2}{2}; \quad \ldots\ldots\ldots \text{(XIII.)}$$

an expression in which the upper sign — must be taken when the projection is *upwards*, the lower sign + when the projection is *downwards*.

4. When only an approximate result is required with reference to bodies falling vertically, 32 may be put for g, instead of $32\frac{1}{6}$: there would then result, in motions from quiescence,

$$s = 16\,t^2 = \frac{v^2}{64} = \frac{tv}{2} \quad \ldots \quad \text{(XIV.)}$$

$$t = \frac{v}{32} = \frac{\sqrt{s}}{4} = \frac{2s}{v} \quad \ldots \quad \text{(XV.)}$$

$$v = 8\sqrt{s} = \frac{2s}{t} = 32\,t. \quad \ldots\ldots \text{(XVI.)}$$

Thus, if the space descended were 64 feet, we should have $v = 8 \times 8 = 64$ feet per second, and $t = \frac{8}{4} = 2$ seconds.

If the space descended were 400; then $v = 8 \times 20 = 160$ feet per second, and $t = \frac{20}{4} = 5$ seconds.

5. The force of gravity differs a little at different latitudes; the law of the variation is not as yet *precisely* ascertained; but the following

theorems are known to represent it very nearly. That is, if g denote the force of gravity at latitude 45°, g_1 the force at the poles, g_2 the force at the equator, and g_3 the force at any other place: then

$$g_1 = g\,(1 + \cdot 002837) \quad . \quad . \quad . \quad . \quad . \quad \text{(XVII.)}$$
$$g_2 = g\,(1 - \cdot 002837) \quad . \quad . \quad . \quad . \quad . \quad \text{(XVIII.)}$$
$$g_3 = g\,(1 \cdot - \cdot 002837 \cos 2 \text{ lat.}) \quad . \quad . \quad \text{(XIX.)}$$

MOTION OVER A FIXED PULLEY.

In this case let the two weights which are connected by the cord that goes over the pulley be denoted by W_1 and W_2: then $\frac{W_1 - W_2}{W_1 + W_2} \cdot g = Wf$ in formula (III.); so that

$$s = \frac{W_1 - W_2}{W_1 + W_2} \cdot \frac{g t^2}{2}$$

Or, if the resistance caused by the rigidity of the rope, and the friction and inertia of the pulley, be represented by r; then

$$s = \frac{W_1 - W_2}{W_1 + W_2 + r} \cdot \frac{g t^2}{2} \cdot \quad . \quad . \quad . \quad . \quad . \quad \text{(XX.)}$$

Example 1. Suppose the two weights to be 5 and 3 lbs. respectively, what will be the space descended in 4 seconds?

Here $\frac{W_1 - W_2}{W_1 + W_2} \cdot \frac{g t^2}{2} = \frac{5 - 3}{5 + 3} \times 16\frac{1}{12} \times 16 = \frac{1}{4} \times 16\frac{1}{12} \times 16 = 16\frac{1}{12} \times 4 = 64\frac{1}{3}$ feet.

Example 2. But suppose that, in an actual experiment with the weights of 5 and 3 lbs. over a pulley, the heavier weight descended only 50 feet in 4 seconds.

Then $\frac{W_1 - W_2}{W_1 + W_2 + r} \cdot \frac{g t^2}{2} = 50$ feet: and, as W_1, W_2, g, and t, are the same in both examples,

we have $W_1 + W_2 + r : W_1 + W_2 :: 64\frac{1}{3} : 50$
or, dividendo $r : W_1 + W_2 :: 14\frac{1}{3} : 50$
that is, $r : 5 + 3 :: 14\frac{1}{3} : 50$
whence $r = \frac{8 + 14\frac{1}{3}}{50} = \frac{114\frac{2}{3}}{50} = 2{\cdot}2933$ lbs.

the measure of the resistance and the inertia.

2. Similar principles are applicable in a variety of other cases; and by varying the relations of W_1, W_2, and r, the force may have

any assigned ratio to that of gravity; which is, indeed, the foundation of Mr. Atwood's elegant apparatus for experiments on accelerating forces; an account of which may be seen in the 2nd volume of my *Mechanics*, or in almost any of the general dictionaries of arts and sciences.

3. If, instead of pulleys, small wheels and axles, as in figure 181, be employed, to raise weights by the preponderance of equal weights: then, if the diameter of the wheel and axle A be as 3 to 2; those of the wheel and axle B, as 5 to 2; and those of C, as 8 to 2; it will be found that the weight b will be elevated more rapidly than either a or c: the proportion of 5 to 2, (or, more accurately, of $\sqrt{1+\sqrt{2}}$ to 1) being in that respect the most favourable.

MOTION ON INCLINED PLANES.

1. When bodies move down inclined planes, the accelerating force (independently of the modification occasioned by the position of the *centre of gyration*) is equal to g multiplied by the quotient of the height of the plane divided by its length, or what is equivalent, by the sine of the inclination of the plane. In this case, therefore, putting s for the inclination of the plane, the formulæ become

$$s = \frac{g t^2 \cdot \sin s}{2} = \frac{v^2}{2g \cdot \sin s} = \frac{t v}{2} \quad \text{(XXI.)}$$

$$v = g t \cdot \sin s = \sqrt{(2 g s \cdot \sin s)} = \frac{2 s}{t} \quad \text{(XXII.)}$$

$$t = \sqrt{\frac{2 s}{g \cdot \sin s}} = \frac{2 s}{v}. \quad \text{(XXIII.)}$$

Further, if v_1 be the velocity with which a body is projected up or down a plane, then

$$v = v_1 \mp g t \cdot \sin s \quad \text{(XXIV.)}$$

$$s = v_1 t \mp \frac{g t \cdot \sin s}{2} = \frac{{v_1}^2 - v^2}{2 g \cdot \sin s}. \quad \text{(XXV.)}$$

Making $v_1 = 0$. in equation (XXIV.), and the latter member of equation (XXV.), the first will give the *time* at which the body will cease to rise, the latter the *space*.

Example. Suppose a body be projected up a smooth inclined plane whose height is 12 and length 193 feet, with a velocity of 20 feet per second, how high will it rise up the plane before its motion is extinguished?

Then substituting these values in formula (XXV.), we have

$$s = \frac{400 - 0}{64\frac{1}{3} \times \frac{12}{193}} = \frac{400}{\frac{193}{3} \times \frac{12}{193}} = \frac{400}{4} = 100 \text{ feet, the space required.}$$

2. With regard to the *velocities* acquired by bodies in falling down planes of the same height, this proposition holds; viz. that they are all equal, estimated in their respective directions. Thus, if A D, B E, C F, (fig. 182), be planes of different inclinations, and A C, D F, horizontal lines, the balls A, B, C, after descending along those planes will have equal velocities when they arrive at the points D, E, F, respectively.

3. Also, all the chords, such as A D, B D, C D, (fig. 183), that terminate either in the upper or the lower extremity of the vertical diameter of a circle, will be described in the same time by heavy bodies A, B, C, running down them; and that time will be equal to the time of vertical descent through the diameter D E.

4. If three weights, as A, B, C, (fig. 184), be drawn up three planes of different inclinations, by three equal weights hanging from cords over pulleys at P, then if the *length* of the middle plane be *twice* its *height*, the body B will be drawn up that plane, *quicker* than either of the other weights A or C. Or, generally, to ensure an ascent up a plane in the least time, the *length* of the plane must be to its *height*, as *twice* the *weight* to the *power* employed.

5. If it be proposed to construct a roof over a building of a given width, so that the rain shall run quickest off it, then each side of the roof must be inclined 45° to the horizon, or the angle at the ridge must be a right angle.

6. The force by which spheres, cylinders, &c. are caused to revolve as they move down an inclined plane (instead of sliding) is the adhesion of their surfaces occasioned by the pressure against the plane: this pressure is part of the body's weight; for the weight being resolved into its components, one in the direction of the plane, and the other perpendicular to it, the latter is the force of the pressure upon the plane; and, while the same body rolls down the plane, will be expressed by the cosine of the plane's elevation. Hence, since the cosine decreases while the arc or angle increases, after the angle of elevation arrives at a certain magnitude, the adhesion may become less than what is necessary to make the circumference of the body revolve fast enough; in this case the body descends partly by sliding and partly by rolling. And the same may happen in smaller elevations, if the body and plane are very smooth. But at all elevations the body may be made to roll by the uncoiling of a thread or riband wound about it.

If W denote the weight of a body, s the space described by a body falling freely by the action of gravity, or sliding freely down an inclined plane, then the spaces (s_1) described by rotation in the same time by the following bodies, will be in these proportions.

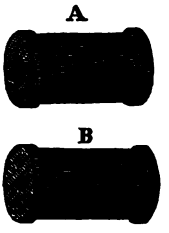
(1.) In a hollow cylinder, or cylindrical surface, $s_1 = \frac{1}{2}\, s$ and the tension of the cord, in the first case $= \frac{1}{2}$ W.

(2.) In a solid cylinder, $s_1 = \frac{2}{3}\, s$, and the tension $= \frac{1}{3}$ W.

(3.) In a spheric surface, or thin spherical shell, $s_1 = \frac{3}{5}\, s$, and the tension $= \frac{2}{5}$ W.

(4.) In a solid sphere, $s_1 = \frac{5}{7}\, s$, and the tension $= \frac{2}{7}$ W.

If two cylinders be taken of equal size and weight, and with equal protuberances upon which to roll, as in the marginal figures: then, if lead be coiled uniformly over the curve surface of B, and an equal quantity of lead be placed uniformly from one end to the other near the axis in the cylinder A, that cylinder will roll down any inclined plane *quicker* than the other cylinder B. The reason is that *each particle of matter in a rolling body resists motion in proportion to the* SQUARE *of its distance from the axis of motion;* and the particles of lead which most resist motion are placed at a *greater* distance from the axis in the cylinder B than in A.



7. The friction between the surface of any body and a plane, may be easily ascertained by gradually elevating the plane until the body upon it *just begins to slide*. The friction of the body is to its weight as the height of the plane is to its base, or as the tangent of the inclination of the plane is to the radius. Thus, if a piece of stone in weight 8 pounds, just begins to slide when the height of the plane is 2 feet, and its base $2\frac{1}{2}$; then the friction will be $\frac{4}{5}$ the weight, or $\frac{4}{5}$ of 8 lbs. $= 6\frac{2}{5}$ lbs.

8. After motion has commenced upon an inclined plane, the friction is usually much diminished. It may easily be ascertained experimentally, by comparing the time occupied by a body in *sliding* down a plane of given height and length, or given inclination, with that which the simple theorem for t, (XXIII.) would give. For, if f be the value of the friction in terms of the pressure, the theorem for the time will be

$$t_1 = \sqrt{\frac{2s}{g\ (\sin \iota - f)}}, \text{ instead of } t = \sqrt{\frac{2s}{g\ .\ \sin \iota}}. \quad \text{Hence}$$

$$t_1^2 : t^2 :: \sin \iota : \sin \iota - f \quad . \quad . \quad . \quad \text{(XXVI.)}$$

Example. Suppose that a body slides down a plane in length 30 feet, height 10, in $2\frac{1}{3}$ seconds, what is the amount of the friction?

$$\text{Here } t = \sqrt{\frac{2s}{g\ .\ \sin \iota}} = \sqrt{\frac{60}{32\frac{1}{6} \times \frac{1}{3}}} = 2{\cdot}366 \text{ nearly.}$$

$$\text{Hence } (2{\cdot}6)^2 : (2{\cdot}366)^2 :: \tfrac{1}{3} : {\cdot}27603 = \sin \iota - f$$

Consequently, ·33333 − ·27603 = ·0573 value of the friction, the weight being unity.

9. When a weight is to be moved either up an inclined plane, or along an horizontal plane, the angle of traction P W B (fig. 185) that the weight may be drawn with least effort, will vary with the value of f. The magnitude of that angle P W B for several values of f are exhibited in the following table: —

f	PWB	f	PWB	f	PWB	f	PWB	f	PWB	f	PWB
1	45° 0′	$\frac{1}{2}$	26°34′	$\frac{1}{3}$	18°26′	$\frac{1}{4}$	14° 2′	$\frac{1}{5}$	11°19′	$\frac{1}{6}$	9° 28′
$\frac{4}{5}$	38 40	$\frac{4}{9}$	23 58	$\frac{3}{13}$	16 54	$\frac{3}{17}$	13 15	$\frac{4}{21}$	10 47	$\frac{1}{7}$	8 8
$\frac{2}{3}$	33 41	$\frac{2}{5}$	21 48	$\frac{2}{7}$	15 57	$\frac{2}{9}$	12 32	$\frac{3}{11}$	10 18	$\frac{1}{8}$	7 8
$\frac{4}{7}$	29 45	$\frac{4}{11}$	19 59	$\frac{4}{15}$	14 56	$\frac{4}{19}$	11 53	$\frac{4}{23}$	9 52	$\frac{1}{9}$	6 20

10. If, instead of seeking the line of traction so that the *moving force* should be a minimum, we required the position such that the *suspending force* to keep a load from descending should be a minimum, or a given force should oppose motion with the greatest energy; then the angles in the preceding table will be still applicable, only the angle in any assigned case must be taken below, as B W p. This will serve in the building and securing walls, in the construction of banks of earth, fortifications, &c., and in arranging the position of *land-ties*, &c.

SECT. III. *Motions about a fixed Center or Axis.*

Centers of Oscillation and Percussion.

1. *The center of oscillation* is that point in the axis of suspension of a vibrating body in which, if all the matter of the system were collected, any force applied there would generate the same angular velocity in a given time as the same force at the center of gravity, the parts of the system revolving in their respective places.

Or, since the force of gravity upon the whole body may be considered as a single force (equivalent to the weight of the body) applied at its center of gravity, the *center of oscillation* is that point in a vibrating body in which, if the whole were concentrated and attached to the same axis of motion, it would then vibrate in the same time that the body does in its natural state.

2. From the first definition it follows that the center of oscillation is situated in a right line passing through the center of gravity, and perpendicular to the axis of motion. It is always farther from the point of suspension than the center of gravity.

3. *The center of percussion* is that point in a body revolving about an axis, at which, if it struck an immovable obstacle, the whole of its motion would be destroyed, or it would not incline either way.

4. When an oscillating body vibrates with a given angular velocity, and strikes an obstacle, the effect of the impact will be the greatest if it be made at the center of percussion. For, in this case the obstacle receives the whole revolving motion of the body; whereas, if the blow be struck in any other point, a part of the motion of the body will be employed in endeavouring to continue the rotation.

5. If a body revolving on an axis strike an immovable obstacle

at the center of percussion, the point of suspension will not be affected by the stroke. We can ascertain this property of the point of suspension when we give a smart blow with a stick. If we give it a motion round the joint of the wrist only, and, holding it at one extremity, strike smartly with a point considerably nearer or more remote than $\frac{2}{3}$ of its length, we feel a painful wrench in the hand: but if we strike with that point which is precisely at $\frac{2}{3}$ of the length (that being the situation of the center of percussion), no such disagreeable strain will be felt. If we strike the blow with one end of the stick, we must make its center of motion at $\frac{1}{3}$ of its length from the other end; and then the wrench will be avoided.

6. The distance of the center of percussion from the axis of motion is equal to the distance of the center of oscillation from the same: supposing that the center of percussion is required in a plane passing through the axis of motion and the center of gravity.

SIMPLE AND COMPOUND PENDULUMS.

1. A *simple pendulum*, theoretically considered, is a single weight, regarded as a point, or as a very small globe hanging at the lower extremity of an inflexible right line, void of weight, and suspended from a fixed point or center, about which it oscillates.

2. A *compound pendulum* is one that consists of several weights movable about one common center of motion, but so connected together as to retain the same distance both from one another and from the center about which they vibrate.

Or any body, as a cone, a cylinder, or of any shape, whether regular or irregular, so suspended as to be capable of vibrating, may be regarded as a compound pendulum; and the distance of its *center of oscillation* from any assumed point of suspension, is considered as the length of an equivalent simple pendulum.

3. If O represent the distance of the center of oscillation from the point of suspension, and G the distance of the center of gravity from the same point, it has been found that G O is a constant quantity, for the same body and the same plane of vibration.

4. Any such vibrating body will have as many *centers of oscillation* as you give it points of suspension: but when any one of those centers of oscillation is determined, either by theory or experiment, the rest are easily found by means of the foregoing property that O G is a constant product, or of the same value for the same body.

5. When a body either revolves about an axis, or oscillates, the sum of the products of each of the material elements, or particles of that body, into the squares of their respective distances from the axis of rotation, is called the *momentum of inertia* of that body.

6. A point, or very small body, on descending along the successive sides of a polygon in a vertical plane, loses at each angle a part of its actual velocity equal to the product of that velocity into the versed sine of the angle made by the side which it has just quitted, and the

prolongation of the side upon which it is just entering. Therefore, that loss is indefinitely small in *curves*.

7. A heavy body which descends by the force of gravity along a curve situated in a vertical plane, has, in any point whatever, the same velocity as it would have had if it had fallen through a vertical line equal to that between the top and the bottom of the arc run over: and when it has arrived at the bottom of any such curve, if there be another branch either similar or dissimilar, rising on the opposite side, the body will rise along that branch (apart from the consideration of friction) until it has reached the horizontal plane from which it first set out. Thus, after having descended from A to V (fig. 186), it will have the same velocity as that acquired by falling through D V, and it will ascend up the opposite branch until it arrives at B.

8. If the body describe a curve by a pendulous motion, the same property will be found to obtain, setting aside the effects of friction. Thus, let a ball hang by a flexible cord S D (fig. 187) from a pin at S: then, after it has descended through the arc D E, it will pass through an equal and similar arc E A, going up to A in the same horizontal line with D, and ascending from E to A in an interval of time equal to that which it descended from D to E. But, if a pin projecting from P or p stop the cord in its course, the ball will still rise to B or to C, in the same horizontal line with A and D; but will describe the ascending portions of the curve in shorter intervals of time than the descending branch.

9. When a pendulum is drawn from its vertical position, it will be accelerated in the direction of the tangent of the curve it would describe, by a force which is as the sine of its angular distance from the vertical position. Thus, the accelerating force at A (fig. 188), would be to the accelerating force at B, as A F to B E. (See art. 5, on the *Center of Gravity*.) This admits of an easy experimental proof.

10. If the same pendulous body descend through different arcs, its velocity at the lowest point will be proportional to the chords of the whole arcs described. Thus, the velocity at D, after passing through A B D, will be to the velocity at D after descending through the portion B D only, as A D to B D.

11. Farther, the velocity after describing A B D, is to the velocity after describing B D, as $\sqrt{\text{F D}}$ is to $\sqrt{\text{E D}}$. If, therefore, we would impart to a body a given velocity V, we have only to compute the height F D, such that $\text{F D} = \frac{\text{V}^2}{2g} = \frac{\text{V}^2}{64\frac{1}{3}}$ feet, and through the point F draw the horizontal line F A; then, letting the body descend as a pendulum through the arc A B D, when it arrives at D it will have acquired the required velocity. This property is extremely useful in experiments on the *collision* of bodies.

12. The oscillations of pendulums in any arcs of a cycloid are *isochronal*, or performed in equal times.

13. Oscillations in *small* portions of a circular arc are nearly *isochronal*.

14. The numbers of oscillations of two different pendulums, in the same time, and at the same place, are in the inverse ratio of the square roots of the length of those pendulums.

15. If l be the length of a single pendulum, or the distance from the point of suspension to the center of oscillation in a compound pendulum, g = the measure of the force of gravity ($32\frac{1}{6}$ feet, or 386 inches at the level of St. Paul's* in the latitude of London), t the time of one oscillation in an indefinitely small circular arc, and $\pi = 3{\cdot}141593$: then

$$t = \pi \sqrt{\frac{l}{g}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(I.)}$$

16. Conformably with this we have

$39\frac{1}{8}$	inches, length of the second	)	pendulum
$9\frac{25}{32}$	inches ——— half second		in the
$4\frac{25}{72}$	inches ——— third of second		latitude
$2\frac{25}{128}$	inches ——— quarter second	)	of London.

17. Putting g_1 for the force of gravity in any latitude and at any altitude, we have also $l = {\cdot}20264 \times \frac{1}{2} g_1$

and $\frac{1}{2} g_1 = 4{\cdot}9348\, l$ ..... (II.)

In other words, whatever be the force of gravity, the length of a second pendulum, and the space descended freely by a falling body in 1 second, *are in a constant ratio.*

18. If l_1 be the length of a pendulum, g_1 the force of gravity, and t_1 the time of oscillation at any other place, then

$$t : t_1 :: \sqrt{\frac{l}{g}} : \sqrt{\frac{l_1}{g_1}} \quad \ldots\ldots\ldots\ldots\ldots \text{(III.)}$$

If the force of gravity be the same,

$$t : t_1 :: \sqrt{l} : \sqrt{l_1} \ldots\ldots\ldots\ldots\ldots\ldots \text{(IV.)}$$

If the same pendulum be actuated by different gravitating forces, we have

$$t : t_1 :: \sqrt{\frac{1}{g}} : \sqrt{\frac{1}{g_1}} :: \sqrt{g_1} : \sqrt{g} \ldots\ldots \text{(V.)}$$

* At the level of *the sea*, in the latitude of London, g is 386·289 inches, and the corresponding length of the second pendulum is 39·1393 inches, according to the determination of *Major Kater*. Conformably with this result are the numbers in the Table (in the Appendix), computed at the expense of Messrs. *Bramah* and *Donkin*, and obligingly communicated by them for this work. It has been suspected by M. *Bessel*, and demonstrated by Mr. *Francis Baily*, that, in the refined computations relative to the pendulum, the formulæ for the reduction to a vacuum are inaccurate, and that, in consequence, we do not yet *precisely* know the length of a second pendulum. See Phil. Transac. 1832.

When pendulums oscillate in equal times in different places, we have

$$g : g_1 :: l : l_1.$$

For the variations of gravity in different latitudes, see formulæ (XVII., XVIII., XIX.) in the preceding section.

18. If the arcs are not indefinitely short, let v denote the versed sine of the semi-arc of vibration; then

$$t = \pi \sqrt{\frac{l}{g}} (1 + \tfrac{1}{8} v + \tfrac{9}{256} v^2 + \&c.) \ldots . \text{(VI.)}$$

In which, when the semi-arc of vibration does not exceed 4 or 5 degrees, the third term of the series may be omitted.

If the time of an oscillation in an indefinitely small arc be 1 second, the augmentation of the time will be

for a semi-arc of	30°	0·01675
	15°	0·00426
	10°	0·00190
	5°	0·00012
	$2\frac{1}{2}$°	0·00003

So that for oscillations of $2\frac{1}{2}$° on each side of the vertical, the augmentation would not occasion more than 2″ difference in a day.

19. If D denote the degrees in the semi-arc of an oscillating pendulum, the time lost in each second by vibrating in a circle instead of the cycloid, is $\frac{D^2}{52524}$; and consequently the time lost in a whole day of 24 hours, or $24 \times 60 \times 60$ seconds, is $\frac{5}{3} D^2$ nearly. In like manner, the seconds lost per day by vibrating in the arc of Δ degrees, is $\frac{5}{3} \Delta^2$. Therefore, if the pendulum keep true time in one of these arcs, the seconds lost or gained per day, by vibrating in the other, will be $\frac{5}{3} (D^2 - \Delta^2)$. So, for example, if a pendulum measure true time in an arc of 3 degrees, it will lose $11\frac{2}{3}$ seconds a day by vibrating 4 degrees; and $26\frac{2}{3}$ seconds a day by vibrating 5 degrees: and so on.

20. If a clock keep true time very nearly, the variation in the length of the pendulum necessary to correct the error will be equal to twice the product of the length of the pendulum, and the error in time divided by the time of observation in which that error is accumulated.

If the pendulum be one that should beat seconds, and t_1 the daily variation be given in minutes, and n be the number of threads in an inch of the screw which raises and depresses the bob of the pendulum, then $\lambda = \frac{\pm 2 \times 39\frac{1}{8} \times n t_1}{24 \times 60} = \cdot 05434\, n t_1 = \frac{2}{37} n t_1$, nearly, for the number of threads which the bob must be raised or lowered, to make the pendulum vibrate truly.

21. For civil and military engineers, and other practical men, it is

highly useful to have a *portable pendulum*, made of painted tape with a brass bob at the end, so that the whole, except the bob, may be rolled up within a box, which may be enclosed in a shagreen case. The tape is marked 200, 190, 180, 170, 160, &c., 80, 75, 70, 65, 60, at points, which being assumed respectively as points of suspension, the pendulum will make 200, 190, &c., down to 60 vibrations in a minute. Such a portable pendulum may be readily employed in experiments relative to falling bodies, the velocity of sound, &c.

22. If the *momentum of inertia* (§ 5, page 221) of a pendulum, whether simple or compound, be divided by the product of the pendulum's weight or mass into the distance of its center of gravity from the point of suspension (or axis of motion), the quotient will express the distance of the center of oscillation from the same point (or axis).

23. Whatever the number of separate masses or bodies which constitute a pendulum, it may be considered as a single pendulum, whose center of gravity is at the distance d from the axis of suspension, or of rotation: then, if K^2 denote the momentum of inertia of that body divided by its mass, the distance O from the axis of rotation to the center of oscillation, or the length of an equivalent simple pendulum, will be

$$O = \frac{d^2 + K^2}{d}.^* \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(VII.)}$$

24. To find the distance of the center of oscillation from the point or axis of suspension, *experimentally*. Count the number, n, of oscillations of the body in a very short arc in a minute; then

$$O = \frac{140850}{n^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(VIII.)}$$

Thus, if a body so oscillating made 50 vibrations in a minute; then $O = \frac{140850}{2500} = 56{\cdot}34$ inches.

Or, $O = 39\frac{1}{8} t^2$, in inches, t being the time of one oscillation in a very small arc.

If the arc be of finite appreciable magnitude, the time of oscillation must be reduced in the ratio of 8 + versin of semi-arc to 8, before the rule is applied.

25. From the foregoing principles are derived the following expressions for the distances of the centers of oscillation for the several figures, suspended by their vertices and vibrating flatwise, viz.:—

(1.) Right line or very thin cylinder, $O = \frac{2}{3}$ of its length.
(2.) Isosceles triangle, $O = \frac{3}{4}$ of its altitude.
(3.) Circle, $O = \frac{5}{4}$ radius.
(4.) Common parabola, $O = \frac{4}{5}$ of its altitude.

* For some curious and valuable theorems, by Professor *Airy*, for the reduction of vibrations in the air to those in a vacuum, see Mr. *F. Baily's* paper referred to in the preceding note.

(5.) Any parabola, $O = \frac{2m+1}{3m+1} \times$ its altitude.

Bodies vibrating laterally or sideways, or in their own plane:

(6.) In a circle, $O = \frac{3}{4}$ of diameter.

(7.) In a rectangle suspended by one angle, $O = \frac{8}{9}$ of diagonal.

(8.) Parabola suspended by its vertex, $O = \frac{5}{7}$ axis $+ \frac{1}{3}$ parameter.

(9.) Parabola suspended by middle of its base, $O = \frac{4}{7}$ axis $+ \frac{1}{2}$ parameter.

(10.) In a sector of a circle, $O = \frac{3 \text{ arc} \times \text{rad}}{4 \text{ chord}}$.

(11.) In a cone, $O = \frac{1}{2}$ axis $+ \frac{(\text{rad of base})^2}{5 \text{ axis}}$.

(12.) In a sphere, $O = \text{rad} + d + \frac{2 \text{ rad}^2}{5(d + \text{rad})}$; where d is the length of the thread by which it is suspended.

(13.) If the weight of the thread is to be taken into the account, we have the following distance between the center of the ball and that of oscillation, where B is the weight of the ball, d the distance between the point of suspension and its center, r the radius of the ball, w the weight of the thread or wire, and δ the distance of the center of oscillation from the center of gravity, we have

$$\delta = \frac{(\frac{1}{3}w + \frac{2}{5}B)4r^2 - \frac{1}{6}w(2dr + d^2)}{(\frac{1}{2}w + B)d - rw} \quad \text{..... (IX.)}$$

Or, if B be expressed in terms of w considered as a unit, then

$$\delta = \frac{\frac{1}{6}d}{B + \frac{1}{2}} \quad \text{..... (X.)}$$

(14.) If two weights W, W_1, be fixed at the two extremities of a rod of given length, S being the center of motion between W and W_1; then, if d equal the distance of the weight W from S, D equal the distance of W_1 from S, and m the weight of a unit in length of the rod, we shall have

$$O = \frac{mD^3 + 3W_1D^2 + md^3 + 3Wd^2}{mD^2 + 2W_1D - md^2 - 2Wd} \quad \text{..... (XI.)}$$

the radii of the balls being supposed very small in comparison with the length of the rod.

(15.) In the bob of a clock pendulum, supposing it two equal spheric segments joined at their bases, if the radii of those bases be each $= \rho$, the height of each segment v, and d the distance from the point of suspension to the center of the bob, then is

$$\delta = \frac{1}{3}d \cdot \frac{\rho^4 + \frac{1}{2}\rho^2v^2 + \frac{1}{10}v^4}{\rho^2 + \frac{1}{3}v^2} \quad \text{..... (XII.)}$$

which shows the distance of the center of oscillation below the center of the bob.

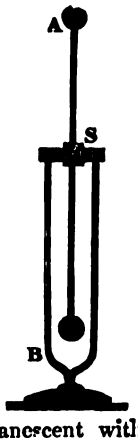
If r the radius of the sphere be known, the latter expression becomes

$$\lambda = \frac{\frac{2}{5} l^2 v \frac{1}{2} r v - \frac{1}{10} v^3}{d(r - \frac{1}{3} v)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(XIII.)}$$

(16.) Let the length of a rectangle be denoted by l, its breadth by $2x$, the distance (along the middle of the rectangle) from one end to the point of suspension by d, then the distance O, from the point of suspension to the center of oscillation, will be

$$O = \frac{d^2 - dl + \frac{1}{3} l^2 + \frac{1}{3} x^2}{\frac{1}{2} l - d} = \frac{1}{2} l - d + \frac{\frac{1}{12} l^2 + \frac{1}{3} x^2}{\frac{1}{2} l - d} \ldots\ldots \text{(XIV.)}$$

whether the figure be a mere geometrical rectangle, or a prismatic metallic plate of uniform density. It follows from this theorem, that a plate of 1 foot long and $\frac{2}{3}$ of a foot broad, suspended at a fourth of a foot from either end, would vibrate as a half second pendulum. Also, that a plate a foot long, $\frac{1}{15}$ of a foot wide, and suspended at $\frac{1}{8}$ of a foot from the middle, would vibrate 36,469 times in 5 hours. *And hence, the length of a foot may be determined experimentally by vibrations.*



(17.) If a thin rod, say of a foot in length, have a ball of an inch diameter at each end, A and B, and a moveable point of suspension, S; then *the time of oscillation of such a pendulum may be made as long as we please*, by bringing the point of suspension nearer to the middle of the rod.

Or, if the point of suspension be fixed, the distance O (and consequently the time of oscillations which is as $\sqrt{O}$) may be varied by placing A nearer to or farther from S. And this is the principle of the *Metronome*, by which musicians sometimes regulate their time.

(18.) If the weight of the connecting rod be evanescent with regard to the weight of the balls A and B; then if R equal the radius of the larger ball, r that of the smaller, D and d the distances of their respective centers from S: we shall have

$$O = \frac{R^3(5D^2 + 2R^2) + r^3(5d^2 + 2r^2)}{5(DR^3) - dr^3)} \quad \ldots\ldots\ldots\ldots\ldots \text{(XV.)}$$

When R and r are equal, this becomes

$$O = (D + d) + \frac{4}{5} \cdot \frac{r^2}{D - d} \quad \ldots\ldots\ldots\ldots\ldots \text{(XVI.)}$$

(19.) If the minor and major axes of an ellipse (or of an elliptical plate of wood or metal) be as 1 to $\sqrt{3}$, or as 1000 to 1732; then,

if it be suspended at one extremity of the minor axis, the center of oscillation will be at the other extremity of that axis, or its oscillations will be performed in the same time as those of a simple pendulum whose length is equal to the minor axis.

The same ellipse also possesses this curious and useful property, viz.:—That any segment or any zone of the ellipse cut off by lines parallel to the major axis, whether it be taken near the upper part of the minor axis, near the middle, or near the bottom of the same, will vibrate in the same time as the whole ellipse, the point of suspension being at an extremity of the minor axis.

26. It is evident from § 17, page 228, that pendulums in different latitudes require to be of different lengths, in order that they may perform their vibrations in the same time; but besides this there is another irregularity in the motion of a pendulum in the same place, arising from the different degrees of temperature. Heat expanding, and cold contracting the rod of the pendulum, certain small variations must necessarily follow in the time of its vibration; to remedy which, various methods have been invented for constructing what are commonly called *compensation pendulums*, or such as shall always preserve the same distance between the center of oscillation and the point of suspension; and of these we shall describe two or three.

Compensation pendulums have received different denominations, from their form and materials, as the *gridiron pendulum*, *mercurial pendulum*, &c.

27. The *gridiron* pendulum consists of five rods of steel, and four of brass, placed in an alternate order, the middle rod being of steel, by which the pendulum ball is suspended; these rods of brass and steel are placed in an alternate order, and so connected with each other at their ends, that while the expansion of the steel rods has a tendency to lengthen the pendulum, the expansion of the brass rods acting upwards tends to shorten it. And thus, when the lengths of the brass and steel rods are duly proportioned, their expansions and contractions will exactly balance and correct each other, and so preserve the pendulum invariably of the same length. Sometimes 3, 7, or 9 rods, are employed in the construction of the gridiron pendulum; and zinc, silver, and other metals may be used instead of brass and steel.

28. The *mercurial pendulum* was invented by Mr. Graham, an eminent clockmaker, about the year 1715. Its rod was made of brass, and branched towards its lower end, so as to embrace a cylindric glass vessel 13 or 14 inches long, and about 2 inches diameter; which, being filled about 12 inches deep with mercury, forms the weight or ball of the pendulum. The height of the mercury in the glass being so proportioned to the length of the rod that its expansion and contraction exactly balanced the expansion and contraction of the pendulum rod, and preserved the distance of the center of oscillation from the point of suspension invariably the same.

This kind of pendulum fell entirely into disuse soon after Graham's

time; but it has lately been re-adopted with considerable success by practical astronomers. A very instructive paper on its principles, construction, and use, has been published by Mr. *F. Baily*, in vol. i. part 2, *Memoirs of the Astronomical Society of London;* in which paper is also contained an extensive and valuable table of the *expansion* of different substances by heat.

29. Reid's *compensation* pendulum is a recent invention of Mr. *Adam Reid*, of Woolwich, the construction of which is as follows:—A N (fig. 189) is a rod of wire, and Z Z a hollow tube of zinc, which slips on the wire, being stopped from falling off by a nut N, on which it rests; and on the upper part of this cylinder of zinc rests the heavy ball B: now the length of the tube Z Z being so adjusted to the length of the rod A N, that the expansions of the two bodies shall be equal with equal degrees of temperature; that is, by making the length of the zinc tube to that of the wire, as the expansion of the wire is to that of zinc, it is obvious that the ball B will in all cases preserve the same distance from A; for just so much as it would descend by the expansion of the wire downwards, so much will it ascend by the expansion of the zinc upwards, and consequently its vibrations will in all temperatures be equal in equal times.

30. Drummond's *compensation* pendulum was proposed by an artist of that name, in Lancashire, more than 70 years ago. A bar of the same metal with the rod of the pendulum, and of the same thickness and length, is placed against the back part of the clock case; from the top of this a piece projects, to which the upper part of the pendulum is connected by two fine pliable chains or silken strings, which just below pass between two plates of brass whose lower edges will always terminate the length of the pendulum at the upper end. These plates are supported on a foot fixed to the back of the case. This bar rests upon an immoveable base on the lower part of the case, and is braced into a proper groove, which admits of no motion any way but that of expansion and contraction in length by heat and cold. In this construction, since the two bars are of equal magnitude and of the same material, their expansions and contractions will always be equal and in opposite directions; so that one will serve to correct and annihilate the effects of the other.

CENTER OF GYRATION AND THE PRINCIPLES OF ROTATION.

1. The *center of gyration* is that point in which, if all the matter contained in a revolving system were collected, the same angular velocity would be generated in the same time by a given force acting at any place as would be generated by the same force acting similarly in the body or system itself.

When the axis of motion passes through the center of gravity, then is the center called the *principal* center of gyration.

2. The distance of the center of gyration from the point of suspension or the axis of motion, is a mean proportional between the distances of the centers of oscillation and gravity from the same point or axis.

If G equal the distance of the center of gravity from the point of suspension, O the distance of the center of oscillation, and R the distance of the center of gyration from the same point, then we have

$$R = \sqrt{G\,O} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(XVII.)}$$

3. The distance R of the center of gyration, from the center or axis of motion, in some of the most useful cases, is given below.

In a circular wheel of uniform thickness ... $R = \text{rad} \sqrt{\frac{1}{2}}$.
In the periphery of a circle revolving about the diameter..... $R = \text{rad} \sqrt{\frac{1}{2}}$.
In the plane of a circle ..... ditto ..... $R = \frac{1}{2}\,\text{rad}$.
In the surface of a sphere ..... ditto ..... $R = \text{rad} \sqrt{\frac{2}{3}}$.
In a solid sphere ..... ditto $R = \text{rad} \sqrt{\frac{2}{5}} = \frac{7}{11}\,r$ nearly.
In a plane ring formed of circles whose radii are R, r, revolving about its center..... $R = \sqrt{\frac{R^2 + r^2}{2}}$.
In a cone revolving about its vertex ..... $R = \frac{1}{2} \sqrt{\frac{12}{5}\,a^2 + \frac{3}{5}\,r^2}$.
In a cone revolving about its axis ..... $R = r \sqrt{\frac{3}{10}}$.
In a paraboloid ..... $R = r \sqrt{\frac{1}{3}}$.
In a straight lever whose arms are R and r, $R = \sqrt{\frac{R^3 + r^3}{3(R + r)}}$.

4. If the matter in any gyrating body were actually to be placed as if in the center of gyration, it ought either to be disposed in the circumference of a circle whose radius is R, or at two points R, R′, diametrically opposite, and each at the distance R from the center.

5. By means of the theory of the center of gyration, and the values of $R = \varrho$, thence deduced, the phenomena of rotation on a fixed axis become connected with those of accelerating forces: for then, if a weight or other moving power P act at a radius r to give rotation to a body whose weight equals W, and the distance of whose center of gyration from the axis of motion equals ϱ, we shall have for the accelerating force, the expression

$$f = \frac{P\,r^2}{P\,r^2 + W\,\varrho^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(XVIII.)}$$

and consequently for the space described by the actuating weight or power P, in a given time t, we shall have the usual formulæ (V.) and (VII.), page 214,

$$s = \frac{f\,t^2}{2}, \text{ and } t = \sqrt{\frac{2\,s}{f}}, \text{ \&c.}$$

introducing the above value of f.

6. In the more complex cases, the distance of the center of gyration from the axis of motion may best be computed from an experiment. Let motion be given to the system, turning upon a horizontal axis, by a weight P acting by a cord over a pulley or wheel whose radius equals r, fixed upon the same axis, and let s be the space through which the weight P descends in the time t, the proposed body whose weight is W turning upon the same axis with the same angular velocity: then

$$R = \rho = \sqrt{\frac{g P t^2 r^2 - 2 s P r^2}{2 s W}} \quad \ldots\ldots\ldots\ldots \text{(XIX.)}$$

Example. A body which weighs 100 lbs. turns upon a horizontal axis, motion being communicated to it by a weight of 10 lbs. hanging from a very light wheel of 1 foot diameter. The weight descends 2 feet in 3 seconds. Required the distance of the center or circle of gyration from the axis of motion.

Here, putting $g = 32$, instead of $32\frac{1}{6}$, we obtain as an approximative result,

$$R = \sqrt{\frac{32 \times 10 \times 9 \times \frac{1}{4} - 4 \times 10 \times \frac{1}{4}}{4 \times 100}} = 1{\cdot}3323 \text{ feet.}$$

7. When the impulse communicated to a body is in a line passing through its center of gravity, all the points of the body move forward with the same velocity, and in lines parallel to the direction of the impulse communicated. But when the direction of that impulse does not pass through the center of gravity, the body acquires a rotation on an axis, and also a progressive motion, by which its center of gravity is carried forward in the same straight line, and with the same velocity, as if the direction of the impulse had passed through the center of gravity.

The progressive and rotatory motion are independent of one another, each being the same as if the other had no existence.

8. When a body revolves on an axis, and a force is impressed, tending to make it revolve on some other, it will not revolve on either, but on a line in the same plane with them, dividing the angle which they contain, so that the sines of the parts are in the inverse ratio of the angular velocities with which the body would have revolved about the said axes separately.

9. A body may begin to revolve on any line as an axis that passes through its center of gravity, but it will not continue to revolve permanently about that axis, unless the opposite rotatory forces exactly balance one another.

This admits of a simple experimental illustration. Suspend a thin circular plate of wood or metal by a cord tied to its edge, from a hook to which a rapid rotation can be given. The plate will at first turn upon an axis which is in the continuation of the cord of rotation; but as the velocity augments, the plane will soon quit that axis, and revolve permanently upon a vertical axis passing through its center of gravity, itself having assumed a horizontal position.

The same will happen if a *ring* be suspended, and receive rotation in like manner. And if a flexible *chain* of small links be united at its two ends, tied to a cord and receive rotation, it will soon adjust itself so as to form a ring, and spin round in a horizontal plane.

Also, if a flattened spheroid be suspended from *any* point, however remote from its minor axis, and have a rapid rotation given it, it will ultimately turn upon its shorter axis posited vertically. This evidently serves to confirm the motion of the earth upon its shorter axis.

10. In every body, however irregular, there are three axes of permanent rotation, at right angles to one another. These are called the *principal axes of rotation;* and they have this remarkable property, that the momentum of inertia with regard to any of them is either a *maximum* or a *minimum*.

CENTRAL FORCES.

1. *Centripetal force* is a force which tends constantly to solicit or to impel a body towards a certain fixed point or center.

2. *Centrifugal force* is that by which it would recede from such center, were it not prevented by the centripetal force.

3. These two forces are, jointly, called *central* forces.

4. If W denote the weight of a body moving in a circle whose radius equals r, with the velocity v, its centrifugal force $= f$ will be

$$f = \frac{v^2 \, W}{g \, r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (XX.)$$

where g equals the force of gravity.

If t equal the time of one revolution, and $\pi = 3{\cdot}14159$, then

$$f = \frac{4 \, \pi^2 \, r \, W}{g \, t^2} = \frac{1{\cdot}2273 \, r}{t^2} \, W \quad . \quad . \quad . \quad . \quad (XXI.)$$

5. When a body describes a *circle* by means of a force directed its center, its actual velocity is every where equal to that which would acquire in falling by the same uniform force through half the radius.

6. This velocity is the same as that which a second body would acquire by falling through half the radius, whilst the first describes a portion of the circumference equal to the whole radius.

7. In equal circles the forces are as the squares of the times inversely.

8. If the times are equal, the velocities are as the radii, and the forces are also as the radii.

9. In general, the forces are as the distances or radii of the circle directly, and the squares of the times inversely.

10. The squares of the times are as the distances directly, and the forces inversely.

11. Hence, if the forces are inversely as the squares of the distances, the squares of the times are as the cubes of the distance That is,

$$\text{if } F : f :: d^2 : D^2, \text{ then } T^2 : t^2 :: D^3 : d^3. \quad . \quad . \quad (XXII.)$$

12. The right line that joins a revolving body and its center of attraction, called the radius *vector*, always describes equal areas in equal times, and the velocity of the body is inversely as the perpendicular drawn from the center of attraction to the tangent of the curve the place of the revolving body.

13. If a body revolve in an elliptic orbit by a force directed to one of the foci, the force is inversely as the square of the distance: a

the mean distances and the periodic times have the same relation as in Art. 11. *This comprehends the case of the planetary motions.*

14. If the force which retains a body in a curve increase in the simple ratio as the distance increases, the body will still describe an ellipse; but the force will in this case be directed to the center of the ellipse; and the body in each revolution will twice approach towards it, and again twice recede from that point.

15. On the principles of central forces depends the operation of the conical pendulum applied as a *governor* or regulator to steam engines, water mills, &c.

This contrivance will be readily comprehended from fig. 190, where A *a* is a vertical shaft capable of turning freely upon the sole *a*. C D, C F, are two bars which move freely upon the center C, and carry at their lower extremities two equal weights, P, Q; the bars C D, C F, are united, by a proper articulation, to the bars G, H, which latter are attached to a ring I, capable of sliding up and down the vertical shaft A *a*. When this shaft and connected apparatus are made to revolve, in virtue of the centrifugal force, the balls P Q fly out more and more from A *a*, as the rotatory velocity increases: if, on the contrary, the rotatory velocity slackens, the balls descend and approach A *a*. The ring I *ascends* in the former case, *descends* in the latter: and a lever connected with I may be made to regulate the energy of the moving power. Thus, in the steam engine, the ring may be made to act on the valve by which the steam is admitted into the cylinder; to augment its opening when the motion is slackening, and reciprocally diminish it when the motion is accelerated.

The construction is often so modified, that the flying out of the balls causes the ring I to be depressed, and *vice versâ;* but the general principle is the same.

Here, if the vertical distance of P or Q below C, be denoted by d, the time of one rotation of the regulator by t, and 3·141593 by π, the theory of central forces gives

$$t = 2\pi\sqrt{\frac{d}{32\frac{1}{6}}} = 1{\cdot}10784\sqrt{d} \quad . \quad . \quad \text{(XXIII.)}$$

Hence, the periodic time varies as the square root of the altitude of the conic pendulum, let the radius of the base be what it may. Also, when I C Q = I C P = 45°, the centrifugal force of each ball is equal to its weight.

16. As the practical utility of the conical pendulum depends in a great degree upon its sensibility, or the change which must take place in its velocity before it will move the ring I to the required extent, we subjoin the following formulæ for determining the weight of the balls P and Q, for any degree of sensibility which may be required.

Let W equal the weight of both the balls, P the power required to move the valve (or produce whatever effect may be required) when applied to the ring in the direction of the spindle, N_1 the number of revolutions which the governor is intended to make per second, N_2 the number of revolutions which the same must make to move the

ring I, d as before the vertical distance from the plane of the balls P and Q to the point C, b the distance C D or C F, l the distance C P, a the distance of the point D from the axis of the spindle, c the vertical distance from the plane joining the points D and F from the point I, and r the radius of the circle described by the balls, or their distance from the axis of the spindle; then

$$d = \frac{1}{1{\cdot}22731\ N_1^2} \quad \ldots\ldots\ldots \text{(XXIV.)}$$

$$W = \frac{a\,b\,P}{1{\cdot}22731\ c\,l\,r\,(N_2^2 - N_1^2)} \quad \ldots\ldots \text{(XXV.)}$$

If the distance C D is made equal to D I, so that the four rods form a parallelogram, this last formula becomes

$$W = \frac{b\,P}{1{\cdot}22731\ d\,l\,(N_2^2 - N_1^2)} \quad \ldots\ldots \text{(XXVI.)}$$

17. In the foregoing formulæ the center of oscillation of the balls and system of rods is assumed to coincide with the center of the balls, an assumption not strictly correct, although sufficiently so for all practical purposes. Should, however, greater accuracy be required, the true center of oscillation having been found, the following substitutions must be made, when the preceding formulæ will give an accurate result; viz., for l, the distance from C to the center of oscillation, and for r the distance of the center of oscillation from the vertical axis of the spindle.

INQUIRIES CONNECTED WITH ROTATION AND CENTRAL FORCES.

1. Suppose the diameter of a grindstone to be 44 inches, and its weight half a ton; suppose also that it makes 386 revolutions in a minute. What will be the centrifugal force, or its tendency to burst?

$$\text{Here the velocity} = \frac{44}{12} \times 3{\cdot}1416 \times \frac{386}{60} = 74{\cdot}106;$$

then by substituting these values in formula (XX.) we have

$$f = \frac{74{\cdot}106^2 \times {\cdot}5}{32 \times \frac{22}{12}} = 46{\cdot}8 \text{ tons.}$$

the measure of the required tendency.

2. If a fly wheel 12 feet diameter, and 3 tons in weight, revolves in 8 seconds: and another of the same weight revolves in 6 seconds, what must be the diameter of the last, when their centrifugal force the same?

By formula (XXI.) $F : f :: \frac{D}{T^2} : \frac{d}{t^2}$. Therefore, since F is = f, $\frac{D}{T^2} = \frac{d}{t^2}$, or $d = \frac{D\,t^2}{T^2} = \frac{12 \times 36}{64} = 6\frac{3}{4}$ feet, the answer.

3. If a fly of 12 feet diameter revolve in 8 seconds, and another of the same diameter in 6 seconds: what is the ratio of their weights when their central forces are equal?

By § 7, page 232, the forces are as the squares of the times inversely when the weights are equal: therefore, when the weights are unequal, they must be directly as the squares of the times, that the central forces may be equal.

Hence $w : W :: 36 : 64 :: 1 : 1\frac{7}{9}$

That is, the weight of the more rapidly to that of the more slowly revolving fly, must be as 1 to $1\frac{7}{9}$, in the case proposed.

4. If a fly 2 tons weight and 16 feet diameter, is sufficient to regulate an engine when it revolves in 4 seconds; what must be the weight of another fly of 12 feet diameter revolving in 2 seconds, so that it may have the same power upon the engine?

Here, by § 9, page 232, we must have $\frac{W D}{T^2} = \frac{w d}{t^2}$; therefore

$w = \frac{W D t^2}{d T^2} = \frac{40 \text{ cwt.} \times 16 \times 4}{12 \times 16} = \frac{160}{12} = 13\frac{1}{3}$ cwt., the weight of the smaller fly.

Note.—A fly should always be made to move rapidly. If it be intended for a mere regulator, it should be near the *first mover*. If it be intended to accumulate force in the *working point*, it must not be far separated from it.

5. Given the radius R of a wheel, and the radius r of its axle, the weight of both, w, and the distance of the center of gyration from the axis of motion, ϱ; also a given power P acting at the circumference of the wheel; to find the weight W raised by a cord folding about the axle, so that its momentum shall be a maximum. Here

$$W = \frac{\sqrt{(R^4 P^2 + 2 R^2 P \varrho^2 w + \varrho^4 w^2 + P w R r \varrho^2 + P^2 R^3 r)} - R^2 P - \varrho^2 w}{r^2}.$$

Cor. 1. When $R = r$, as in the case of the single fixed pulley, then

$$W = \sqrt{(2 P^2 R^3 + 2 R P \varrho^2 w + \frac{\varrho^4}{R} w^2 + P w R \varrho^2)} - \frac{\varrho^2}{R^2} w - P.$$

Cor. 2. When the pulley is a cylinder of uniform matter $\varrho^2 = \frac{1}{2} R^2$, and the expression becomes

$$W = \sqrt{\{R^3 (2 P^2 + \tfrac{3}{2} P w + \tfrac{1}{4} w^2)\}} - \tfrac{1}{2} w - P.$$

6. Let a given power P be applied to the circumference of a wheel, whose radius equals R, to raise a weight W at its axle, whose radius is r, it is required to find the ratio of R and r when W is raised with the greatest momentum; the characters W and ϱ denoting the same as in the last proposition.

$$\text{Here } r = \frac{R\sqrt{\{P^2 W^2 + P^2 (q + W)\}}\, P W}{P (q + W)}.$$

Cor. When the inertia of the machine is evanescent, with respe to that of P + W, then is $r = R\sqrt{\left(1 + \frac{P}{W}\right)} - 1$.

7. In any machine whose motion accelerates, the weight will b moved with the greatest velocity when the velocity of the power is that of the weight as $1 + P\sqrt{\left(1 + \frac{P}{W}\right)}$ is to 1; the inertia the machine being disregarded.

8. If in any machine whose motion accelerates, the descent of o weight causes another to ascend, and the descending weight be give the operation being supposed continually repeated, the effect will greatest in a given time when the ascending weight is to the d scending weight, as 1 to 1·618, in the case of equal heights; and other cases when it is to the exact counterpoise in a ratio which always between 1 to $1\frac{1}{2}$ and 1 to 2.

9. The following general proposition with regard to rotatory m tion will be of use in the more recondite cases.

If a system of bodies be connected together and supported at point which is not the center of gravity, and then left to descend that part of their weight which is not supported, $2g$ multiplied i the sum of all the products of each body into the space it has p pendicularly descended, will be equal to the sum of all the produ of each body into the square of its velocity.

SECT. IV. *Percussion or Collision of Bodies in motion.*

1. In the ordinary theory of percussion, or collision, bodies regarded as either *hard*, *soft*, or *elastic*. A *hard* body is that wh parts do not yield to any stroke or percussion, but retains its fig unaltered. A *soft* body is that whose parts yield to any stroke impression, without restoring themselves again, the shape of the b remaining altered. An *elastic* body is that whose parts yield to stroke, but presently restore themselves again, so that the body gains the same figure as before the stroke. When bodies which h been subjected to a stroke or pressure return only in part to th original form, the *elasticity* is then *imperfect:* but if they rest themselves entirely to their primitive shape, and employ just as m time in the restoration as was occupied in the compression, then the elasticity *perfect.*

It has been customary to treat only of the collision of bodies p fectly hard or perfectly elastic: but as there do not exist in nat any bodies (which we know) of either the one or the other of th kinds, the usual theories are but of little service in practical

chanics, except as they may suggest an extension to the actual circumstances of nature and art.

2. The general principle for determining the motions of bodies from percussion, and which belongs equally to both elastic and non-elastic bodies, is this: viz. that there exists in the bodies the same momentum, estimated in any one and the same direction, both before the stroke and after it. And this principle is the immediate result of the law of nature or motion, that reaction is equal to action, and in a contrary direction; from whence it happens, that whatever motion is communicated to one body by the action of another, exactly the same motion does this latter lose in the same direction, or exactly the same does the former communicate to the latter in the contrary direction.

From this general principle too it results, that no alteration takes place in the common center of gravity of bodies by their actions upon one another; but that the said common center of gravity perseveres in the same state, whether of rest or of uniform motion, both before and after the impact.

3. If the impact of two perfectly hard bodies be direct, they will, after impact, either remain at rest, or move on uniformly together with different velocities, according to the circumstances under which they met.

Let B and b represent two perfectly hard bodies, and let the velocity of B be represented by V, and that of b by v, which may be taken either positive or negative, according as b moves in the same direction as B, or contrary to that direction, and it will be zero when b is at rest. This notation being understood, all the circumstances of the motions of the two bodies, after collision, will be expressed by the formula:

$$\text{velocity} = \frac{\mathrm{B}\,\mathrm{V} \pm b\,v}{\mathrm{B} + b}$$

which being accommodated to the three circumstances under which v may enter, become

$$\text{(I.) velocity} = \frac{\mathrm{B}\,\mathrm{V} + b\,v}{\mathrm{B} + b} \left\{ \begin{array}{l} \text{when both bodies moved in} \\ \text{the same direction} \end{array} \right.$$

$$\text{(II.) velocity} = \frac{\mathrm{B}\,\mathrm{V} - b\,v}{\mathrm{B} + b} \left\{ \begin{array}{l} \text{when the bodies moved in} \\ \text{contrary directions} \end{array} \right.$$

$$\text{(III.) velocity} = \frac{\mathrm{B} + b}{\mathrm{B}\,\mathrm{V}} \left\{ \begin{array}{l} \text{when the body } b \text{ was at} \\ \text{rest.} \end{array} \right.$$

These formulæ arise from the supposition of the bodies being perfectly hard, and consequently that the two after impact move on uniformly together as one mass. In cases of perfectly elastic bodies, other formulæ have place which express the motion of each body separately; as in the following proposition.

4. If the impact of two perfectly elastic bodies be direct, their *relative* velocities will be the same both before and after impact, or

they will recede from each other with the same velocity with which they met; that is, they will be equally distant, in equal times, both before and after their collision, although the absolute velocity of each may be changed. The circumstances attending this change of motion in the two bodies, using the above notation, are expressed in the two following formulæ:

$$\frac{2bv + (B - b)V}{B + b} = \text{velocity of B} \ldots \text{(IV.)}$$

$$\frac{2BV + (B - b)v}{B + b} = \text{velocity of } b \ldots \text{(V.)}$$

which needs no modification, when the motion of b is in the same direction with that of B.

5. In the other case of b's motion, the general formulæ become—

$$\frac{-2bv + (B - b)V}{B + b} = \text{velocity of B} \ldots \text{(VI.)}$$

$$\frac{2BV - (B - b)v}{B + b} = \text{velocity of } b \ldots \text{(VII.)}$$

when b moves in a contrary direction to that of B, which arises from taking v negative. And

$$\frac{(B - b)V}{B + b} = \text{velocity of B} \ldots \ldots \text{(VIII.)}$$

$$\frac{2Bv}{B + b} = \text{velocity of } b \ldots \ldots \ldots \text{(IX.)}$$

when b was at rest before impact, that is, when $v = 0$.

6. If a perfectly hard body B, (fig. 191,) impinge obliquely upon a perfectly hard and immoveable plane A D, it will after collision move along the plane in the direction C A.

And its velocity before impact
Is to its velocity after impact
As radius
Is to the cosine of the angle B C D.

But if the body be elastic it will rebound from the plane in the direction C E, with the same velocity, and at the same angle with which it met it, that is, the angle A C E will be equal to the angle B C D.

7. The force with which a body impinging obliquely strikes a plane, is to the same if it had acted perpendicularly, as the sine of the angle (B C D) of incidence, is to radius.

8. In the case of direct impact, if B be the striking body, b the body struck, V and v their respective velocities before impact, V_1 and v_1 their velocities afterwards; then the two following are general formulæ: viz.

$$V_1 = V - n\left(\frac{V - v}{B + b}\right)b \quad \ldots\ldots\ldots \text{(X.)}$$

$$v_1 = v + n\left(\frac{V - v}{B + b}\right)B \quad \ldots\ldots\ldots \text{(XI.)}$$

In these, if $n = 1$, they serve for non-elastic bodies; if $n = 2$, for bodies perfectly elastic. If the bodies be imperfectly elastic, n has some intermediate value.

When the body struck is at rest, the preceding equations become

$$V_1 = V - \frac{n\,V\,b}{B + b} \quad \ldots\ldots\ldots\ldots \text{(XII.)}$$

$$v_1 = \frac{n\,V\,B}{B + b} \quad \ldots\ldots\ldots\ldots\ldots \text{(XIII.)}$$

$$n = \frac{v_1\,(B + b)}{B\,V} \quad \ldots\ldots\ldots\ldots \text{(XIV.)}$$

from which the value of n may be determined experimentally.

9. In the usual apparatus for experiments on collision, balls of different sizes and of various substances are hung from different points of suspension on a horizontal bar. M A N (fig. 192) is an arc of a circle whose center is S; and its graduations, 1, 2, 3, 4, 5, &c., indicate the lengths of chord lines measured from the lowest point D. Any ball, therefore, as P, may be drawn from the vertical, and made to strike another ball hanging at the lowest point, with any assigned velocities, the height to which the ball struck ascends on the side A M furnishing a measure of its velocity; and from that the value of n may be found from the last equation. Balls not required in an individual experiment, may be put behind the frame as shown at A and B.

The cup C may be attached to a cord, and carry a ball of clay, &c. when required.

Example. Suppose that a ball weighing 4 ounces strikes another ball of the same substance weighing 3 ounces, with a velocity of 10, and communicates to it a velocity of $8\frac{1}{4}$: what, in that case, will be the value of n?

Here $n = \frac{v_1\,(B + b)}{B\,V} = \frac{8\frac{1}{4} \times 7}{4 \times 10} = \frac{57\frac{3}{4}}{40} = 1{\cdot}44375$ the index of the degree of elasticity; *perfect* elasticity being indicated by 2.

SECT. V. *On the Mechanical Powers.*

1. The most complicated machinery is nothing more than a combination, or constant repetition, of a few simple mechanical expedients for modifying and changing the direction of the several forces

or pressures, transmitted by them through the machine. A certain pressure or force, being communicated to what is termed the first or *prime mover* of the machinery, is thus transmitted through every part of the machinery, being regulated and modified in such a manner as to produce the effect required, at what is termed the *working point*. Thus, by means of the machine termed a crane, a man, by applying the pressure of his hand (amounting to perhaps about 30lbs.) to the handle of the crane, is enabled, through the intervention of the machinery, to raise an enormous weight, as, for instance, say 12,000 lbs., which, without some such expedient, would defy all his efforts to move.

2. These elementary parts of which more complicated machinery is composed, have been called the *mechanical powers*, a term which is liable to lead to a misapprehension of the effects which they are really capable of producing. For, (continuing the use of the previous illustration,) although the man, through the instrumentality of the crane, by exerting a force of 30lbs., is enabled to lift 12,000lbs., or four hundred times the amount, it will be found that his hand will move four hundred times as fast as the weight, and, therefore, what advantage he may appear to gain in the weight which he is enabled to lift, he really loses again by the length of time which it takes him to raise it to any given height; and which would be found by experiment to be just what he would require to raise separately four hundred weights, each of 30lbs., to the same height.

3. The number of the mechanical powers is usually reckoned to be six: viz. the *lever*, the *wheel and axle*, the *pulley*, the *inclined plane*, the *wedge*, and the *screw*.

4. In treating of these machines, we use the word *power*, to denote the force which is supposed to be exerted at the origin of the machine, and the word weight to denote the effect which that force produces at the working point of the machine: and we shall express them by the letters P and W respectively.

LEVERS.

1. A *lever* is an inflexible bar, whether straight or bent, and supposed capable of turning upon a fixed, unyielding point, called *a fulcrum*. There are three kinds of levers.

2. When the *fulcrum* is between the power and the weight, as in fig. 193, the lever is said to be of the *first* kind.

3. When the *weight* is between the power and the fulcrum, as in fig. 194, the lever is of the *second* kind.

4. When the *power* is between the weight and the fulcrum, as in fig. 195, the lever is of the *third* kind.

The *bent lever*, as employed in the operation of drawing a nail with a hammer, is sometimes considered as a fourth kind, but is really a lever of the first kind.

5. In all these cases, where there is an equilibrium, it is indicated by this general property, that the product of the weight into the distance at which it acts, is equal to the product of the power into the

distance at which it acts: *the distances being estimated in directions perpendicular to those* in which the weight and power act respectively. Thus, in each of the three preceding figures,

$$P \cdot AF = W \cdot BF,$$

or the power and weight are reciprocally as the distances at which they act.

If, in fig. 193, for example, the arm A F were 4 times F B, 4 lbs. hanging at B would be balanced by 1 lb. at A; and if A F were 5 times F B, 1 lb. at A would balance 5 lbs. at B; and so on.

6. If several weights hang upon a lever, some on one side of the fulcrum, some on the other, then there will be an equilibrium, when the sum of the products of the weights into their respective distances on one side, is equal to the several products of weights and distances on the other side.

For, the product of the weight into its distance from the fulcrum is, is the same as its *moment* about the point F, and therefore the foregoing proposition is a direct consequence of the principle of the equality of the moments of any forces in equilibrium, about any fixed point, as explained at § 5, page 189.

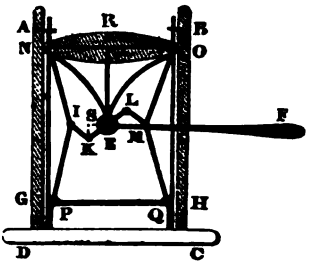
7. When the *weight* of the lever is to be taken into the account, proceed just as though it were a separate weight suspended at the place of its center of gravity.

8. When two, three, or more levers act one upon another in succession, then the entire mechanical advantage which they afford, is found by taking, not the *sum*, but the *product* of their separate advantages. Thus, if the arms of three levers, acting thus in connection, are as 3 to 1, 4 to 1, and 5 to 1, then the joint advantage is that of $3 \times 4 \times 5$ to 1, or 60 to 1: so that 1 lb. would, through their intervention, balance 60.

9. In the *first* kind of lever the pressure upon the fulcrum $= P + W$: in the other two it is $= P \sim W$.

10. Upon the foregoing principles depends the nature of scales and beams for weighing all bodies. For, if the distances be equal, then will the weights be equal also; which gives the construction of the common scales. And the Roman statera, or steel-yard, is also a lever, but of unequal arms or distances, so contrived that one weight only may serve to weigh a great many, by sliding it backwards and forwards to different distances upon the longer arm of the lever. In the common *balance*, or scales, if the weight of an article when ascertained in one scale is not the same as its weight in the other, *the square root of the product of those two weights will give the true weight.*

11. From numerous examples of the power and use of the lever, one which shows its manner of application in the printing-presses of the late Earl Stanhope may be advantageously introduced.



In the adjoining figure, let A B C D be the general frame of the press, connected by the cross pieces N O, D C. E is a center connected with the frame by the bars E N, E R, E O. To this center are fixed a bar K L, and a lever E F, to which the hand is applied when the press is used.

There are also several other pieces connected by joints at N, G, I, K, L, M, O, H, which are so adjusted to each other, that when the hand is applied to the lever E F at F, by pressing it downwards K L is brought into a horizontal line or parallel to G H or D C, in which situation N I G, O M H, also form each a straight line. It is evident that the nearer these different pieces, a above mentioned, are to a straight line the greater is the lever E F. in proportion to the perpendicular K S at the other end of the lever E K, formed by a perpendicular from K falling on F E produced Consequently a small force applied at F will be sufficient to produce *very great effect* at K, when I K, K E are nearly in a straight line, an so on, for the other pieces above mentioned.

Hence the force applied by hand at F must be very considerabl in forcing down G H, which slides on iron cylindrical bars, or in pressing any substance placed in the aperture P Q, between the b or plate and the frame D C.

This contrivance is now often introduced into mechanism, und the name of the *toggle*, or *knee-joint*.

WHEEL AND AXLE.

1. The nature of this machine is suggested by its name. To it may be referred all turning or wheel-machines containing wheels of different radii; as well-rollers and handles, cranes, capstans, wi lasses, &c.

2. In the wheel and axle the mechanical property is the same as in the lever: viz. P . A C = W . B C (fig 196); and the reason is evident, because the wheel and axle is only a kind of perpetual lever.

3. When a series of wheels and axles act upon each other, so as to transmit and accumulate a mechanical advantage, whether the communication be by means of cords and bèlts, or of teeth and pinions, the weight will be to the power, not as the *sum*, but as the continual *product* of the radii of the wheels to the continual product of the radii of the axles. Thus, if the radii of the axles, *a*, *b*, *c*, *d*, *e*, (fig. 197,) be each three inches, while the radii of the wheels, A, B, C, D, E, be 9, 6, 9, 10, and 12 inches respectively: then W : P :: 9 × 6 × 9 × 10 × 12 : 3 × 3 × 3 × 3 × 3 :: 240 : 1. A computation, however, in which the effect of friction is disregarded.

pulley is a small wheel, commonly made of wood or brass, of turning upon an iron axis passing through its center, and a block, and the use of which is, that by means of a cord ound its circumference, we are enabled to alter the *direction* rce in any way that we choose. The pulley is either single purpose, or combined with others to obtain a mechanical ad-. It is also either fixed or moveable, according as it is fixed lace, or moves up and down with the weight or power *.

a power sustain a weight by means of a fixed pulley, as in , the power and weight are equal.

iere are several different methods of combining pulleys to- or forming as they are termed separate systems of pulleys, for g a mechanical advantage.

first system is shown in fig. 199, in which there are two f pulleys, the upper fixed, and the lower attached to the and rising with it; only one continuous rope is employed, ay be attached to either block, and passed successively round lley. In this system, the weight sustained at W, is to the sustaining it at P, as the number of ropes *engaged* between blocks, is to 1; thus in fig. 199 *a*, the mechanical advantage, , and in fig. 199 *b*, it is 4 to 1.

he second system is when there are as many separate ropes are pulleys, each rope being attached to a fixed point at one , passing under one of the pulleys, has its other end attached lock of the next pulley above it. This system is represented

seful combination of the wheel and axle, a fixed and a moveable pulley

ted in the marginal dia- The load, as of stones or build a wall, is raised from thus: a rope B P L is fixed

in fig. 200, in which the weight is to the power as 2^n is to 1, n being the number of pulleys or ropes; thus in the figure it is 16 to 1.

5. The third system (fig. 201) is very similar to the second, there being as many separate ropes as there are pulleys, but the arrangement is somewhat different, the ropes being all attached at one end to the weight to be raised, and the upper pulley being fixed; in this system the weight is to the power, as $2^n - 1$ is to 1; or, in the figure, as 31 is to 1.

INCLINED PLANE.

1. A body which touches a plane only in one point, can only remain in equilibrio so long as the forces which act upon it are reducible to a single force which shall act in a direction perpendicular to the plane at the point of contact.

2. When a power sustains a heavy body in equilibrio upon an inclined plane, then the power, the weight, and the pressure upon the plane, will be respectively, as the sine of the plane's inclination, the cosine of the angle which the direction of the power makes with the plane, and the cosine of the angle which the direction of the power makes with the horizon.

3. Or, putting P for the pressure on the plane, we have from fig. 202,

$$W : P :: \cos abc : \sin cbd$$

$$W : p :: \cos abc : \cos abd.$$

4. When the direction of the power is parallel to the plane, the power, weight, and pressure on the plane, are respectively as the height, length, and base of the plane; or as the sine of inclination, radius, and cosine of inclination.

Thus, suppose a plank 12 feet long were laid aslant from the ground to a window 4 feet high: then, since the length is three times the height, a power of 50 lbs. would sustain three times as much, or 150 lbs. upon the plank; and a greater power, as 55 or 60 lbs., would cause that weight to ascend.

The truth of these propositions may be confirmed most readily, by attaching the weight to a cord tied to a spring steel-yard, by which the relations between the entire weight, and that supported by a cord either parallel to the inclined plane or in any other direction, may at once be measured.

5. If two weights W, W_1, sustain each other upon two inclined planes A C, C B (fig. 203), which have a common altitude C D, by means of a cord which runs freely over a pulley and is parallel to both planes, then will $W : W_1 :: A C : C B$.

6. When a heavy body is supported by two planes, as in fig. 204, then, if the weight of the body be represented by the sine of the angle comprehended between the two planes, the pressures upon them are reciprocally as the sines of the inclinations of those planes to the horizon: viz.,

If the weight equal sin A B D
the pressure on A B will equal sin D B I
and the pressure on B D will equal sin A B H.

Thus, suppose the angle A B H was 30°, D B I 60°, and consequently A B D 90°: since the natural sines of 90°, 60°, and 30°, are 1, ·866, and ·5 respectively, or nearly as 100, 86·6, and 50; if the heavy body weigh 100 lbs., the pressure upon A B would be 86·6 lbs., and upon B D 50 lbs.

This proposition is of very extensive utility, comprehending the pressure of arches on their piers, of buttresses against walls, or upon the ground, &c., because the circumstance of one of the planes becoming either horizontal, or vertical, will not affect the general relation above exhibited.

WEDGE.

1. A *wedge* is a triangular prism, or a solid conceived to be generated by the motion of a plane triangle parallel to itself upon a straight line which passes through one of its angular points. The wedge is called *isosceles*, *rectangular*, or *scalene*, according as the generating triangle is isosceles, right-angled, or scalene. It is very frequently used in cleaving wood, as represented in the figure, and often in raising great weights.

2. When a resisting body is sustained against the face of a wedge, by a force acting at right angles to its direction; in the case of equilibrium, the power is to the resistance as the sine of the semi-angle of the wedge, is to the sine of the angle which the direction of the resistance makes with the face of the wedge; and the sustaining force will be as the cosine of the latter angle.

3. When the resistance is made against the face of a wedge by a body which is not sustained, but will adhere to the place to which it is applied without sliding, the power is to the resistance, in the case of equilibrium, as the cosine of the difference between the semi-angle of the wedge and the angle which the direction of the resistance makes with the face of the wedge, is to radius.

4. When the resisting body is neither sustained nor adheres to the point to which it is applied, but slides freely along the face of the wedge, the power is to the resistance as the product of the sines of the semi-angle of the wedge and the angle in which the resistance is inclined to its face is to the square of radius.

SCREW.

1. The screw is a spiral thread or groove cut round a cylinder, and every where making the same angle with the length of it. So that if the surface of the cylinder, with this spiral thread on it, were unfolded or *developed* into a plane, the spiral thread would form a straight inclined plane, whose length would be to its height, as the circumference of the cylinder is to the distance between two threads of the screw: as is evident by considering that, in making one round, the spiral rises along the cylinder the distance between the two threads.

2. The energy of a power applied to turn a screw round, is to the force with which it presses upward or downward (setting aside

the friction), as the distance between two threads is to th ference where the power is applied: viz., as the circumfere is to the distance B I (fig. 205).

3. The *endless screw*, or *perpetual screw*, is one which and turns a toothed wheel D F (fig. 206), without a concave screw; being so called because it may be turned for eve coming to an end. From the diagram it is evident that screw turns once round, the wheel only advances the d one tooth.

4. If the power applied to the lever, or handle of a screw, A B, be to the weight, in a ratio compounded of the of the axis of the wheel, E H, to the periphery describe power in turning the handle, and of the revolutions of the to the revolutions of the screw C B, the power will be weight. Hence,

5. As the motion of the wheel is very slow, a small p raise a very great weight by means of an endless screw. A fore the chief use of such a screw is, either where a great to be raised through a little space, or where only a sl motion is wanted. For which reason it is very serviceable and watches.

The screw is of admirable use in the mechanism of mi and in the adjustments of astronomical and other instrun refined construction.

6. The mechanical advantage of a compound machin determined by analyzing its parts, finding the mechanical of each part severally, and then blending or compoundi ratios*. Thus, if m to 1, n to 1, r to 1, and s to 1, separate advantages; then $m n r s$ to 1, will measure the ad the system.

* The marginal representation of a common construction of a *crane* to raise heavy loads, will serve to illustrate this. By human energy at the handle *a*, the pinion *b* is turned; that gives motion to the wheel W, round whose axle, *c*, a cord is coiled; which cord passes over the fixed pulley, *d*, and thence over the fixed triple block, B, and the moveable triple block, P, below which the load, L, hangs. Now, if the radius of the handle be 6 times that of the pinion, the radius of the wheel W 10 times that of its axle, and a power equivalent to 30 lbs. be exerted at *a*; then, since a triple moveable pulley gives a mechanical advantage of 6 to 1, we shall have

$$30 \times 6 \times 10 \times 6 = 10800 \text{ lbs.}$$

and such would be the load, L, that might be raised by a powe applied at *a*, were it not for the loss occasioned by friction.

making such a calculation, the subjoined table, exhibiting at one the ratio of the power to the weight, in all the simple mechaniwers, will be of service.

ption of Power.	Ratio of P : W.	Symbol.	Observation.
. . . . .	$1 : \frac{L}{l}$	L equals the leverage of the power, l that of the weight.	In all the cases both the friction and weight of the machines themselves are neglected.
and axle . .	$1 : \frac{R}{r}$	R = rad. of the wheel, r that of the axle.	
ed wheels .	$1 : \frac{N}{n}$	N = No. of teeth in the wheel, n the No. in the pinion.	
s:—			
tem (fig. 199)	$1 : n$	n equals the No. of pulleys, both fixed and moveable.	The strings are all supposed to be parallel.
lo. (fig. 200)	$1 : 2^n$		
lo. (fig. 201)	$1 : 2^n - 1$		
ed plane . .	$1 : \frac{\lambda}{h}$	λ = the length of the plane, h its height.	When the power acts in a direction parallel to the plane.
e.	$1 : \frac{\lambda}{b}$	λ = the length of the side of the wedge, b the thickness of its back.	When the resistance is perpendicular to the side of the wedge, the wedge being single.
. . . . .	$1 : \frac{c}{d}$	c = the circumference of the circle described by the power, and d the vertical distance between two threads.	

CHAP. IV.

HYDROSTATICS.

SECT. I. *General Definitions.*

1. HYDROSTATICS comprises the doctrine of the pressure and the equilibrium of non-elastic fluids, as water, mercury, &c., and that of the weight and pressure of solids immersed in them.

2. A *perfect fluid* is a body whose parts are very minute, yielding to any force impressed upon it (however small), and by so yielding are easily moved among themselves.

Some attempt to give mechanical ideas of a fluid body, by comparing it to a heap of sand; but the impossibility of giving fluidity by any kind of mechanical comminution, will appear by considering two of the circumstances necessary to constitute a fluid body: 1. That the parts, notwithstanding any compression, may be moved in relation to each other, with the smallest conceivable force, or will give no *sensible resistance* to motion within the mass in any direction. 2. That the parts shall gravitate to each other, whereby there is a constant tendency to arrange themselves about a common center, and form a spherical body; which, as the parts do not resist motion, is easily effected in small bodies. Hence the appearance of drops always takes place when a *fluid* is in proper circumstances. It is obvious that a body of sand can by no means conform to these circumstances.

Different fluids have different degrees of fluidity, according to the facility with which the particles may be moved amongst each other. Water and mercury are classed among the most perfect fluids. Many fluids have a very sensible degree of tenacity, and are therefore called *viscous* or *imperfect* fluids.

3. Fluids may be divided into *compressible* and *incompressible*, or *elastic* and *non-elastic* fluids. A *compressible* or *elastic* fluid is one whose apparent magnitude is diminished as the pressure upon it is increased, and increased by a diminution of pressure. Such is air, and the different vapours. An *incompressible* or *non-elastic* fluid (called also a *liquid*) is one whose dimensions are not sensibly affected by any augmentation of pressure. Water, mercury, oil, &c, are generally ranged under this class.

It has been of late years proposed to limit the application of the term *fluids* to those which are *elastic*, and to apply the word *liquid* to such as are *non-elastic*.

4. The *specific gravity* or density of any solid or fluid body, is the absolute weight of a known volume of that substance; namely, of that volume which we take for unity in measuring the capacities of bodies.

SECT. II. *Pressure and Equilibrium of Non-elastic Fluids.*

1. Fluids press *equally in all directions*, upwards, downwards, aslant, or laterally.

This constitutes one essential difference between fluids and solids, solids pressing only *downwards*, or in the direction of gravity.

2. The upper surface of a gravitating fluid at rest is horizontal.

3. The pressure of a fluid on every particle of the vessel containing it, or of any other surface, real or imaginary, in contact with it, is equal to the weight of a column of the fluid, whose base is equal to that particle, and whose height is equal to its depth below the upper surface of the fluid.

4. If, therefore, any portion of the upper part of a fluid be replaced by a part of the vessel, the pressure against this from below will be the same which before supported the weight of the fluid removed, and every part remaining in equilibrium, the pressure on the bottom will be the same as it would be if the vessel were a prism or a cylinder.

5. Hence, the smallest given quantity of a fluid may be made to produce a pressure capable of sustaining any proposed weight, either by diminishing the diameter of the column and increasing its height, or by increasing the surface which supports the weight.

6. The perpendicular pressure of a fluid on any surface, whether vertical, oblique, or horizontal, is equal to the weight of a column of the fluid whose base is equal to the surface pressed, and height equal to the distance of the center of gravity of that surface below the upper horizontal surface of the fluid.

7. Fluids of different specific gravities that do not mix, will counterbalance each other in a bent tube, when their heights above the surface of junction are inversely as their specific gravities.

A portion of fluid will be quiescent in a bent tube, when the upper surface in both branches of the tube is in the same horizontal plane, or is equidistant from the earth's center. And water poured down one branch of such a tube (whether it be of uniform bore throughout, or not) will rise to its own level in the other branch.

Thus, water may be conveyed by pipes from a spring on the side of a hill, to a reservoir of equal height on another hill.

8. The ascent of a body in a fluid of greater specific gravity than itself, arises from the pressure of the fluid upwards against the under surface of the body.

9. The *center of pressure* is that point of a surface against which any fluid presses, to which if a force equal to the whole pressure

were applied it would keep the surface at rest, or balance
ency to turn or move in any direction.

10. If a plane surface which is pressed by a fluid be pr
the horizontal surface of it, and their common intersection be
as the axis of suspension, *the centers of percussion and of*
will be at the same distance from the axis.

11. The center of pressure of a parallelogram, whose upp
in the plane of the horizontal level of the liquid, is at $\frac{2}{3}$ o
(measuring downwards) that joins the middles of the two l
sides of the parallelogram.

12. If the base of a triangular plane coincides with t
surface of the water, then *the center of pressure is at the*
the line drawn from the middle of the base to the verte
triangle. But, if the vertex of the triangle be in the uppe
of the water, while its base is horizontal, *the center of pr*
at $\frac{3}{4}$ *of the line drawn from the vertex to bisect the base.*

13. If in any closed vessel containing a fluid suppos
without weight, an opening or orifice be made and any pr
applied, that pressure will be equally distributed over t
interior surface of the vessel; and if the fluid has a wei
own, the pressure upon any point will equal the sum of t
buted pressure and the pressure occasioned by the weight of
at that point.

ILLUSTRATIONS AND APPLICATIONS.

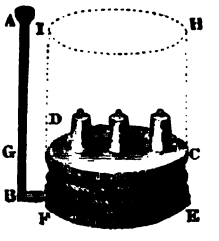
1. If several glass tubes of different shapes and sizes be put into a larger glass vessel containing water, the tubes being all open at top; then the water will be seen to rise to the same height in each of them, as is marked by the upper surface *a c*, of the liquid in the larger vessel.

2. If three vessels of equal bases, one cylindrical, the sec
siderably larger at top than at bottom, the third considerab
top than at bottom, and with the sides of the two latt
regularly or irregularly sloped, have their bottoms moveable,
close by the action of a weight upon a lever; then it will l
that when the same weight acts at the same distance upon
water must be poured in to the same height in each vessel t
pressure will force open the bottom.

3. Let a glass tube open at both ends (whether cylindric
does not signify) have a piece of bladder tied loosely over
so as to be capable of hanging below that end, or of rising u
it, when pressed from the outside. Pour into this tube son
tinged red, so as to stand at the depth of seven or eight incl

then immerse the tube with its coloured water vertically into a larger glass vessel nearly full of colourless water, the bladder being downwards, serving as a flexible bottom to the tube. Then, it will be observed that when the depth of the water in the tube exceeds that in the larger vessel, the bladder will be forced *below* the tube, by the excess of the interior over the exterior pressure: but when the exterior water is deeper than the interior, the bladder will be thrust up *within* the tube, by the excess of exterior pressure: and when the water in the tube and that in the larger vessel have their upper surfaces in the same horizontal plane, then the bladder will adjust itself into a *flat* position, just at the bottom of the tube. The success of this experiment does not depend upon the actual depth of the water in the tube, but upon the relation between the depths of that and the exterior water; and proves that in all cases the deeper water has the greater pressure at its bottom, tending equally upward or downward.

4. The *hydrostatical paradox*, as it is usually denominated, results from the principle that any quantity of a non-elastic fluid, however small, may be made to balance another quantity; or any weight, as large as we please (§ 5, page 249). It may be illustrated by a machine, the *hydrostatic bellows*, which consists of two thick boards DC, FE, each about 16 or 18 inches diameter, more or less, covered or connected firmly with leather round the edges, to open and shut like a common bellows, but without valves; only a pipe AB, about 3 feet high, is fixed into the bellows above F. Now, let water be poured into the pipe at A, and it will run into the bellows, gradually separating the boards by raising the upper one. Then, if several weights, as three hundred-weights, be laid upon the upper board, and water be poured in at the pipe till it is full, it will sustain all the weights, though the water in the pipe should not weigh a quarter of a pound: for the pipe or tube may be as small as we please, provided it be but long enough, the whole effect depending upon the height, and not at all on the width of the pipe, for the proportion is always this:—



> As the area of the orifice of the pipe
> is to the area of the bellows board,
> so is the weight of water in the pipe, above DC,
> to the weight it will sustain on the board.

5. In lieu of the bellows part of the apparatus, the leather of which would be incapable of resisting any very considerable pressure, the late Mr. Joseph Bramah used a very strong metal cylinder, in which a piston moved in a perfectly air and water tight manner, by passing through leather collars, and as a substitute for the high column of water he adopted a very small forcing pump, to which any power can be applied; and thus the pressing column becomes indefinitely long,

although the whole apparatus is very compact and takes but little room. Figure 207 is a section of one of these presses, in which *t* is the piston of the large cylinder, formed of a solid piece of metal turned truly cylindrical, and carrying the lower board *v* of the press upon it: *u* is the piston of the small forcing pump, being also a cylinder of solid metal moved up and down by the handle or lever *w*. The whole lower part of the press is sometimes made to stand in a case *x x*, containing more than sufficient water, as at *y*, to fill both the cylinders; and the suction pipe of the forcing pump *u* dipping into this water will be constantly supplied. Whenever, therefore, the handle *w* is moved upwards, the water will rise through the conical metal valve *z*, opening upwards into the bottom of the pump *u*; and when the handle is depressed that water will be forced through another similar valve *a*, opening in an opposite direction in the pipe of communication between the pump and the great cylinder *b*, which will now receive the water, by which the piston rod *t* will be elevated at each stroke of the pump *u*. Another small conical valve *c* is applied by means of a screw to an orifice in the lower part of the large cylinder, the use of which is to release the pressure whenever it may be necessary; for, on opening this valve, any water which was previously contained in the large cylinder *b*, will run off into the reservoir *y* by the passage *d*, and the piston *t* will descend so that the same water may be used over and over again. The power of such a machine is enormously great; for, supposing the hand to be applied at the end of the handle *w*, with a force of only 10 pounds, and that this handle or lever be so constructed as to multiply that force but 5 times, then the force with which the piston *u* descends will be equal to 50 pounds: let us next suppose that the magnitude of the piston *t* is such, that the area of its horizontal section shall contain a similar area of the smaller piston *u* 50 times; then 50 multiplied by 50 gives 2500 pounds, for the force with which the piston *t* and the lower board *v* of the press will rise. A man can, however, exert ten times this force for a short time, and could therefore raise 25,000 pounds; and would do more if a greater disproportion existed between the two pistons *t* and *u*, and the lever *w* were made more favourable to the exertion of his strength.

This machine not only acts as a press, but is capable of many other useful applications, such as a jack for raising heavy loads, even buildings; to the purpose of drawing up trees by their roots or the piles used in bridge-building.

To find the thickness of the metal in Bramah's press, to resist certain pressures, Mr. Barlow gives this theorem, $t = \frac{p\,r}{c - p}$ where p = pressure in lbs. per square inch, r = radius of the cylinder, t = its thickness, and $c = 18000$ lbs. the cohesive power of a square inch of cast iron.

Ex. Suppose it were required to determine the thickness of metal in two presses, each of 6 inches radius, in one of which the pressure

may extend to 4278 pounds, in the other to 8556 pounds per square inch.

Here in the first case,

$$t = \frac{4278 \times 6}{18000 - 4278} = 1{\cdot}87 \text{ inches, thickness.}$$

In the second,

$$t = \frac{8556 \times 6}{18000 - 8556} = 5{\cdot}43 \text{ inches, thickness.}$$

The usual rules, explained below (Art. 10), would make the latter thickness double the former: extensive experiments are necessary to tell which method deserves the preference.

6. If b equals the breadth, and d the depth of a rectangular gate, or other surface exposed to the pressure of water from top to bottom; then the entire pressure is equal to the weight of a prism of water whose content is $\frac{1}{2}\, b\, d^2$. Or, if b and d be in feet, then the whole pressure $= 31\frac{1}{4}\, b\, d^2$, in lbs., or nearly $= \frac{1}{11}\, b\, d^2$, in cwts.

7. If the gate be in the form of a trapezoid, widest at top, then, if B and b be the breadths at the top and bottom respectively, and d the depth, the

$$\text{whole pressure in lbs.} = 31\tfrac{1}{4}\left\{\tfrac{1}{3}\,(B - b) + b\right\} d^2$$
$$\text{whole pressure in cwts.} = \tfrac{1}{11}\left\{\tfrac{1}{3}\,(B - b) + b\right\} d^2 \text{ nearly.}$$

8. The weight of a cubic foot of rain or river water, is nearly equal to $\frac{6}{11}$ cwt.

The pressure on a square inch, at the depth of THIR-*ty* feet is very nearly THIR-*teen* pounds.

The pressure on a *square foot* is nearly a ton at the depth of thirty-six feet. [The true depth is 35·84 feet.]

The weight of an *ale gallon* of rain water is nearly $10\frac{1}{2}$ lbs., that of an *imperial* gallon 10 lbs.

The weight of a cubic foot of *sea-water* is nearly $\frac{1}{4}$ of a cwt.

These are all useful approximations; the actual weight of a cubic foot of distilled water is 6·25 lbs.

Thus, the pressure of rain water upon a square *inch* at the depth of 3000 feet, is 1300 lbs.

And the pressure upon a square *foot* at the depth of 108 feet is nearly three tons.

9. In the construction of dykes or embankments, both faces or slopes should be planes, and the exterior and interior slopes should make an angle of not less than 90°. For, if A D′ (fig. 208) be the exterior slope, and the angle D′ A B be acute, E D′ perpendicular to A B is the direction of the pressure upon it; and the portion D′ A E will probably be torn off. But when D A is the exterior face, making with A B an obtuse angle, the direction of the pressure falls *within* the base, and therefore augments its stability.

10. The strength of a circular bason confining water requires the consideration of other principles.

The perpendicular pressure against the wall depends merely on the altitude of the fluid, without being affected by the volume. But, as Professor Leslie remarks, the longitudinal effort of the thrust, or its tendency to open the joints of the masonry, is measured by the radius of the circle. To resist that action in very wide basons, the range or course of stones along the inside of the wall must be proportionally thicker. On the other hand, if any opposing surface present some convexity to the pressure of water, the resulting longitudinal strain will be exerted in closing the joints and consolidating the building. Such reversed incurvation is, therefore, often adopted in the construction of dams.

Upon similar principles, the thickness of *pipes* to convey water, must vary in proportion to $\frac{h\,d}{c}$, where h is the height of the head of water, d the diameter of the pipe, and c the measure of the cohesion of a bar of the same material as the pipe, and an inch square.

A pipe of *cast iron*, 15 inches diameter, and $\frac{3}{4}$ of an inch thick, will be strong enough for a head of 600 feet.

A pipe of *oak* of the same diameter, and 2 inches thick, would sustain a head of 180 feet.

Where the cohesion is the same, t varies as $h\,d$; or as $\mathrm{H\,D} : \mathrm{T} :: h\,d : t$, in the comparison of two cases*.

Example. What, then, must be the respective thicknesses of pipes of cast iron and oak, each 10 inches diameter, to carry water from a head of 360 feet?

Here, 1st, for *cast iron:*

$\mathrm{H\,D}\ (= 600 \times 15) : \mathrm{T}\ (= \frac{3}{4}) :: h\,d\ (= 360 \times 10) : t$

$$\frac{360 \times 10 \times 3}{600 \times 15 \times 4} = \frac{10800}{36000} = \tfrac{3}{10} \text{ of an inch.}$$

2ndly, for *oak:*

$\mathrm{H\,D}\ (= 180 \times 15) : \mathrm{T}\ (= 2) :: h\,d\ (= 360 \times 10) : t$

$$\frac{360 \times 10 \times 2}{180 \times 15} = \tfrac{40}{15} = \tfrac{8}{3} = 2\tfrac{2}{3} \text{ inches.}$$

SECT. III. *Floating Bodies.*

1. If any body float on a fluid, it displaces a quantity of the fluid equal to itself in weight.

* To ascertain whether or not a pipe is strong enough to sustain a proposed pressure, it is a good custom amongst practical men to employ a *safety-valve*, usually of an inch in diameter, and load it with the proposed weight, and a *surplus* determined by practice. Then, if the proposed pressure be applied interiorly, by a forcing pump, or in any other way, if the pipe remain *sound* in all its parts after the safety-valve has yielded, such pipe is regarded as sufficiently strong.

The *actual* pressures upon a pipe of any proposed diameter and head, may evidently be determined by a similar method.

2. Also, the centers of gravity of the body and of the fluid displaced, must, when the body is at rest, be in the same vertical line.

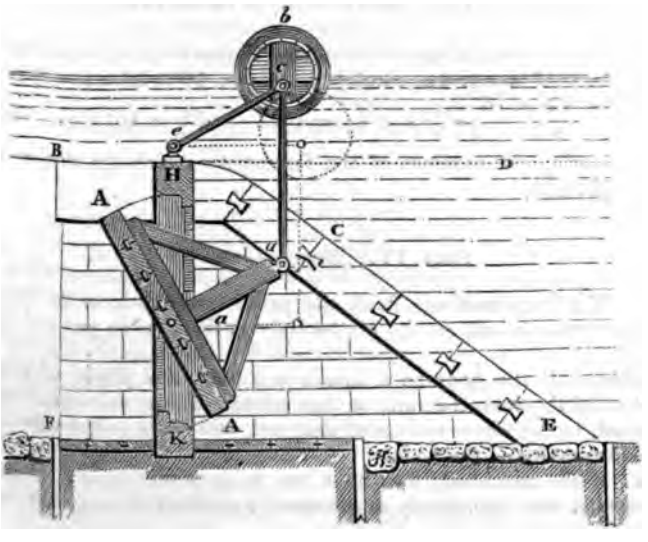
3. If a vessel contain two fluids that will not mix (as water and mercury), and a solid of some intermediate specific gravity be immersed under the surface of the lighter fluid and float on the heavier; the part of the solid immersed in the heavier fluid, is to the whole solid as the difference between the specific gravities of the solid and the lighter fluid, is to the difference between the specific gravities of the two fluids.

4. The buoyancy of casks, or the load which they will carry without sinking, may be estimated by reckoning 10 lbs. avoirdupois to the ale gallon, or $8\frac{1}{3}$ lbs. to the wine gallon.

5. The buoyancy of *pontoons* may be estimated at about *half a hundred-weight* for each cubic foot.

Thus a pontoon which contained 96 cubic feet, would sustain a load of 48 cwt. before it would sink. This is an approximation, in which the difference between $\frac{6}{11}$ and $\frac{1}{2}$, that is, $\frac{1}{22}$ of the whole weight, is allowed for that of the pontoon itself.

6. The principles of buoyancy are very ingeniously applied in Mr. Farey's *self-acting flood-gate*. In the case of common sluices to a mill-dam, when a sudden flood occurs, unless the miller gets up in the night to open the gate or gates, the neighbouring lands may become inundated; and, on the contrary, unless he be present to shut them up when the flood subsides, the mill-dam may be emptied and the water lost which he would need the next day. To prevent either of these occurrences, Mr. John Farey, whose talent and ingenuity are well known, has proposed *a self-acting flood-gate*, the following description of which has been given in the *Mechanics' Weekly Journal*.



A A represents a vertical section of a gate poised upon a horizontal axis passing rather *above* the center of pressure of the gate, so as to give it a tendency to shut close: *a a* is a lever, fixed perpendicular to the gate, and connected by an iron rod with a cask, *b*, which is floated whenever the surface of the water rises to the line B, D, which is assumed as the level of the wear, or mill-dam, B, C, E, F, in which the flood-gate is placed: by this arrangement it will be seen that when the water rises above the dam, it floats the cask, opens the gate, and allows the water to escape until its surface subsides to the proper level at B, D; the cask now acts by its weight, when unsupported by the water, to close the gate and prevent leakage. The gate should be fitted into a frame of timber, H, K, which is set in the masonry of the dam. The upper beam H, of the frame being just level with the crown of the dam, so that the water runs over the top of the gate at the same time that it passes through it: to prevent the current disturbing the cask, it is connected by a small rod, *e*, at each end, to the upper beam, H, of the frame, and jointed in such a manner as to admit of motion in a vertical direction.

7. By means of the same principle of buoyancy it is, that a hollow ball of copper attached to a metallic lever of about a foot long, is made to rise with the liquid in a water-tub, and thus to close the cock and stop the supply from the pipe, just before the time when the water would otherwise run over the top of the vessel.

8. This property, again, has been successfully employed in pulling up old piles in a river where the tide ebbs and flows. A barge of considerable dimensions is brought over a pile as the water begins to rise: a strong chain which has been previously fixed to the pile by a ring, &c. is made to gird the barge and is then fastened. As the tide rises the vessel rises too, and by means of its buoyant force draws up the pile with it.

In an actual case, a barge 50 feet long, 12 feet wide, 6 deep, and drawing 2 feet of water, was employed. Here, $50 \times 12 \times (6 - 2) \times \frac{4}{7}$ $= \frac{50 \times 12 \times 16}{7} = 192 \times 7\frac{1}{7} = 1344 + 27\frac{3}{7} = 1371\frac{3}{7}$ cwt. $= 66\frac{1}{2}$ tons nearly, the measure of the force with which the barge acted upon the pile.

SECT. IV. *Specific Gravities.*

1. If a body float on a fluid, the part immersed is to the whole body, as the specific gravity of the body to the specific gravity of the fluid.

Hence, if the body be a square or a triangular prism, and it be laid upon the fluid, the ratio of that portion of one end which is immersed, to the whole surface of that end, will serve to determine the specific gravity of the body.

2. If the same body float upon two fluids in succession, the parts immersed will be inversely as the specific gravities of those fluids.

3. The weight which a body loses when wholly immersed in a fluid is equal to the weight of an equal bulk of the fluid.

When we say that a body loses part of its weight in a fluid, we do not mean that its *absolute* weight is less than it was before, but that it is partly supported by the reaction of the fluid under it, so that it requires a less power to sustain or to balance it.

4. A body immersed in a fluid ascends or descends with a force equal to the difference between its own weight and the weight of an equal bulk of fluid; the resistance or viscidity of the fluid not being considered.

5. *To find the specific gravity of a fluid, or of a solid.*—On one arm of a balance suspend a globe of lead by a fine thread, and to the other fasten an equal weight, which may just balance it in the open air. Immerse the globe into the fluid, and observe what weight balances it then, and consequently what weight is lost, which is proportional to the specific gravity as above. And thus the proportion of the specific gravity of one fluid to another is determined by immersing the globe successively in all the fluids, and observing the weights lost in each, which will be the proportions of the specific gravities of the fluids sought.

This same operation determines also the specific gravity of the solid immerged, whether it be a globe or of any other shape or bulk, supposing that of the fluid known. For the specific gravity of the fluid is to that of the solid, as the weight lost is to the whole weight.

Hence also may be found the specific gravity of a body that is lighter than the fluid, as follows:

6. *To find the specific gravity of a solid that is lighter than the fluid, as water, in which it is put.*—Annex to the lighter body another that is much heavier than the fluid, so that the compound mass may sink in the fluid. Weigh the heavier body and the compound mass separately, both in water and out of it; then find how much each loses in water, by subtracting its weight in water from its weight in air; and subtract the less of these remainders from the greater.

Then, As this last remainder,

: the weight of the light body in air,
:: the specific gravity of the fluid,
: the specific gravity of that body.

7. The specific gravities of bodies of equal weight are reciprocally proportional to the quantities of weight lost in the same fluid. And hence is found the ratio of the specific gravities of solids, by weighing in the same fluids masses of them that weigh equally in air, and noting the weights lost by each.

8. Instead of a hydrostatic *balance*, a hydrostatic *steelyard* is now frequently employed. It is contrived to balance exactly by making the shorter end wider, and with an enlargement at the extremity. The shorter arm is undivided, but the longer arm is divided into short equal divisions: thus, if that longer arm be 8 inches long, it

may be divided into 400 parts, the divisions *commencing* at A (fig. 209). Then, in using this instrument, any convenient weight is suspended by a hook from a notch at the end of the scale A. The body whose specific gravity is to be determined, is suspended from the other arm by a horse-hair, and moved to and fro till an equilibrium is produced. Then, without altering its situation at D on the beam, it is immersed in water, and balanced a second time by sliding the counterpoise from A, say to C.

Here, evidently, the weight in water : weight in air :: B C : B A, and the loss of weight in water : weight in air :: A C : A B.

$$\text{Consequently, } \frac{\text{weight in air}}{\text{loss}} = \frac{AB}{AC} = \text{specific gravity.}$$

With such an instrument nicely balanced upon a convenient pedestal, I find that the specific gravities of solids are ascertainable both with greater facility and correctness than with any hydrostatic *balance* which I have seen *.

A copious table of the specific gravities of various substances will be found in the Appendix, Table X.

SECT. V. *On Capillary Attraction.*

1. If two plates of glass be set up vertically in a shallow vessel, containing a coloured liquid, in such a way as to touch along the edges A B, fig. 210, and forming a very acute angle with each other, the coloured liquid will rise between the two plates to a height inversely as the distance between them; from which it follows, that its surface will form a rectangular hyperbola, whose two asymptotes are the edge of the plate A B and the surface of the liquid. And if a glass tube, the bore of which is small, be immersed in the liquid, it will rise within the tube to a greater height than the surface of the liquid without the same.

2. The cause of this phenomenon is an attractive force existing between the liquid and the glass, and which is termed *capillary* attraction, from *capillus*, the Latin for hair, because it only takes place in tubes, the bores of which are small, resembling hairs.

3. The heights to which fluids rise in capillary tubes are inversely as their diameters: and they are found to rise to the same height in a vacuum, as in the air.

4. In a tube $\frac{1}{100}$th of an inch in diameter, water rises 5·3 inches; and as the height is inversely as the diameter, their product will be a constant quantity, therefore $5{\cdot}3 \times {\cdot}01 = {\cdot}053 = q$, which may be taken to represent the attractive force between water and glass. The following are the values of q, according to different experimenters:—

* We owe this contrivance to Dr. *Coates*, of Philadelphia.

Sir I. Newton	·020
Sir D. Brewster	·033
M. M. Gay Lussac	·046
Mr. Atwood	·053

5. The height at which the fluid is supported by capillary attraction in a tube of varying diameter, is that due to its size at the surface of the liquid, without any regard to the dimensions or shape of the lower portion. Thus, in a vessel of the form shown in figure 211, terminating in a capillary tube, the liquid will be supported at the same height in this tube (and also fill the whole of the space *a b c* of the vessel) as if the tube had been of uniform dimensions throughout its whole length, and every where equal to its upper portion. The water, however, contained in the lower portion of the vessel *a b c*, is in this case supported by the pressure of the atmosphere, as it ceases to be supported in a vacuum.

6. Different liquids rise to different heights in capillary tubes of the same bore, depending upon the attraction between them and glass. The following are Dr. Brewster's results for several fluids, with a tube ·0561 inches in diameter:—

	Height of ascent in inches.	Value of the constant *q*.
Water	·587	·0327
Water, very hot	·537	·0301
Muriatic acid	·442	·0248
Nitric acid	·395	·0222
Spermaceti oil	·392	·0220
Olive oil	·387	·0215
Oil of turpentine	·333	·0187
Alcohol	·317	·0178
Æther	·285	·0160
Sulphuric acid	·200	·0112

7. The internal diameter of a uniform capillary, or other small tube, may be found in the following manner. Let the tube be weighed when empty, and again when filled with mercury, and let w be the difference of those weights in troy grains, l the length of the tube in inches; and d its diameter, then $d = ·019252 \sqrt{\frac{w}{l}}$

Thus, if the difference of the weights were 500 grains, and the length of the tube were 20 inches: we should have $d = ·019252 \sqrt{\frac{500}{20}} = ·019252 \times 5 = ·09626$ of an inch *.

* The same thing may easily be accomplished thus:—Let a cone of box wood, or of brass, be very accurately turned, of about 6 inches in length, and the diameter of its base about a quarter of an inch; and let its curve surface be very accurately marked with a series of parallel rings, about a twentieth of an inch asunder, from its vertex to its base. Insert this cone carefully in the cylinder (so that their axes shall coincide) as in fig. 212: then it will be as V A : V *a* :: A B : *a b*; where, as the ratio of V A to V *a* is known by means of the equidistant rings on the surface, and A B is known, *a b* becomes determined.

CHAP. V.

HYDRODYNAMICS.

Hydrodynamics is that part of mechanical science which relates to the motion of non-elastic fluids, and the forces with which they act upon bodies.

This branch of mechanics is the most difficult, and the least advanced: whatever we know of it is almost entirely due to the researches of the moderns.

Could we know with certainty the mass, the figure, and the number of particles of a fluid in motion, the laws of its motion might be determined by the resolution of this problem, viz. to find the motion of a proposed system of small free bodies acting one upon the other in obedience to some given exterior force. We are, however, very far from being in possession of the data requisite for the solution of this problem. We shall, therefore, simply present a few of the most usually received theoretical deductions; and then proceed to state those rules which have resulted from a judicious application of theory to experiment.

SECT. I. *Motion and Effluence of Liquids.*

1. A jet of water, issuing from an orifice of a proper form, and directed upwards, rises, under favourable circumstances, nearly to the height of the head of water in the reservoir; and since the particles of such a stream are but little influenced by the neighbouring ones, they may be considered as independent bodies, moving initially with the velocity which would be acquired in falling from the height of the reservoir. And the velocity of the jet will be the same whatever may be its direction.

2. Hence, if a jet issue horizontally from any part of the side of a vessel standing on a horizontal plane, and a circle be described having the whole height of the fluid for its diameter, the fluid will reach the plane at a distance from the vessel, equal to that chord of the circle in which the jet initially moves.

Thus, if AS (fig. 213) be the upper surface of the fluid in the vessel, B the place of the orifice, CF the horizontal plane on which the fluid spouts, then CF is equal to ED, the horizontal chord of the

ıe diameter is A C, passing through B. It is therefore evi-
the orifice from which the fluid will spout to the greatest
ı situate at G, half the height of the fluid, and also, that if
ıres be made at equal distances above and below G, the jet
om both will strike the plane C E in the same point.
ın a cylindrical or prismatic vessel empties itself by a small
ə velocity at the surface is uniformly retarded; and in the
nptying itself, twice the quantity would be discharged if it
: full by a new supply.
the quantity discharged is by no means equal to what would
ıole orifice, with this velocity. If the aperture is made sim-
ıin plate, the lateral motion of the particles towards it tends
t the direct motion, and to contract the stream which has
ifice, nearly in the ratio of two to three. So that, in order
ıe quantity discharged, the section of the orifice must be
to be diminished from 100 to 62 for a simple aperture, to
ipe of which the length is twice the diameter, and in other
ırding to circumstances.
ın a syphon, or bent tube, is filled with a fluid, and its
ımersed in the fluids of different vessels, if both surfaces of
are in the same level, the whole remains at rest; but if
the longer column of fluid in the syphon preponderates,
essure of the atmosphere forces up the fluid from the higher
il the equilibrium is restored; and the motion is the more
he difference of the levels is greater: provided that the
ight of the tube above the upper surface be not more than
oise to the pressure of the atmosphere.
ıe lower vessel be allowed to empty itself, the syphon will
unning as long as it is supplied from the upper, and the
it descends the further below the vessel. In the same
e discharge of a pipe, descending from the side or bottom
ı vessel, would be increased almost without limit by length-

notch or sluice in form of a rectangle be cut in the ver-
of a vessel full of water, or any other fluid, the quantity
ough it will be $\frac{2}{3}$ of the quantity which would flow through
ifice, placed horizontally at the whole depth, in the same
essel being kept constantly full.

rovement in the construction of the syphon has been lately proposed
ew *Mechanics' Magazine*, and by M. *Buntem* at Paris. It might
ntageously used if constructed on a large scale, for lowering the
l-dams or canals. The improvement in the present syphon is, that
ng pipe is enlarged to the same diameter as that of the syphon, and
widened out to something of a funnel shape, as in fig. 214. In
nto action, the short arm is immersed in the water as in the usual
bottom of the long arm is closed, the exhausting pipe is then filled
y the funnel-shaped mouth. On the bottom of the long arm being
ater flows out, and exhausts the air from the syphon, when the
is wished to be emptied flows out in a continual stream.

8. If a short pipe elevated in any direction from an aperture in a conduit, throw the water in a parabolic curve to the distance or range R, on a board, or other horizontal plane passing through the orifice, and the greatest height of the spouting fluid above that plane, be H, then the height of the *head* of water above that conduit pipe, may be found *nearly:* viz. by taking 1st, $2 \cot E = \frac{R}{2H}$; and 2ndly, the altitude of the head $A = \frac{1}{4} R \times \text{cosec } 2E$.

Example. Suppose that R = 40 feet, and H = 18 feet. Then $\frac{R}{2H} = \frac{40}{36} = 1{\cdot}1111111 = 2 \cot 60° 57'$: and $A = \frac{1}{4} R \times \text{cosec } 2E$ $= 20 \times \text{cosec } 121° 54' = 20 \times 1{\cdot}177896 = 23{\cdot}55792$ feet, height required.

Note. This result of theory will usually be found about $\frac{4}{5}$ of that which is furnished by experiment.

SECT. II. *Motion of Water in Conduit Pipes and Open Canals, over Weirs, &c.*

1. When the water from a reservoir is conveyed in long horizontal pipes of the same aperture, the discharges made in equal times are nearly in the inverse ratio of the square roots of the lengths.

It is supposed that the lengths of the pipes to which this rule is applied, are not very unequal. It is an approximation not deduced from principle, but derived immediately from experiment. [Bossut, tom. 11, § 647, 648. At § 673, he has given a table of the actual discharges of water-pipes, as far as the length of 2340 toises, or 14,950 feet English.]

2. Water running in open canals, or in rivers, is accelerated in consequence of its depth and of the declivity on which it runs, till the resistance, increasing with the velocity, becomes equal to the acceleration, when the motion of the stream becomes uniform.

It is evident that the amount of the resisting forces can hardly be determined by principles already known, and therefore nothing remains but to ascertain, by experiment, the velocity corresponding to different declivities, and different depths of water, and to try, by multiplying and extending these experiments, to find out the law which is common to them all.

The Chevalier Du Buat has been successful in this research, and has given a formula for computing the velocity of running water, whether in close pipes, open canals, or rivers, which, though it may be called *empirical*, is extremely useful in practice. *Principes d'Hydraulique.* Professor Robison has given an abridged account of this book, in his excellent article on Rivers and Water-works, in the *Encyclopædia Britannica.*

Let V be the velocity of the stream, measured by the feet it moves over in a second; R a constant quantity, viz., the quotient obtained

by dividing the area of the transverse section of the stream, expressed in square feet, by the boundary or perimeter of that section, *minus* the superficial breadth of the stream expressed in linear feet.

The mean velocity is that with which, if all the particles were to move, the discharge would be the same with the actual discharge.

The line R is called by Du Buat, the *radius;* and by Dr. Robison, the *hydraulic mean depth.* As its affinity to the radius of a circle seems greater than to the depth of a river, we shall call it, with the former, the *radius of the section.*

Lastly, let S be the denominator of a fraction which expresses the slope, the numerator being unity, that is, let it be the quotient obtained by dividing the length of the stream, supposing it extended in a straight line, by the difference of level of its two extremities: or, which is nearly the same, let it be the co-tangent of the inclination or slope.

3. The above denominations being understood, and the section, as well as the velocity, being supposed uniform, we have

$$V = \frac{307\ \sqrt{(R - \frac{1}{10})}}{S^{\frac{1}{2}} - \frac{1}{2}\log(S + \frac{6}{10})} - \frac{3}{10}\ \sqrt{(R - \frac{1}{10})};$$

$$\text{or } V = \sqrt{R - \tfrac{1}{10}}\left\{\frac{307}{S^{\frac{1}{2}} - \frac{1}{2}\log(S + \frac{6}{10})} - \frac{3}{10}\right\} \quad \ldots\ldots\ldots \text{(I.)}$$

When R and S are very great,

$$V = R^{\frac{1}{2}}\left\{\frac{307}{S^{\frac{1}{2}} - \frac{1}{2}\log s} - \frac{3}{10}\right\}, \text{ nearly.} \quad \ldots\ldots\ldots\ldots \text{(II.)}$$

The logarithms understood here are the hyperbolic, and are found by multiplying the common logarithms by 2·3025851; or more easily by the method described at page 94.

The slope remaining the same, the velocities are as $\sqrt{R - \frac{1}{10}}$.

The velocities of two great rivers that have the same declivity, are as the square roots of the radii of their sections.

If R is so small, that $\sqrt{R - \frac{1}{10}} = 0$, or $R = \frac{1}{10}$, the velocity will be nothing; which is agreeable to experience; for in a cylindric tube $R = \frac{1}{2}$ the radius; the radius, therefore, equals two-tenths; so that the tube is nearly capillary, and the fluid will not flow through it.

The velocity may also become nothing by the declivity becoming so small, that

$$\frac{307}{S^{\frac{1}{2}} - \frac{1}{2}\log(S + \frac{6}{10}} - \frac{3}{10} = 0; \text{ but}$$

if $\frac{1}{S}$ is less than $\frac{1}{500000}$, or than $\frac{1}{10}$th of an inch to an English mile, the water will have sensible motion.

4. In a river, the greatest velocity is at the surface, and in the middle of the stream, from which it diminishes towards the bottom

and the sides, where it is least. It has been found by experiment that if from the square root of the velocity in the middle of the stream, expressed in inches per second, unity be subtracted, the square of the remainder is the velocity at the bottom.

Hence, if the former velocity be $= v$, the velocity at the bottom $= v - 2\sqrt{v} + 1$. ..... (III.)

5. The mean velocity, or that with which, were the whole stream to move, the discharge would be the same with the real discharge, is equal to half the sum of the greatest and least velocities, as computed in the last proposition.

The mean velocity is, therefore, $= v - \sqrt{v} + \frac{1}{2}$. ..... (IV.)

This is also proved by the experiments of Du Buat.

6. Suppose that a river having a rectangular bed, is increased by the junction of another river equal to itself, the declivity remaining the same; required the increase of the depth and velocity.

Let the breadth of the river $= b$, the depth before the junction d, and after it x; and in like manner, v and v_1 the mean velocities before and after; then $\frac{bd}{b+2d}$ is the radius before, and $\frac{bx}{b+2x}$ the radius after, so $v = \frac{307\,R^{\frac{1}{2}}}{S^{\frac{1}{2}}}$, supposing the breadth of the river to be such, that we may reject the small quantity subtracted from R, in formulæ (I. and II.); and, in like manner, $v_1 = \frac{307\,R_1^{\frac{1}{2}}}{S^{\frac{1}{2}}}$.

Then, substituting for R and R_1, we have

$$v = \frac{307}{S^{\frac{1}{2}}} \times \sqrt{\frac{bd}{v+2d}},$$

$$\text{and } v_1 = \frac{307}{S^{\frac{1}{2}}} \times \sqrt{\frac{vx}{v+2x}}.$$

Multiplying these into the areas of the sections bd and bx, we have the discharges, viz.,

$$bdv = \frac{307}{S^{\frac{1}{2}}} \times \frac{bd\sqrt{bd}}{\sqrt{b+2d}}; \text{ and } bxv_1 = \frac{307}{S^{\frac{1}{2}}} \times \frac{bx\sqrt{bx}}{\sqrt{b+2x}}.$$

Now the last of these is double of the former; therefore,

$$\frac{bx\sqrt{bx}}{\sqrt{b+2x}} = \frac{2bd\sqrt{bd}}{\sqrt{b+2d}}, \text{ or, } \frac{x^3}{b+2x} = \frac{4d^3}{b+2d}, \text{ and}$$

$x^3 - \frac{3d^3}{b+2d}x = \frac{4bd^3}{b+2d}$, a cubic equation which can always be resolved by Cardan's rule, or by the approximating method given at page 82.

As an example, let $b = 10$ feet, and $d = 1$, then $x^3 - \frac{1}{2}x = \frac{10}{3}$, and $x = 1{\cdot}4882$, which is the depth of the increased river. Hence we have $1{\cdot}488 \times v_1 = 2v$, and $1{\cdot}488 : 2 :: v : v_1$, or $v : v_1 :: 37$ to 50 nearly.

When the water in a river receives a permanent increase, the depth and the velocity, as in the example above, are the first things that are augmented. The increase of the velocity increases the action on the sides and bottom, in consequence of which the width is augmented, and sometimes also, but more rarely, the depth. The velocity is thus diminished, till the tenacity of the soil or the hardness of the rock affords a sufficient resistance to the force of the water. The bed of the river then changes only by insensible degrees, and, in the ordinary language of hydraulics, is said to be permanent, though in strictness this epithet is not applicable to the course of any river.

7. When the sections of a river vary, the quantity of water remaining the same, the mean velocities are inversely as the areas of the sections. This must happen, in order to preserve the same quantity of discharge. (*Playfair's Outlines.*)

8. The following table, abridged from *Du Buat*, serves at once to compare the surface, bottom, and mean velocities in rivers, according to the formulæ (III. and IV.)

Velocities of Rivers.

VELOCITY IN INCHES.			VELOCITY IN INCHES.		
Surface.	Bottom.	Mean.	Surface.	Bottom.	Mean.
4	1·	2·5	56	42·016	49·008
8	3·342	5·67	60	45·509	52·754
12	6·071	9·036	64	49·	56·5
16	9·	12·5	68	52·505	60·252
20	12·055	16·027	72	56·025	64·012
24	15·194	19·597	76	59·568	67·784
28	18·421	23·210	80	63·107	71·553
32	21·678	26·839	84	66·651	75·325
36	25·	30·5	88	70·224	79·112
40	28·345	34·172	92	73·788	82·894
44	31·742	37·871	96	77·370	86·685
48	35·151	41·570	100	81·	90·5
52	38·564	45·282			

9. The knowledge of the velocity at the bottom is of the greatest use for enabling us to judge of the action of the stream on its bed.

Every kind of soil has a certain velocity consistent with the stability of the channel. A greater velocity would enable the waters to tear it up, and a smaller velocity would permit the deposition of more movable materials from above. It is not enough, then, for the stability of a river, that the accelerating forces are so adjusted to the

size and figure of its channel that the current may be in train: it must also be in equilibrio with the tenacity of the channel.

We learn from the observations of Du Buat, and others, that a velocity of three inches per second at the bottom will just begin to work upon the fine clay fit for pottery, and however firm and compact it may be, it will tear it up. Yet no beds are more stable than clay when the velocities do not exceed this: for the water soon takes away the impalpable particles of the superficial clay, leaving the particles of sand sticking by their lower half in the rest of the clay, which they now protect, making a very permanent bottom, if the stream does not bring down gravel or coarse sand, which will rub off this very thin crust, and allow another layer to be worn off; a velocity of six inches will lift fine sand; eight inches will lift sand as coarse as linseed; twelve inches will sweep along fine gravel; twenty-four inches will roll along rounded pebbles an inch diameter; and it requires three feet per second at the bottom to sweep along shivery angular stones of the size of an egg. (*Robison on Rivers.*)

10. In the elbow or bend of a river, the velocity is always greater near the concave than the convex side.

11. The swell occasioned by the piers of a bridge, or the sides of a cleaning sluice which contract the passage by a given quantity, for a short length only of the channel, may be determined when the vertical section of the river and the velocity of the stream are known, in the following manner. Let v be the velocity of the stream, independently of the effect of the bridge, r the section of the river, and a the amount of the sections between the piers; let $2g$, instead of being taken at $64\frac{1}{3}$, be reckoned 58·6, to accord with the results of experimental contractions through arches of bridges, &c., and let s be the slope of the bed of the river, or the sine of its angle with the horizon; then *Du Buat* (tom. i. p. 225) gives for the swell or rise (R) of the stream in feet, which will be occasioned by the obstruction.

$$R = \left(\frac{v^2}{58{\cdot}6} + s\right)\left(\left[\frac{r}{a}\right]^2 - 1\right) \quad \text{(V)}$$

12. The value of s will, of course, be different in different cases; but if we assume $\frac{1}{20}$ or ·05, as a mean value, it will enable us to compute and tabulate results, which, though they cannot be presented as perfectly correct, may be regarded as exhibiting a medium between those that will usually occur; and will serve to anticipate the consequences of floods of certain velocities, when constrained to pass through bridges which more or less contract the stream.

Table of the Rise of Water occasioned by Piers of Bridges, or other Contractions.*

Veloc. of current in feet, per second.	Amount of obstructions, compared with the vertical section of the River.									
	1-10th.	2-10ths.	3-10ths.	4-10ths.	5-10ths.	6-10ths.	7-10ths.	8-10ths.	9-10ths.	
	Proportional Rise of Water, in feet and decimals.									
	feet.	feet.	feet.	feet.	feet.	feet.	feet.	feet.	feet.	
1	0·0157	0·0377	0·0698	0·1192	0·2012	0·3521	0·678	1·6094	6·6389	
2	0·0277	0·0665	0·1231	0·2102	0·3548	0·6208	1·1995	2·8378	11·7058	Ordinary floods.
3	0·0477	0·1144	0·2118	0·3618	0·6107	1·0687	2·0580	4·8850	20·1504	
4	0·0760	0·1822	0·3372	0·5759	0·9719	1·7008	3·2755	7·7750	32·0720	
5	0·1165	0·2793	0·5168	0·8782	1·4895	2·6066	5·0202	11·9160	49·1535	Violent floods.
6	0·1558	0·3736	0·6912	1·1807	1·9925	3·4868	6·7154	15·9398	65·7518	
7	0·2078	0·4983	0·9221	1·5750	2·6578	4·6511	8·9578	21·2686	87·7080	
8	0·2678	0·6423	1·1884	2·0299	3·4255	5·9947	11·5454	27·4042	113·0422	Unusually violent floods.
9	0·3359	0·8054	1·4903	2·5566	4·2956	7·5172	14·4777	34·3646	141·7541	
10	0·4119	0·9877	1·8276	3·1218	5·2680	9·2190	17·7551	42·1440	173·8440	

13. It will be evident, from an inspection of this table, that even in the case of ordinary floods, old bridges with piers and starlings, occupying 6 or 7-tenths of the section of the river, will produce a swell of 2, 3, or more feet, often overflowing the river's banks and occasioning much mischief. Also, that in violent floods, an obstruction amounting to 7-tenths of the channel, will cause a rise of 7 or 8 feet, probably choking up the arches and occasioning the destruction of the bridge. Greater velocities and greater contractions produce a rapid augmentation of danger and mischief; as the table obviously shows.

14. The same principles and tabulated results serve to estimate the fall from the higher to the lower side of a bridge, on account of an ebbing tide, &c. Thus, for old London Bridge, where the breadth of the Thames is 926 feet, and the sum of the water ways at *low water* only 236 feet, the amount of the obstructions was 690 feet, about $7\frac{1}{2}$ tenths of the entire section: so that a velocity of $3\frac{1}{2}$ feet per second would give a fall of nearly $4\frac{3}{4}$ feet, agreeing with the actual result.

At Rochester Bridge, before the opening of the middle arches, the piers and starlings presented an obstruction of 7-tenths, and at the time of greatest fall, the velocity 100 yards above bridge exceeded 6 feet per second. This, from the table, would occasion a fall of more than 6-7 feet; and the recorded results vary from $6\frac{1}{2}$ to 7 feet.

At Westminster Bridge, where the obstructions are about 1-sixth of the whole channel, when the velocity is $2\frac{1}{3}$ feet, the fall but little exceeds half an inch: a result which the table would lead us to expect.

* A similar table was computed by Mr. Wright of Durham, more than fifty years ago, and inserted in the first edition of Dr. Hutton's treatise on Bridges; but it is not constructed upon a correct theory.

15. Mr. *Eytelwein*, a German mathematician, has devoted much time to inquiries in hydrodynamics. In his investigations he has paid attention to the mutual cohesion of the liquid moleculæ, their adherence to the sides of the vessel in which the water moves, and to the contraction experienced by the liquid vein when it issues from the vessel under certain circumstances. He obtains formulæ of the utmost generality, and then applies them to the motion of water; 1st, in a cylindric tube; 2ndly, in an open canal.

16. Let d be the diameter of the cylindric tube E F (fig. 215), h the total height F G of the head of water in the reservoir above the orifice F, and l the length E F of the tube, all in inches: and v equal the velocity in inches per second with which the fluid will issue from the orifice F; then

$$v = 23\tfrac{1}{3}\sqrt{\frac{57\,h\,d}{l + 57\,d}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(VI.)}$$

which, multiplied into the area of the orifice, will give the quantity discharged per second.

17. Let d = the diameter of the pipe in inches, Q = the quantity of water in cubic feet discharged through the pipe per minute, l = the length of the pipe in feet, and h = the difference of level between the surface of the water in the reservoir and at the end of the pipe, or the head; then, any three of these quantities being given, the fourth may be determined from the following formulæ:—

$$d = \sqrt[5]{\frac{\cdot 0448\ \mathrm{Q}^2\,(l + 4\cdot 2\,d)}{h}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(VII.)}$$

$$\mathrm{Q} = \sqrt{\frac{h\,d^5}{\cdot 0448\,(l + 4\cdot 2\,d)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(VIII.)}$$

$$l = \frac{h\,d^5}{\cdot 0448\ \mathrm{Q}^2} - 4\cdot 2\,d \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(IX.)}$$

$$h = \frac{\cdot 0448\ \mathrm{Q}^2\,(l + 4\cdot 2\,d)}{d^5} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(X.)}$$

These formulæ are more convenient when expressed logarithmically, and then become

$$\log d = \tfrac{1}{5}\{2\log \mathrm{Q} + \overline{2}\cdot 6515 + \log(l + 4\cdot 2\,d) - \log h\} \ldots\ldots \text{(XI.)}$$

$$\log \mathrm{Q} = \tfrac{1}{2}\{\log h + 5\log d - \overline{2}\cdot 6515 - \log(l + 4\cdot 2\,d)\} \ \ldots \text{(XII.)}$$

$$\log\ l = \log h + 5\log d - \overline{2}\cdot 6515 - 2\log \mathrm{Q}, [\text{neglecting} - 4\cdot 2d] \text{(XIII.)}$$

$$\log\ h = 2\log \mathrm{Q} + \overline{2}\cdot 6515 + \log(l + 4\cdot 2\,d) - 5\log d \ \ldots\ldots \text{(XIV.)}$$

18. When a pipe is bent in one or more places, then if the squares of the sines of the several changes of direction be added into one sum s, the velocity v will, according to *Langsdorf*, be found by the theorem

$$v = \sqrt{\frac{548\,dh}{d + \frac{1}{45}l + \frac{1}{6}ds}} \quad \text{(XV.)}$$

l, h, d, and v, being all in inches.

19. *For open canals.*—Let v be the mean velocity of the current in feet, a area of the vertical section of the stream, p perimeter of the section, or sum of the bottom and two sides, l length of the bed of the canal corresponding to the fall h, all in feet: then

$$v = \sqrt{9582\,\frac{ah}{pl} + 0{\cdot}0111} - 0{\cdot}109 \quad \text{(XVI.)}$$

The experiments of M. *Bidone*, of Turin, on the motion of water in canals, agree within the 80th part of the results of computations from the preceding formulæ.

20. *For apertures in the sides or bottom of vessels.*—If q equals the quantity of water discharged in cubic feet per minute, v the velocity of the effluent water in feet per second, through the aperture, a the area of the aperture in square inches, and h the height from its center to the surface of the water; we have

$$v = c\sqrt{h} \quad \text{(XVII.)}$$

$$q = {\cdot}4167\,ac\sqrt{h} \quad \text{(XVIII.)}$$

In which c is a constant quantity depending upon the nature of the aperture, and the value of which for several different forms is contained in the following table.

Nature of the Orifices employed.	Ratio between the theoretical and real discharges	Coefficients for finding the velocities in Eng. feet.
For the whole velocity due to the height	1 to 1·00	8·0458
For an orifice of the form of the contracted vein.	1 to 0·969	7·8
For wide openings whose bottom is on a level with that of the reservoir; for sluices with walls in a line with the orifice; for bridges with pointed piers.	1 to 0·961	7·7
For narrow openings whose bottom is on a level with that of the reservoir; for smaller openings in a sluice with side walls; for abrupt projections and square piers of bridges.	1 to 0·861	6·9
For openings in sluices without side walls.....	1 to 0·635	5·1
For an orifice in a thin plate	1 to 0·621	5·0
When there is a short cylindrical pipe projecting on the inside of the vessel, length 2 to 4 times the bore, and rim like contracted vein, not full.	1 to 0·510	4·1
Idem, when it runs full bore of water	1 to 0·671	5·4
Idem, when it does not project inside the vessel.	1 to 0·808	6·5

21. *Table, abridged from one by Mr. Smeaton, for showing the height of head necessary to overcome the friction of water in horizontal pipes.*

Bore of the pipe in inches.	Velocities per second of water in the pipe.																				Bore of the pipe in inches.
	Feet	in.	Feet	in.	Feet	in.	Feet	in.	Feet	in.	Feet	in.	Feet	in.	Feet	in.	Feet	in.	Feet	in.	
	0	6	1	0	1	6	2	0	2	6	3	0	3	6	4	0	4	6	5	0	
½	0	4·5	1	4·7	2	11·0	4	9·7	7	1·7	10	1·0	13	8·0	17	10·0	22	6·7	28	0·2	½
¾	0	3·0	0	11·1	1	11·3	3	2·5	4	9·2	6	8·6	9	1·3	11	10·6	15	0·5	18	8·1	¾
1	0	2·2	0	8·4	1	5·5	2	4·9	3	6·9	5	0·5	6	10·0	8	11·0	11	3·4	14	0·0	1
1¼	0	1·8	0	6·7	1	2·0	1	11·1	2	10·3	4	0·4	5	5·6	7	1·6	9	0·3	11	2·5	1¼
1½	0	1·5	0	5·6	0	11·7	1	7·2	2	4·6	3	4·3	4	6·7	5	11·3	7	6·2	9	4·1	1½
1¾	0	1·3	0	4·8	0	10·0	1	4·5	2	0·5	2	10·6	3	10·9	5	1·1	6	5·4	8	0·1	1¾
2	0	1·1	0	4·2	0	8·7	1	2·4	1	9·4	2	6·2	3	5·0	4	5·5	5	7·7	7	0·0	2
2¼	0	1·0	0	3·7	0	7·8	1	0·8	1	7·0	2	2·9	3	0·4	3	11·6	5	0·1	6	2·7	2¼
2½	0	0·9	0	3·3	0	7·0	0	11·5	1	5·1	2	0·2	2	8·8	3	6·8	4	6·1	5	7·2	2½
3	0	0·7	0	2·8	0	5·0	0	9·6	1	2·3	1	8·2	2	3·3	2	11·7	3	9·1	4	8·0	3
3½	0	0·6	0	2·4	0	5·0	0	8·2	1	0·2	1	5·3	1	11·4	2	6·6	3	2·7	4	0·0	3½
4	0	0·6	0	2·1	0	4·4	0	7·2	0	10·7	1	3·1	1	8·5	2	2·7	2	9·8	3	6·0	4
4½	0	0·5	0	1·9	0	3·9	0	6·4	0	9·5	1	1·4	1	6·2	1	11·8	2	6·1	3	1·4	4½
5	0	0·4	0	1·7	0	3·5	0	5·8	0	8·6	1	0·1	1	4·4	1	9·4	2	3·1	2	9·6	5
6	0	0·4	0	1·4	0	2·9	0	4·8	0	7·1	0	10·1	1	1·7	1	5·8	1	10·6	2	4·0	6
7	0	0·3	0	1·2	0	2·5	0	4·1	0	6·1	0	8·6	0	11·7	1	3·3	1	7·3	2	0·0	7
8			0	1·0	0	2·2	0	3·6	0	5·4	0	7·6	0	10·2	1	1·4	1	4·9	1	9·0	8
9					0	1·9	0	3·2	0	4·8	0	6·7	0	9·1	0	11·9	1	3·0	1	6·7	9
10							0	2·9	0	4·3	0	6·0	0	8·2	0	10·7	1	1·5	1	4·8	10
11									0	3·9	0	5·5	0	7·5	0	9·7	1	0·3	1	3·3	11
12											0	5·0	0	6·8	0	8·9	0	11·3	1	2·0	12

Look for the velocity of water in the pipe in the upper row, and in the column below it, and opposite to the given diameter of the pipe standing in the last column, will be found the perpendicular height of a column or head, in feet, inches, and tenths, requisite to overcome the friction of such pipe for 100 feet in length, and obtain the given velocity.

22. *Table containing the quantity of Water discharged over an inch vertical section of a Weir.*

Depth of the upper edge of the waste-board below the surface in English inches.	Cubic feet of water discharged in a minute by an inch of the waste-board, according to Du Buat's formulæ.	Cubic feet of water discharged in a minute by an inch of the waste-board, according to experiments made in Scotland.	Gallons of 282 inches, corresponding with results in col. 3.
1	0·403	0·428	2·621
2	1·140	1·211	7·417
3	2·095	2·226	13·634
4	3·225	3·427	20·990
5	4·507	4·789	29·332
6	5·925	6·295	38·357
7	7·466	7·933	48·589
8	9·122	9·692	59·364
9	10·884	11·564	70·826
10	12·748	13·535	83·164
11	14·707	15·632	95·746
12	16·758	17·805	109·055
13	18·895	20·076	122·965
14	21·117	22·437	137·427
15	23·419	24·883	152·408
16	25·800	27·413	167·905
17	28·258	30·024	183·897
18	30·786	32·710	200·350

To the above table, originally due to *Du Buat*, is added a third column, containing the quantities of water discharged, as inferred from experiments made in Scotland, and examined by Dr. Robison, who found that they in general gave a discharge $\frac{1}{16}$ greater than that which is deduced from *Du Buat's* formulæ. We would recommend it therefore to the engineer to employ the third column in his practice, or the fourth if he wish for the result in gallons.

If they be odd quarters of an inch, look in the table for as many inches as the depth contains quarters, and take the eighth part of the answer. Thus, for $3\frac{3}{4}$ inches, take the eighth part of 24·883, which corresponds to 15 inches. This is 3·110.

23. The quantity discharged increases more rapidly than the width: to obtain a correct measure of it, if l be the width or length of the

waste-board in inches, take $(l + \frac{1}{20} l)$ times the quantity for one inch of wasteboard of the given depth, from the preceding table.

In the preceding table it is supposed that the water from which the discharge is made is perfectly stagnant; but if it should happen to reach the opening with any velocity, we have only to multiply the area of the section by the velocity of the stream.

24. When the quantity of water Q discharged over a weir, is known, the depth of the edge of the wasteboard, or H, may be approximated from the following formulæ, l length of wasteboard.

$$H = \left(\frac{Q}{11{\cdot}4172\, l}\right)^{\frac{2}{3}} = \left(\frac{Q}{11{\cdot}5\, l}\right)^{\frac{2}{3}} \text{ nearly} \quad \ldots\ldots\ldots \text{(XIX.)}$$

$$Q = 11{\cdot}5\, l\, H^{\frac{3}{2}} \text{ nearly} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(XX.)}$$

or, more accurately by adding the correction in article 15,

$$Q = 11{\cdot}5 \left(l + \frac{l}{20}\right) H^{\frac{3}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(XXI.)}$$

26. The quantities discharged for any given width are as the $\frac{3}{2}$ power of the depth, or as $H^{\frac{3}{2}}$.

Hence, to extend the use of the table to greater depths, we have only for

Twice	any depth, take	Q ×	2·828
3 times	.....	Q ×	5·196
4 times	.....	Q ×	8·000
5 times	.....	Q ×	11·180
6 times	.....	Q ×	14·697
7 times	.....	Q ×	18·520
8 times	.....	Q ×	22·627
9 times	.....	Q ×	27·000
10 times	.....	Q ×	31·623

and the results will be *nearly* true.

To make them still more correct, where great accuracy is required, add to them their *thousandth* part.

Examples of the use of the Tables and Rules.

Ex. 1. Let the depth of a weir be 10 inches below the upper surface of the water, and the width 8 inches. How many cubic feet of water will be discharged in a minute?

	cub. feet.
By table § 23. Q for 10 inches deep and 1 inch wide =	13·535
Multiply this by l = ...	8
	106·280
Add $\frac{1}{20}$ of this product	5·314
Discharge in one minute = ...	111·594

Ex. 2. Let the depth be 9 feet, and the width 1 foot. Required the cubic feet discharged in a minute.

By the table Q for 12 inches deep and 1 inch wide =	17·805
Factor for 9 times depth = ...	27
Quantity for 1 inch width	480·735
Multiply by l = ...	12
	5768·802
Add $\frac{1}{10}$ of the product.....	286·441
Total quantity in cubic feet = ...	6055·261

Ex. 3. Let a square orifice of 6 inches each side be placed in a sluice-gate with its top 4 feet below the upper surface of the water: how much will it discharge in a minute?

Here the quantity discharged by a slit in depth 48 inches, must be taken from one in depth 54 inches.

	cubic feet.
For 54, multiply Q for 6 by $3^{\frac{3}{2}}$ or 27	169·965
For 48, ——— Q for 12 by $4^{\frac{3}{2}}$ or 8	142·440
Difference	27·525

$27{\cdot}525 \times (6 + \frac{6}{10}) = 173{\cdot}4$ cubic feet, quantity discharged.

Note.—In an example like this, it is a good approximation, *to multiply continually together the area of the orifice, the number* 336*, *and the square root of the depth in feet of the middle of the orifice.*

Thus, in the preceding example, it will be $\frac{1}{2} \times \frac{1}{2} \times 336 \times \sqrt{4{\cdot}25}$ $= \frac{1}{4} \times 336 \times 2{\cdot}062 = 173{\cdot}2$ cubic feet.

The less the height of the orifice compared with its depth under the water, the nearer will the result thus obtained approach to the truth.

If the height of the orifice be such as to require consideration, the principle of § 7, page 261, may be blended with this rule.

Thus, applying this rule to *Ex.* 2, we shall have area $\times \sqrt{}$ depth $\times 336 \times \frac{2}{3} = 9 \times 3 \times 224 = 6048$, for the cubic feet discharged. This is less than the former result by about its 900th part. It is, therefore, a good approximation, considering its simplicity: it may in many cases supersede the necessity of recurrence to tables.

SECT. III. *Contrivances to measure the velocity of running waters.*

1. For these purposes, various contrivances have been proposed, of which two or three may be here described.

* 336 = 5·6 × 60.

Suppose it be the velocity of the water of a river that is required; or, indeed, both the velocity and the quantity which flows down it in a given time. Observe a place where the banks of the river are steep and nearly parallel, so as to make a kind of trough for the water to run through, and by taking the depth at various places in crossing make a true section of the river. Stretch a string at right angles over it, and at a small distance another, parallel to the first. Then take an apple, an orange, or other small ball, just so much lighter than water as to swim in it, or a pint or quart bottle partly filled with water, and throw it into the water above the strings. Observe when it comes under the first string, by means of a quarter-second pendulum, a stop watch, or any other proper instrument; and observe likewise when it arrives at the second string. By this means the velocity of the upper surface, which in practice may *frequently* be taken for that of the whole, will be obtained. And the section of the river at the second string must be ascertained by taking various depths, as before. If this section be the same as the former, it may be taken for the mean section: if not, add both together, and take half the sum for the mean section. Then the area of the mean section in square feet being multiplied by the distance between the strings in feet, will give the contents of the water in solid feet which passed from one string to the other during the time of observation: and this, by the rule of three, may be adapted to any other portion of time. The operation may often be greatly abridged by taking notice of the arrival of the floating body opposite two stations on the shore, especially when it is not convenient to stretch a string across. An arch of a bridge is a good station for an experiment of this kind, because it affords a very regular section and two fixed points of observation; and in some instances the sea practice of heaving the log may be advantageous. Where a timepiece is not at hand, the observer may easily construct a quarter-second or other pendulum, by means of the rules and table relating to pendulums in the Dynamics.

2. M. Pitot invented a stream measurer of a simple construction, by means of which the velocity of any part of a stream may readily be found. This instrument is composed of two long tubes of glass, open at both ends; one of these tubes is cylindrical throughout; the other has one of its extremities bent into nearly a right angle, and gradually enlarges like a funnel, or the mouth of a trumpet: these tubes are both fixed in grooves in a triangular prism of wood, so that their lower extremities are both on the same level, standing thus one by the side of the other, and tolerably well preserved from accidents. The frame in which these tubes stand is graduated, close by the side of them, into divisions of inches and lines.

To use this instrument, plunge it perpendicularly into the water, in such manner that the opening of the funnel at the bottom of one of the tubes shall be completely opposed to the direction of the current, and the water pass freely through the funnel up into the tube. Then observe to what height the water rises in each tube, and note

the difference of the sides, for this difference will be the height due to the velocity of the stream. It is manifest, that the water in the cylindrical tube will be raised to the same height as the surface of the stream by the hydrostatic pressure, while the water entering from the current by the funnel into the other tube will be compelled to rise above that surface by a space at which it will be sustained by the impulse of the moving fluid: that is, the momentum of the stream will be in equilibrio with the column of water sustained in one tube above the surface of that in the other. In estimating the velocity by means of this instrument, we must have recourse to theory, as corrected by experiments. Thus, if h, the height of the column sustained by the stream, or the difference of heights in the two tubes, be in feet, we shall have $v = 6{\cdot}5 \sqrt{h}$, nearly, the velocity, per second, of the stream; if h be in inches, then $v = 22{\cdot}47 \sqrt{h}$, nearly: or farther experiments made with the same instrument may a little modify these coefficients.

It will be easy to put the funnel into the most rapid part of the stream, if it be moved about to different places until the difference of altitude in the two tubes becomes the greatest. In some cases it will happen, that the immersion of the instrument will produce a little eddy in the water, and thus disturb the accuracy of the observation: but keeping the instrument immersed only a few seconds will correct this. The wind would also affect the accuracy of the experiments; it is, therefore, advisable to make them where there is little or no wind. By means of this instrument a great number of curious and useful observations may easily be made: the velocity of water at various depths in a canal or river may be found with tolerable accuracy, and a mean of the whole drawn, or they may be applied to the correcting of the theory of waters running down gentle slopes. The observations may likewise be applied to ascertain whether the augmentations of the velocities are in proportion to the increase of water passing along the same canal, or what other relations subsist between them, &c.

Where great accuracy is not required, the tube with the funnel at bottom will alone be sufficient, as the surface of the water will be indicated with tolerable precision by that part of the prismatic frame for the tube which has been moistened by the immersion: and the *velocities* may be marked against those altitudes in the tube which indicate them.

3. Another good and simple method of measuring the velocity of water in a canal, river, &c., is that described by the Abbé *Mann*, in his treatise on rivers; it is this:—Take a cylindrical piece of dry light wood, and of a length something less than the depth of the water in the river; about one end of it let there be suspended as many small weights as may keep the cylinder in a vertical or upright position, with its head just above water. To the centre of this end fix a small straight rod, precisely in the direction of the cylinder's axis, in order that, when the instrument is suspended in the water, the devia-

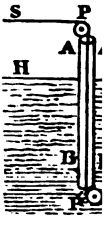
tions of the rod from a perpendicularity to the surface of indicate which end of the cylinder goes foremost, by which discovered the different velocities of the water at different for when the rod inclines forward, according to the direction current, it is a proof that the surface of the water has the velocity; but when it reclines backward, it shows that the current is at the bottom; and when it remains perpendicular sign that the velocities at the top and bottom are equal.

This instrument, being placed in the current of a river o receives all the percussions of the water throughout the whol and will have an equal velocity with that of the whole curre the surface to the bottom at the place where it is put by that means may be found, both with exactness and e mean velocity of that part of the river for any determinate and time.

But, to obtain the mean velocity of the whole section of t the instrument must be put successively both in the mid towards the sides, because the velocities at those places are of different from each other. Having by these means found th velocities, from the spaces run over in certain times, the arit mean proportional of all these trials, which is found by divi common sum of them all by the number of the trials, will mean velocity of the river or canal. And if this medium vel multiplied by the area of the transverse section of the water place, the product will be the quantity running through that a second of time.

The cylinder may be easily guided into that part which to measure, by means of two threads or small cords, wh persons, one on each side of the canal or river, must hold and taking care at the same time neither to retard nor accele motion of the instrument.

4. Let A A′ B B′ be a hollow cylinder, open at both ends, it be capable of being fixed by the side of a platform or of a that its lower extremity B B′ may be placed at any proposed depth below H R, the upper surface of the stream. Let P P′ be pulleys, fixed at opposite sides of the top and bottom of the tube. To G, a globe of specific gravity nearly the same as that of water, let a cord G P′ P S be attached, passing freely over the pulleys P′, P, and having sufficient length towards S to allow of its running off to any convenient distance. Then, the bottom of the tube being immersed to any proposed depth, let the globe G be exposed to the free operation stream; and as it is carried along with it, it will in 1, 2, seconds, or any other interval of time, draw off from a fixe as S, a portion of cord; from which and the time elapsed, the at the assigned depth will become known.



marex invented, in 1720, an instrument called the *Marine* , for the double purpose of measuring a ship's way, and ng the velocities of streams. It is described in the *Phil.* l. 33; and in the succeeding volume a curious example of given in "tables showing the strength and gradual increase se of the tides of flood and ebb in the river Thames, as in Lambeth Reach." They are too extensive to be inserted are very interesting, and may be seen in the *Philosophical ons Abridged*, vol. vii. p. 133.

CHAP. VI.

PNEUMATICS.

SECT. I. *Weight and Equilibrium of Air and Elastic Fluids.*

1. THE fundamental propositions that belong to hydrostatics are common to the compressible and incompressible fluids, and need not, therefore, be repeated here.

2. *Atmospheric air* is the best known of the elastic fluids, and has been defined to be an elastic fluid, having weight, and resisting compression with forces that are directly as its density, or inversely as the spaces within which the same quantity of it is contained.

The correctness of this definition is confirmed by experiment.

3. The weight of air is known from the Torricellian experiment, or that of the barometer. The air presses on the orifice of the inverted tube with a force just equal to the weight of the column of mercury sustained in it.

The weight of a cubic foot of air, with the barometer at 30 inches, and the temperature 55° of Fahr., is 1·2 ounces avoirdupois, or about $\frac{1}{833}$rd part of the weight of a cubic foot of water under the same circumstances.

A bottle, weighed when filled with air, is found heavier than after the air is extracted. The mean pressure of the atmosphere at London is 14·18 lbs. on every square inch of the earth's surface, which is equivalent to the pressure of a column of quicksilver 28·89 inches in height; or a column of water nearly 33 feet in height. Hence the total pressure on the convex surface of the earth amounts to 10,686,000,000 hundreds of millions of pounds.

The elastic force of the air is proved, by simply inverting a vessel full of air in water: the resistance it offers to farther immersion, and the height to which the water ascends within it, in proportion as it is farther immersed, are proofs of the elasticity of the air contained in it*.

* It is in virtue of this property, and ought to be known as extensively as possible, that a man's hat will serve in most cases as a temporary life-preserver, to persons in hazard of drowning, by attending to the following directions:—When a person finds himself in, or about to be in, the water, let him lay hold of his hat between his hands, laying the crown close under his chin, and the mouth under the water. By this means, the quantity of air contained in the cavity of the hat will be sufficient to keep the head above water for several hours, or until assistance can be rendered.

4. When air is confined in a bent tube, and loaded with different weights of mercury, the spaces it is compressed into are found to be inversely as those weights. But those weights are the measures of the elasticity; therefore the elasticities are inversely as the spaces which the air occupies.

The densities are also inversely as those spaces; therefore the elasticity of air is directly as its density. This law was first proved by Mariotte's experiments.

In all this, the temperature is supposed to remain unchanged.— These properties seem to be common to all elastic fluids.

Air resists compression equally in all directions. No limit can be assigned to the space which a given quantity of air would occupy if all compression were removed.

5. In ascending from the surface of the earth, the density of the air necessarily diminishes: for each stratum of air is compressed only by the weight of those above it; the upper strata are therefore less compressed, and of course less dense than those below them.

6. Supposing the same temperature to be diffused through the atmosphere, if the heights from the surface be taken increasing in arithmetical progression, the densities of the strata of air will decrease in geometrical progression. Also, since the densities are as the compressing forces, that is, as the columns of mercury in the barometer, the heights from the surface being taken in arithmetical progression, the columns of mercury in the barometer at those heights will decrease in geometrical progression.

7. Logarithms have, relatively to the numbers which they represent, the same property, therefore if b be the column of mercury in the barometer at the surface, and β at any height h above the surface, taking m for a constant coefficient, to be determined by experiment,

$$h = m(\log b - \log \beta), \text{ or } h = m \log \frac{b}{\beta} \quad \ldots\ldots\ldots\ldots \text{(I.)}$$

where m may be determined by finding trigonometrically the value of h in any case, where b and β have been already ascertained.

8. If b be the height of the mercury in the barometer at the lowest station, β at the highest, t and t' the temperatures of the air at those stations, f the fixed temperature at which no correction is required for the temperature of the air; and if q and q' be the temperatures of the quicksilver in the two barometers, and n the expansion of a column of quicksilver, of which the length is 1, for each degree of heat; h being the perpendicular height (in fathoms) of the one station above the other,

$$h = 10000\left\{1 + \cdot00244\left(\frac{t+t'}{2} - f\right)\right\} \log \frac{b}{\beta \times [1 + n(q-q')]} \quad \text{(II.)}$$

n being nearly $= \frac{1}{10000}$.

If the centigrade thermometer is used, because the beginning of the scale agrees with the temperature f, so that $f = 0$, the formula becomes more simple; and if the expansion for air and mercury be both adapted to the degrees of this scale,

$$h = 10000\left\{1 + \cdot00441\left(\frac{t+t'}{2}\right)\right\} \log \frac{b}{\beta[1 + \cdot00018(q - q')]} \quad \text{(III.)}$$

9. The temperature of the air diminishes on ascending into the atmosphere, both on account of the greater distance from the earth, the principal source of its heat, and the greater power of absorbing heat that air acquires by being less compressed.

10. Professor Leslie, in the notes on his *Elements of Geometry*, p. 495 (edit. 2nd), has given a formula for determining the temperature of any stratum of air when the height of the mercury in the barometer is given. The column of mercury at the lower of two stations being b, and at the upper β, the diminution of heat, in degrees of the centigrade thermometer, is $\left(\frac{b}{\beta} - \frac{\beta}{b}\right) 25$. Which seems to agree well with observation.

11. If the atmosphere were reduced to a body of the same density which it has at the surface of the earth, and of the same temperature, the height to which this homogeneous atmosphere would extend is, in fathoms, equal to $4343\left(1 + \cdot00441\,\frac{t+t'}{2}\right)$, or, taking the expansion according to Laplace, $= 4343\left(1 + \frac{4 \times t}{1000}\right)$ . . . (IV.)

Hence if b be the height of the mercury in the barometer, reduced to the temperature t, the specific gravity of mercury is to that of air, as b to $4343\left(1 + \frac{4t}{1000}\right)$, or the specific gravity of air

$$= \frac{b}{72 \times 4343 \times \left(1 + \frac{4t}{1000}\right)}.$$

The divisor 72 is introduced in consequence of b being expressed in inches.—(*Playfair's Outlines.*)

12. The velocity with which air rushes into a vacuum is equal to that which a heavy body would acquire in falling from a height equal to that of a homogeneous atmosphere equivalent in weight to that of the air at the time. Thus if H be the height of the homogeneous atmosphere, deduced from the formula (IV.) above, expressed in feet, and V the velocity of the air in feet per second, we have from the formula, page 215,

$$V = \frac{g\sqrt{H}}{4} \quad \text{..... (V.)}$$

From which we find the value of V for the mean temperature and pressure to be 1360 feet per second.

13. The velocity with which sound traverses the atmosphere depends not only upon the direction and force of the wind, but also on the temperature and density of the air at the time, and the quantity of moisture which it contains. In the following formula given by Mr. Galbraith, of Edinburgh, the whole of these circumstances are taken into account. Let v be the velocity of sound in feet per second, t the temperature of the air, Θ the elastic force of vapour for the due point, β the barometric pressure, λ the latitude of the place of observation, ω the velocity of the wind, and ϕ the angle which the direction of the wind makes with that of the sound; then

$$v = (102{\cdot}4225 + 0{\cdot}1103\,t)\left(1 + \frac{\Theta}{5\frac{1}{2}\beta - 2\Theta}\right)(10{\cdot}2738 - 0{\cdot}01378 \cdot \cos 2\lambda) + \omega \cdot \cos\phi \quad \ldots\ldots\ldots\ldots \text{(VI.)}$$

SECT. II. *Machines for Raising Water by the Pressure of the Atmosphere.*

1. The term *Pump* is generally applied to a machine for raising water by means of the air's pressure.

2. The *common suction-pump* consists of two hollow cylinders, which have the same axis, and are joined in A C (fig. 216). The lower is partly immersed, perpendicularly, in a spring or reservoir, and is called the *suction-tube;* the upper the *body of the pump.* At A C is a fixed sucker containing a valve which opens upwards, and is less than 34 feet from the surface of the water. In the body of the pump is a piston D made air-tight, movable by a rod and handle, and containing a valve opening upwards. And a spout G is placed at a greater or less distance, as convenience may require, above the greatest elevation of D.

The action of this pump is as follows. Suppose the movable piston D at its lowest depression, the cylinders free from water, and the air in its natural state. On raising this piston, the pressure of the air above it keeping its valve closed, the air in the lower cylinder A B forces open the valve at A C, and occupies a larger space, viz., between B, the surface of the water, and D; its elastic force, therefore, being diminished, and no longer able to sustain the pressure of the external air, this latter forces up a portion of the water into the cylinder A B to restore the equilibrium. This continues till the piston has reached its greatest elevation, when the valve at A C closes. In its subsequent descent, the air below D becoming condensed, keeps the valve at A C closed, and escapes by forcing open that at D till the piston has reached its greatest depression. In the following turns a similar effect is produced, till at length the water rising in the cylinder forces open the valve at A C, and enters the body of the pump;

when, by the descent of D, the valve in AC is kept closed, and the water rises through that in D, which on re-ascending carries it forward, and throws it out at the spout G.

3. *Cor.* 1. The greatest height to which the water can be raised in the common pump by a single sucker is when the column is in equilibrio with the weight of the atmosphere, that is, between 32 and 36 feet.

4. *Cor.* 2. The quantity of water discharged in a given time is determined by considering that at each stroke of the piston a quantity is discharged equal to a cylinder whose base is a section of the pump, and altitude the play of the piston.

5. To determine the force necessary to overcome the resistance experienced by the piston in ascending. Let h = the height HF (fig. 217) of the surface of the water in the body of the pump above EF, the level of the reservoir; and a^2 = the area of the section MN. Let h_1 = the height of the column of water equivalent to the pressure of the atmosphere; and suppose the piston in ascending to arrive at any position mn which corresponds to the height IF. It is evident that the piston is acted upon downwards by the pressure of the atmosphere $= a^2 h_1$, and by the pressure of the column $Bm = a^2 \times HI$; therefore the whole tendency of the piston to descend $= a^2 (h_1 + HI)$.

But the piston is acted upon upwards by the pressure of the air on the external surface EF of the reservoir $= a^2 h_1$; part of which is destroyed by the weight of the column of water having for its base mn, and height FI;

$$\therefore \text{ the whole action upwards} = a^2 \times (h_1 - FI);$$

$$\text{whence } F = a^2 . (h_1 + HI) = a^2 . (h_1 - FI)$$
$$= a^2 . FH = a^2 h,$$

that is, the piston throughout its ascent *is opposed by a force equal to the weight of a column of water having the same base as the piston, and an altitude equal to that of the surface of the water in the body of the pump above that in the reservoir.* In order, therefore, to produce the upward motion of the piston, a force must be employed equal to that determined above, together with the weight of the piston and rod, and the resistance which the piston may experience in consequence of the friction against the inner surface of the tube *.

* Suppose the body of the pump to be 6 inches in diameter, and the greatest height to which the water is raised to be 30 feet; suppose, also, the weight of the piston and its rod to be 10 lbs., and the friction one-fifth of the whole weight. Then, $\frac{1}{10}$ of the square of the diameter gives the ale gallons in a yard in length of the cylinder, and an ale gallon weighs $10\frac{1}{2}$ lbs. Therefore $(6^2 \times 10) + \frac{1}{5}(6^2 \times 10) = 360 + 7{\cdot}4 = 367{\cdot}4$ lbs., weight of the opposing column of water. And $367{\cdot}4 + 10 + \frac{1}{5}(377{\cdot}4) = 452{\cdot}9$ lbs., whole opposing pressure.

If the piston rod be moved by a lever whose arms are as 10 to 1, this pressure will be balanced by a force of 45·29 lbs., and overcome by any greater force.

When the piston begins to descend, it will descend by its own weight; the only resistance it meets with being friction, and a slight impact against the water.

6. *Cor.* 1. If the water has not reached the piston, let its level be in $v\,x$. The under surface of the piston will be pressed by the internal rarefied air. But this air, together with the column of water, $\mathrm{E}\,v$, is in equilibrio with the pressure of the atmosphere $a^2 h_1$; and $\therefore$ its pressure $= a^2 . (h_1 - \mathrm{E}\,v)$. And the pressure downwards $= a^2 h_1$;

$$\therefore \mathrm{F} = a^2 \times \mathrm{E}\,v.$$

Hence the force requisite to keep the piston in equilibrio increases as the water rises, and becomes constant and $= a^2 h$ as soon as the water reaches the constant level B H.

7. *Cor.* 2. If the weight of the piston be taken into the account, let this weight be equal to that of a column of water whose base is $m\,n$ and height p, $= a^2 p$;

$$\therefore \mathrm{F} = a^2 . (\mathrm{E}\,v + p).$$

8. To determine the height to which the water will rise after one stroke of the piston; the fixed valve being placed at the junction of the suction-tube and body of the pump: supposing that after every elevation of the piston there is an equilibrium between the pressure of the atmosphere on the surface of the water in the reservoir, and the elastic force of the rarefied air between the piston and surface of the column of water in the tube, together with the weight of that column.

Let $a\,b$ (fig. 218) be the surface of the water in the suction-tube, after the first stroke of the piston: if the piston were for an instant stationary at D, the pressure of the atmosphere would balance $\mathrm{E}\,b$, and the elastic force of the air in $\mathrm{N}\,a$.

Let A E, the height of the suction-tube, $= a$,
D R, the play of the piston, $= b$,
$h =$ the height of a column of water equivalent to the pressure of the atmosphere,
$y =$ the height of a column equivalent to the pressure of the air in $\mathrm{N}\,a$,
$x = \mathrm{E}\,a$,
$\pi = 3{\cdot}14159$
and R and $r =$ the radii of the body and the suction tube.

Then $x + y = h$,

$$\text{and } y = h . \frac{\mathrm{A\,F}}{\mathrm{N}\,a} = h . \frac{\pi r^2 a}{\pi \mathrm{R}^2 b + \pi r^2 . (a - x)} = \frac{h r^2 a}{\mathrm{R}^2 b + r^2 a - r^2 x};$$

$$\text{whence } h = x + \frac{h r^2 a}{\mathrm{R}^2 b + r^2 a - r^2 x};$$

$$\therefore h\,\mathrm{R}^2 b + h r^2 a - h r^2 x = \mathrm{R}^2 b\,x + r^2 a\,x - r^2 x^2 + h r^2 a,$$

or $x^2 - (h + \frac{R^2}{r^2} \cdot b + a) \cdot x = - h \cdot \frac{R^2}{r^2} \cdot a$,

and $x^2 - p x = - h m b$,

if m be $= \frac{R^2}{r^2}$, and $p = h + m b + a$;

$\therefore x = \frac{1}{2} \cdot \{p \pm \sqrt{p^2 - 4 h m b}\}$,

and $y = \frac{1}{2} \cdot \{2 h - p \mp \sqrt{p^2 - 4 h m b}\}$,

ly one of which values will be applicable, viz. that which answers
the lower sign; since x and y must be less than h; and if the
pper sign be used, x will be found greater than h.

9. Having given the height of the water raised, and that due to
he pressure of air in the pump after the first ascent of the piston
o determine them for the second, third, &c. ascents.

Let E a' represent the height of the water after the second ascent
and let it $= x_1$,

and let $y_1 =$ the height due to the elastic force of the air;

then $x_1 + y_1 = h$;

and $y_1 = y \cdot \frac{A\ b}{N\ a'}$, since the air which occupied C a now occupies
N a';

$\therefore y_1 = \frac{y\ r^2 \cdot (a - x)}{R^2 b + r^2 \cdot (a - x_1)} = \frac{y \cdot (a - x)}{m b + a - x_1}$;

whence $h = x_1 + \frac{y \cdot (a - x)}{m b + a - x_1}$,

and $\therefore x_1 = \frac{1}{2} \cdot \{p - \sqrt{p^2 - 4 h m b - 4 x \cdot (p + a - x)}\}$,

and $y_1 = \frac{1}{2} \cdot \{2 h - p + \sqrt{p^2 - 4 h m b - 4 x \cdot (h + a - x}$

From these are deduced values of x_2, y_2, x_3, y_3, &c.

$x_2 = \frac{1}{2} \cdot \{p - \sqrt{p^2 - 4 h m b - 4 x_1 \cdot (h + a - x_1)}\}$,

$y_2 = \frac{1}{2} \cdot \{2 h - p + \sqrt{p^2 - 4 h m b - 4 x_1 \cdot (h + a -$

and so on. Whence if x_n be taken to represent the height
water after $(n + 1)$ ascents,

$x_n = \frac{1}{2} \cdot \{p - \sqrt{p^2 - 4 h m b - 4 x_{n-1} \cdot (h + a - x_{n-1})}\}$, .

and $y_n = \frac{1}{2} \cdot \{2 h - p + \sqrt{p^2 - 4 h m b - 4 x_{n-1} \cdot (h + a - x_{n-1})}\}$

From which may be determined the height to which the
rise after any given number of ascents of the piston, and
elastic force of the air in the suction-tube. And knowing

tion due to each particular stroke, the differences of those elevations, and the successive differences in the elastic force of the remaining air, may be known.

10. If the weight of the valve c be not considered, it is evident that after a certain number of strokes a vacuum will be produced in the suction-tube, provided it be equal to or not greater than the height due to the pressure of the atmosphere, that is, if a be not greater than h.

For, in this case, $x_n = x_{n-1}$,

and $\therefore x_{n-1} = \frac{1}{2}\{p - \sqrt{p^2 - 4hmb - 4x_{n-1}.(h + a - x_{n-1})}\}$, (IX.)

whence $x_{n-1} = h$, the greatest height of the column of water in the tube. If, therefore, the length of the suction-tube do not exceed the height due to the pressure of the atmosphere, the water will continue to ascend in it after every stroke of the piston, till at length it will into the body of the pump.

But if the altitude of A F be greater than h, the water will continue to ascend without ever reaching its maximum height. For, in this case, an actual vacuum cannot be produced; and as $x_n + y_n = h$, and y_n can never become $= 0$; $\therefore x_n$ can never $= h$*. But, the successive values of y continually decreasing, the corresponding values of x will continually increase.

11. If the weight of the valve c be taken into the account, a column of water must be added equal to the additional pressure to be overcome. Let l = the height of this column, then

$$x + y + l = h;$$
$$\text{and } \therefore x + y = h - l = h_1.$$

If therefore this value of h_1 be substituted for h, the preceding equations are applicable.

12. In the preceding cases, the movable piston has been supposed to descend to A C. If it does not, it may happen that the water may not reach A C, though A C be less than 34 feet from the surface of the water in the reservoir.

After the first elevation of the movable piston to its greatest altitude, c being closed, the elastic force of the air between D N and A C is $(h - x)$, and its magnitude $\pi b R^2$. If, in descending, the piston describes a space b' less than b, so as to stop at a distance $b - b'$ from A C, this magnitude becomes $(b - b').\pi R^2$; $\therefore$ the elastic force is $(h - x).\frac{b}{b - b'}$. Now in order that the pressure upwards may open the valve, this must exceed the elastic force of the atmosphere;

* Hence it appears that it is not *strictly* true, that water will ascend in the suction-tube to a height equal that of a column equivalent to the pressure of the atmosphere. This is a limit to which it approximates, but does not reach in a finite time.

$$\therefore\ (h - x) \cdot \frac{b}{b - b'} > h,$$

$$\text{or } (h - x) \cdot b > h \cdot (b - b');$$

$$\therefore\ bx < hb', \text{ or } \frac{x}{h} < \frac{b'}{b}.$$

If $\therefore\ \frac{b'}{b}$ be less than $\frac{x}{h}$, the valve D N will not open; there will therefore be the same quantity of air between A C and the sucker: which, when the piston has reached its highest elevation, will have the same elastic force as that between A C and $a'\ b'$; and therefore c, being equally pressed on both sides, will remain unmoved, and the water will not ascend.

13. If the fixed valve be placed at the surface of the water; to determine the ascent of the water in the suction-tube.

Let E a, E a' be the successive heights to which the water rises; then, after the first ascent of the piston,

$$x + y = h,$$

$$\text{and } y = \frac{ha}{mb + a - x};$$

$$\text{whence } x = \tfrac{1}{2} \cdot \{p - \sqrt{p^2 - 4hmb}\}$$

$$\text{and } y = \tfrac{1}{2} \cdot \{2h - p + \sqrt{p^2 - 4hmb}\},$$

which equations are the same as were determined for the first ascent of the piston (§ 8). Therefore, in the same manner as before,

$$\text{we shall have } x_n = \tfrac{1}{2} \cdot \{p - \sqrt{p^2 - 4hmb - 4hx_{n-1}}\},$$

$$y_n = \tfrac{1}{2} \cdot \{2h - p + \sqrt{p^2 - 4hmb - 4hx_{n-1}}\}.$$

14. If the water be supposed to stop after $(n + 1)$ ascents of the piston, then $x_n = x_{n-1}$;

$$\text{and } \therefore\ x_{n-1} = \tfrac{1}{2} \cdot \{p - \sqrt{p^2 - 4hmb - 4hx_{n-1}}\},$$

$$\text{whence } x_{n-1} = \tfrac{1}{2} \cdot \{a + mb \pm \sqrt{(a + mb)^2 - 4hmb}\} \quad \text{...(X.)}$$

Hence, therefore, there are two altitudes at which the water may stop in its ascent, if $(a + mb)^2$ is equal to or greater than $4hmb$. In the former case the two values of x_{n-1} are equal, that is, there will be only one altitude $= \frac{1}{2} \cdot (a + mb)$, at which the water will stop. In the latter case there are two which may be ascertained.

If $4hmb$ be greater than $(a + mb)^2$, the water will not stop.

Ex. 1. If $h = 32$ feet, $a = 20$, $b = 4$, and $m = 1$, or the suction-tube and body of the pump be of the same diameter,

$$x_{n-1} = \tfrac{1}{2} \cdot \{20 + 4 \pm \sqrt{(24)^2 - 4 \cdot 1 \cdot 32 \cdot 4}\} = \tfrac{1}{2} \cdot \{24 \pm \sqrt{64}\}$$
$$= 16 \text{ or } 8.$$

Ex. 2. If $h = 32$ feet, $a = 25$, $b = 2$, and $m = 4$,

$$x_{n-1} = \tfrac{1}{2} \cdot \{25 + 8 \pm \sqrt{(33)^2 - 4 \cdot 32 \cdot 4 \cdot 2}\} = \tfrac{1}{2} \cdot \{33 \pm \sqrt{65}\} = 41{\cdot}8062 \text{ or } 24{\cdot}1938.$$

15. If $m = 1$, or the tubes have the same diameter,

$$x_{n-1} = \tfrac{1}{2} \cdot \{a + b \pm \sqrt{(a + b)^2 - 4hb}\},$$

which is imaginary, if $(a + b)^2$ is less than $4hb$, or b greater than $\dfrac{(a + b)^2}{4h}$.

In order, therefore, that this pump may produce its effect, *the play of the piston must be greater than the square of its greatest altitude above the surface of the water in the reservoir divided by four times the height due to the pressure of the atmosphere.*

16. The *lifting-pump* consists of a hollow cylinder, the body of which is immersed in the reservoir. It is furnished with a movable piston, which, entering below, lifts the water up, and is movable by means of a frame which is made to ascend and descend by a handle. The piston is furnished with a valve opening upwards. A little below the surface of the water is a fixed sucker with a valve opening upwards. This is an inconvenient construction, upon the peculiarities of which we need not dwell.

17. The *forcing-pump* consists of a suction-tube A E F C (fig. 219) partly immersed in the reservoir, of the body of the pump A B G C, and of the ascending tube H K. The body is furnished with a movable solid sucker or plunger, D, made air-tight. And at A C and H are fixed suckers with valves opening upwards.

18. To explain the action of this pump. Suppose the plunger D at its greatest depression; the valves closed, and the air in its natural state. Upon the ascent of D, the air in A C D occupying a greater space, its elasticity will be diminished, and consequently the greater elasticity of the air in A F will open the valve at A C, whilst that at H is kept closed by the elasticity of the external air; water therefore will rise in the suction-tube. On the descent of D from its greatest elevation, the elasticity of the air in the body of the pump will keep the valve A C closed, and open that at H, whence air will escape. By subsequent ascents of the piston, the air will be expelled, and water rise into the body. The descending piston will then press the water through the valve at H, which will close, and prevent its return into the body of the pump; D therefore ascending again, the space left void will be filled by water pressing through the valve A C; and this upon the next ascent of D is forced into the ascending tube; and thus, by the ascents and descents of D, water may be raised to the required height.

19. In this pump D must not ascend higher than about 32 feet from the surface of the water in the reservoir.

20. To determine the force necessary to overcome the resistance experienced by the piston: —

Let h = the height of a column of water equivalent to the pressure of the atmosphere, and E B the height to which the water is forced. Let M N be any position of the piston D whose area = A, and the weight of the piston and its appendages = P. Let X = the force necessary to push the piston upwards during the suction, friction not being considered, and Y = that employed to force it down.

When the piston ascends, and H is closed

$$X = P + A h - A . (h - ME)$$
$$= P + A . ME.$$

Let the sucker be in the same position in its descent, and therefore A C closed, and H open,

$$Y = A h + A . MB - (A h + P)$$
$$= A . MB - P.$$

Hence $X + Y = A . EB$; or the whole force exerted, in the case of equilibrium, is equal to the weight of a column of water whose base is equal to that of the piston, and altitude the distance between the surface of the water and the point to which it is to be raised.

21. In this pump the effort is divided into two parts, one opposed to the suction, and the other to the forcing; whereby an advantage is gained over the other pumps where the whole force is exerted at once whilst the water is raised.

22. In order to have the force applied uniform, let $X = Y$;

$$\therefore P + A . ME = A . MB - P;$$
$$\therefore P = \tfrac{1}{2} A . (MB - ME).$$

The piston therefore must play in such a manner that M B may be greater than M E.

23. In the common forcing pump, the stream is intermitting; for there is no force impelling it during the return of the sucker.

One mode of remedying this, is by making an interruption in the ascending tube, which is surrounded by an air vessel T (fig. 220); in which, when the water has risen above Z, the air above it is compressed, and by its elasticity forces the water up through Z; the orifice of which is narrower than that of the tube, and therefore the quantity of water introduced during the descent of the piston will supply its discharge for the whole time of the stroke, producing a continued stream.

24. The *fire-engine* consists of a large receiver A B C D, called the air-vessel, into which water is driven by two forcing-pumps E F, G H (fig. 221), (whose pistons are Q and R), communicating with its lower extremities at I and K, through two valves opening inwards. From the receiver proceeds a tube M L through which the water is thrown, and directed to any point by means of a pipe movable about the extremity L. The pumps are worked by a lever, so that whilst one piston descends the other ascends. The pumps communicate with a reservoir of water at N.

25. To explain the action of this engine.

The tube N being immersed in the reservoir, and the piston R drawn up, the pump G H becomes filled; and the descent of the piston R will, as in the forcing-pump (§ 18), keep the valve H close, and cause the water to pass into the air-vessel by the valve I, whilst, by the weight of the water in the air-vessel, the valve K will be kept shut. In the same manner, when R ascends, Q descending will force the water through K into the air-vessel. By this means the air above the surface of the water becoming greatly compressed will, by its elasticity, force the water to ascend through M L, and to issue with a great velocity from the pipe*.

26. When the air-vessel is half full of water, the air being then compressed into half its natural space will have an elastic force equal to twice the pressure of the atmosphere: therefore, when the stop-cock is turned, the air within pressing on the subjacent water with twice the force of the external air, will cause the water to spout from the engine to the height of $(2-1)\,33$, or 33 feet; except so far as it is diminished by friction.

Or, generally, if $\frac{n-1}{n}$ denote the fractional height of the water in the air-barrel, then $\frac{1}{n}$ will denote the height of the space occupied by the compressed air, n times the pressure of the atmosphere its elastic force, and $(n-1)\,33$, the height in feet to which the water may be projected.

Thus, if $\frac{3}{4}$ of the air-barrel be the height of the water, the elastic force of the air will be four times the pressure of the atmosphere, and $(4-1)\,33 = 99$ feet, the height to which the water may then be thrown by the engine.

27. The modifications in the constructions of pumps with a view to their practical applications are very numerous. Those who wish to acquaint themselves with some of the most useful, may consult the 2nd volume of my *Treatise on Mechanics*, and Nos. 13, 41, 69, and 93, of the *Mechanics' Magazine*.

In addition to these, there may now be presented a short account of a *quicksilver-pump*, which has been recently invented by Mr. Thomas Clark of Edinburgh, and which works almost without friction. It has great power in drawing and forcing water to any height, and is extremely simple in its construction. In fig. 222, *a a* is the main pipe inserted into the well *b*; a valve is situated at *c*, and another at *d*, both opening upwards; a piece of iron tube is then bent into a circular form, as at *f*, again turned off at *g* in an angular direction, so as to pass through a stuffing box at *h*, and from thence bent outwards as at *i*, connecting itself with the ring. A quantity of quick-

* The preceding part of this section is taken from *Bland's Hydrostatics*; a very elegant and valuable work, which I beg most cordially to recommend to those who wish to obtain a comprehensive knowledge of the theory of this department of science.

silver is then put into the ring filling it from q to q', and the ring being made to vibrate upon its axis h, a vacuum is soon effected in the main pipe by the recession of the mercury from g to q, thereby causing the water to rise and fill the vacuum: upon the motion being reversed, the quicksilver slides back to g, forces up the water and expels it at the spout e.

"Mr. Clark calculates that a pump of this description with a ring twelve feet in diameter, will raise water the same height as the common lifting pump, and force it one hundred and fifty feet higher without any friction." (*Mechanics' Register* and *Jamieson's Edinburgh Journal.*)

28. It is usual to class with pumps, the machine known by the name of *Archimedes' screw*, or the *water-snail*. This consists either of a pipe wound spirally round a cylinder, or of one or more spiral excavations formed by means of spiral projections from an internal cylinder, covered by an external cylindrical case, so as to be water tight. The cylinder which carries the spiral is placed aslant, so as to be inclined to the horizon in an angle of from 30° to 45°, and capable of turning upon pivots in the direction of its axis posited at each extremity. The lower end of the spiral canal being immersed in the river or reservoir from which water is to be raised, the water descends at first in the said canal solely by its gravity; but the cylinder being turned, by human or other energy, the water moves on in the canal, and at length it issues at the upper extremity of the tube.

Several circumstances tend to make this instrument imperfect and inefficacious in its operation. The adjustments necessary to insure a maximum of effectual work are often difficult to accomplish. It seldom happens, therefore, that the measure of the work done exceeds *a third* of the power employed: so that this apparatus, notwithstanding its apparent ingenuity and simplicity, is very sparingly introduced by our civil engineers.

29. *Spiral pump.* This machine is formed by a spiral pipe of several convolutions, arranged either in a single plane, as in fig. 223, or upon a cylindrical or conical surface, and revolving round an axis. The curved pipe is connected at its inner end, by a central water-tight joint, to an ascending pipe, r P, while the other end, S, receives during each revolution, nearly equal quantities of air and water. This apparatus is usually called the *Zurich machine*, because it was invented, about 1746, by *Andrew Wirtz*, an inhabitant of Zurich. It has been employed with great success at Florence, and in Russia; and the late Dr. Thomas Young states, that he employed it advantageously for raising water to a height of forty feet. The outer end of the pipe is furnished with a spoon, S, which contains as much water as will half fill one of its coils. The water enters the pipe a little before the spoon has reached its highest position, the other half remaining full of air. This air communicates the pressure of the column of water to the preceding portion; and in this manner the effect of nearly all the water in the wheel is united, and becomes capable of

supporting the column of water, or of water mixed with air, in the ascending pipe. The air nearest the joint is compressed into a space much smaller than that which it occupied at its entrance; so that, when the height is considerable, it becomes advisable to admit a larger portion of air than would naturally fill half the coil. This, while it lessens the quantity of water raised, lessens also the force requisite to turn the machine. The loss of power, supposing the machine well constructed, arises only from the friction of the water against the side of the pipes, and that of the wheel on its axis: and where a large quantity is to be raised to a moderate height, both of these sources of resistance may be rendered very inconsiderable.

30. *Schemnitz vessels*, or the *Hungarian machine*. The mediation of a portion of air is employed for raising water, not only in the spiral pump, but also in the air-vessels of Schemnitz, as shown in fig. 224. A column of water, descending through a pipe, C, into a closed reservoir, B, containing air, obliges the air to act, by means of a pipe, D, leading from the upper part of the air-vessel, or reservoir, on the water in a second reservoir, A, at any distance either above it or below it, and forces this water to ascend through a third pipe, E, to any height less than that of the first column. The air-vessel is then emptied, the second reservoir filled, and the whole operation repeated. The air, however, must acquire a density equivalent to the requisite pressure before it can begin to act: so that, if the height of the columns were thirty-four feet, it must be reduced to half its natural space before any water could be raised, and thus half of the force would be lost. But where the height is small, the height lost in this manner is not greater than what is usually spent in overcoming friction, and other imperfections of the machinery employed. The force of the tide, or of a river rising and falling with the tide, might easily be applied to the purpose of raising water by a machine of this kind. Thus, if at low tide the vessel A were filled with air, then, at high tide, the water flowing down the tube E, would cause the water in the vessel B to ascend in the pipe C.

31. The *hydraulic ram*. In this hydraulic arrangement, the momentum of a stream of water flowing through a long pipe is employed to raise a small quantity of water to a considerable height. The passage of the pipe being stopped by a valve which is raised by the stream, as soon as its motion becomes sufficiently rapid, the whole column of fluid must necessarily concentrate its action almost instantaneously upon the valve. In these circumstances it may be regarded as losing the characteristic property of hydraulic pressure, and to act almost as though it were a single solid: so that, supposing the pipe to be perfectly elastic and inextensible, the impulse may overcome almost any pressure that may be opposed to it. If another valve opens into a pipe leading to an air-vessel, a certain quantity of the water will be forced in, so as to condense the air, more or less rapidly, to the degree that may be required for raising a portion of the water contained in it to a given height.

The late Mr. Whitehurst appears to have been the first who

employed this method: it was afterwards improved by Mr. Boulton. But, like many English inventions, it never was adequately estimated, until it was brought into public notice by a Frenchman. M. Montgolfier, its re-inventor, gave to it the name which it now bears of the *Hydraulic Ram*, in allusion to the battering ram.

The essential parts of this machine are represented in figure 225. When the water in the pipe A B (moving in the direction of the arrows) has acquired sufficient velocity, it raises the valve B, which immediately stops its farther passage. The momentum which the water has acquired then forces a portion of it through the valve, C, into the air-vessel, D. The condensed air in the upper part of D causes the water to rise into the pipe E, as long as the effect of the horizontal column continues. When the water becomes quiescent, the valve B will open again by its own weight, and the current along A B will be renewed, until it acquires force enough to shut the said valve B, open C, and repeat the operation.

The motion in the horizontal tube arises from the acceleration of the velocity of a liquid mass falling down another tube, and communicating with this.

In an experiment made upon an hydraulic ram, at Avilly, near Senlis, by M. Turquet, bleacher, the expense of power was found to be to the produce, as 100 to 62. In another, as 100 to 55; in two others, as 100 to 57. So that a hydraulic ram placed in favourable circumstances, may be reckoned to employ usefully rather more than half its force.

*** For more full accounts of the three last contrivances, the reader may consult the 2nd volume of my *Mechanics*.

SECT. III. *Force of the Wind.*

1. Air, when in continuous motion in one direction, becomes a very useful agent of machinery, of greater or less energy, according to the velocity with which it moves. Were it not for its variability in direction and force, and the consequent fluctuations in its supply, scarcely any more appropriate first mover could generally be wished for. And even with all its irregularity, it is still so useful as to require a separate consideration.

2. The force with which air strikes against a moving surface, or with which the wind strikes against a quiescent surface, is nearly as the square of the velocity: or, more correctly, the exponent of the velocity, determined according to the rule given in Example 6, page 96, varies between 2·03 and 2·05; so that, in most practical cases, the exponent 2, or that of the square, may be employed without fear of error. If β be the angle of incidence, s the surface struck in square feet, and v the velocity of the wind, in feet, per second; then, if f equals the force in avoirdupois pounds, either of the two following approximations may be used, viz.:—

$$f = \frac{v^2 s \sin^2 \beta}{440} \quad \text{..... (XI.)}$$

$$\text{or, } f = \cdot 002288\, v^2 s \sin^2 \beta \quad \text{..... (XII.)}$$

Of these, the first is usually the easiest in operation, requiring only two lines of short division, viz., by 40 and by 11.

If the incidence be perpendicular, $\sin^2 \beta = 1$, and these become,

$$f = \frac{v^2 s}{440} = \cdot 002288\, v^2 s \quad \text{..... (XIII.)}$$

3. The table in the margin exhibits the force of the wind when blowing perpendicularly upon a surface of one foot square, at the several velocities stated. The velocity of 80 miles per hour, is that by which the aëronaut Garnerin was carried in his balloon from Ranelagh to Colchester, in June, 1802. It was a strong and boisterous wind, but did not assume the character of a *hurricane*, although a wind with that velocity is so characterized in Rouse's table. In Mr. Green's aërial voyage from Leeds, in September, 1823, he travelled 43 miles in 18 minutes, although his balloon rose to the height of more than 4000 yards.

Velocity of the Wind.		Perpendicular force on one sq. foot in avoirdupois pounds.
Miles in one hour.	= feet in one second.	
1	1·47	·005
2	2·93	·020
3	4·40	·044
4	5·87	·079
5	7·33	·123
10	14·67	·492
15	22·00	1·107
20	29·34	1·968
25	36·67	3·075
30	44·01	4·429
35	51·34	6·027
40	58·68	7·873
45	66·01	9·963
50	73·35	12·300
60	88·02	17·715
80	117·36	31·490
100	146·70	49·200

4. Borda found by experiment, in the year 1762, that the force of the wind is very nearly as the square of the velocity, but he assigned that force to be greater than what Rouse found (as expressed in the above table) in the ratio of 111 to 100. Borda ascertained also, as was natural to expect, that, upon different surfaces with the same velocity, the force increased more rapidly than the surface. M. Valz, applying the method of the minimum squares to Borda's results, ascertained that the whole might be represented by the formula

$$y = 0{\cdot}001289\, x^2 + 0{\cdot}000050541\, x^3 \quad \text{..... (XIV.)}$$

and nearly as correctly by

$$y = 0{\cdot}00108\, x^{2{\cdot}2} \quad \text{..... (XV.)}$$

x^2 representing the surface in square inches (French), and y the force corresponding to the velocity of 10 feet per second expressed in French pounds.

CHAP. VII.

MECHANICAL AGENTS.

SECT. I. *Water as a Mechanical Agent.*

1. THE impulse of a current of water, and sometimes its weight and impulse jointly, are applied to give motion to mills for grinding corn and for various other purposes. Sometimes the impulse is applied obliquely to float-boards in a manner which may be comprehended at once by reference to a *smoke-jack*, in which the ascending smoke strikes the vanes obliquely, and communicates a rotatory motion. If we imagine the whole mechanism to be inverted, and water to fall upon the vanes, rotation would evidently be produced; and that with greater or less energy in proportion to the quantity of water and the height from which it falls.

Water-wheels of this kind give motion to mills in Germany, and some other parts of the Continent of Europe. I have also seen mills of the same construction in Balta, the northernmost Shetland isle. But wherever they are to be found, they indicate a very imperfect acquaintance with practical mechanics; as they occasion considerable loss of power.

2. Water frequently gives motion to mills, by means of what technically denominated an *undershot* wheel. This has a number of planes disposed round its circumference, nearly in the direction of its radii, and these *float-boards* (as they are called) dipping into the stream, are carried round by it, as shown in fig. 226. The axle of the wheel, of course, by the intervention of proper wheels and pinions, turns the machinery intended to be moved. Where the stream is large and unconfined, the pressure on each float-board is that corresponding to the head due to the relative velocities (or difference between the velocities of stream and float-board): this pressure is, therefore, a *maximum* when the wheel is at *rest;* but the *work performed* is then *nothing*. On the other hand, the *pressure* is nothing when the velocity of the wheel equals that of the stream. Consequently, there is a certain intermediate velocity, which causes the work performed to be a maximum.

The weight equal to the pressure is $Q\,(\sqrt{H} - \sqrt{h})^2$, Q being the quantity of water passing in a second, H the height due to V the velocity of the water, and h that due to U the velocity of the float-board. Considering this as a mass attached to the wheel, its moving force is obtained by multiplying it into U: and as $\sqrt{H} - \sqrt{h}$ varies

as V − U, this moving force varies as $(V - U)^2$. U which is a *maximum* when $U = \frac{1}{3} V$. In this case, then, the rim of the wheel moves with $\frac{1}{3}$ of the velocity of the stream; and the effect which it produces is

$$Q \times (\tfrac{2}{3} V)^2 \times \tfrac{1}{3} V = \tfrac{4}{27} Q V^3:$$

so that the work performed by an *undershot wheel*, according to the usual theory, equals $\frac{4}{27}$ of the moving force.

Friction, and the resistance of fluids, modify these results; but Smeaton and others have found that the maximum work is always obtained when U is between $\frac{1}{2}$ V and $\frac{1}{3}$ V.

3. Where the floats are not totally immersed, the water is heaped upon them; and in this case the pressure is that due to 2 H.

4. When the float-boards move in a circular sweep close fitted to them, or, in general, when the stream cannot escape without acquiring the same velocity as the wheel, the circumstances on which the investigation turns become analogous to what happens in the collision of non-elastic bodies. The stream has the velocity V before the stroke which is reduced to U, and the quantity of motion corresponding to the difference, or to V − U, is transferred to the wheel; this turns with the velocity U; and therefore the effect of the wheel is as $\left(\frac{V - U}{V}\right) U$, or $\frac{V U - U^2}{V}$; which is a maximum when V = 2 U; being then $\frac{1}{4}$ of the moving power.

Hence *appears the utility of constraining the water to move in a narrow channel.*

5. The undershot wheel is used where a large quantity of water can be obtained with a moderate fall. But where the fall is considerable the *overshot* (fig. 227) is almost always employed. Its circumference is formed into angular buckets, into which the water is delivered either at the top or within 60° of it: 52° $\frac{3}{4}$ is the most advantageous distance. In that case, if r = the full radius of the wheel, H the whole, and h the effective height of the fall, $h = r (1 + \sin 37° \frac{1}{2}) = 1{\cdot}605\ r$, and $r = {\cdot}623\ h$. If the friction be about $\frac{2}{3}$ of the moving power, the velocity of the circumference of the wheel to produce a maximum effect, will = $2{\cdot}67 \sqrt{\text{H}}$. Here, too, a fall of $\frac{1}{9}$ H will give the water its due velocity of impact upon the wheel: and $122{\cdot}176\ s\ \text{H}^{\frac{3}{2}}$ equals the mechanical effect in pounds, s being the section, in feet, of the stream that supplies the buckets.

Mr. Smeaton's experiments led him to conclude that overshot wheels do most work when their circumferences move at the rate of 3 feet in a second, and that when they move considerably slower than this, they become unsteady and irregular in their motion. This determination is, however, to be understood with some latitude. He mentions a wheel 24 feet in diameter, that seemed to produce nearly its full effect though the circumference moved at the rate of 6 feet in a second; and another of the diameter of 33 feet, of which the cir-

cumference had only a velocity of 2 feet in a second, without any considerable loss of power. The first wheel turned round in 12·6 seconds, the latter in 51·9 seconds.

6. Where the fall is too small for an overshot wheel, it is most advisable to employ a *breast-wheel* (Plate VI.) which partakes of its properties; its float-boards meeting at an angle, so as to be assimilated to buckets, and the water being considerably confined within them by means of an arched channel fitting moderately close, but not so as to produce unnecessary friction. But when the circumstances do not admit of a breast-wheel, then recourse must be had to the undershot. For such a wheel it is best that the float-boards be so placed as to be perpendicular to the surface of the water at the time they rise out of it; that only one half of each should ever be below the surface, and that from 3 to 5 should be immersed at once. The Abbé Mann proposed that there should not be more than six or eight float-boards on the whole circumference.

7. Mills moved by the *re-action* of water are usually denominated *Barker's Mills;* sometimes, however, *Parent's;* at others, *Segner's.* But the invention is doubtless Dr. Barker's. Their construction is shown in fig. 228, where C D is a vertical axis, moving on a pivot D, and carrying the upper millstone *m*, after passing through a opening in the fixed millstone C. Upon this axis is fixed a vertical tube T T communicating with a horizontal tube A B, at the extremities of which A, B, are two apertures in opposite directions. When water from the mill-course M N is introduced into the tube T T, it flows out of the apertures A, B, and by the re-action or counter-pressure of the issuing water the arm A B, and consequently the whole machine, is put in motion.

In order to understand how this motion is produced, let us suppose both the apertures shut, and the tube T T filled with water up to T. The apertures A and B, which are shut up, will be pressed outwards by a force equal to the weight of a column of water whose height is T T, and whose area is the area of the apertures. Every part of the tube A B sustains a similar pressure; but as these pressures are balanced by equal and opposite pressures, the arm A B is at rest. By opening the aperture at A, however, the pressure at that place is removed, and consequently the arm is carried round by a pressure equal to that of a column T T, acting upon an area equal to that of the aperture A. The same thing happens on the arm T B; and these two pressures drive the arm A B round in the same direction. This machine may evidently be applied to drive any kind of machinery, by fixing a wheel upon the vertical axis C D.

8. Mr. Rumsey, an American, and Mr. Segner, improved this machine, by conveying the water from the reservoir, not by a pipe, in greater part of which the spindle turns, but by a pipe which descends from a reservoir, as F, until it reaches lower than the arms A B, and then turns up by a curvilinear neck and collar, entering between the arms at the lower part, as shown in fig. 229. This greatly diminishes the friction.

9. Professor Playfair has correctly remarked that the moving force becomes greater after the machine has begun to move; for the water in the horizontal arms acquires a centrifugal force, by which its pressure against the sides is increased. When the machine works to the greatest advantage, the center of the perforations should move with the velocity $\frac{r}{r_1}\sqrt{hg}$, where r is the radius of the horizontal arm, measured from the axis of motion to the center of the perforation, and r_1 the radius of the perpendicular tube, g being put for the force of gravity, or $32\frac{1}{6}$ feet.

As $2\pi r$ is the circumference described by the center of each perforation, $\frac{2\pi r}{\sqrt{hg}}$ is the time of a revolution in seconds.

The quantity $\frac{r}{r_1}\sqrt{hg}$ is also the velocity of the effluent water; therefore, when the machine is worked to the greatest advantage, the velocity with which water issues is equal to that with which it is carried horizontally in an opposite direction; so that, on coming out, it falls perpendicularly down.

10. The following dimensions have been successfully adopted; viz. radius of the arms from the center of the pivot to the center of the discharging holes, 46 inches; inside diameter of the arms, 3 inches; diameter of the supplying pipe, 2 inches; and height of the working head of water, 21 feet above the point of discharge. When the machine was not loaded, and had but one orifice open, it made 115 turns in a minute. This agrees to a velocity of 46 feet in a second for the orifice, being greater than the full velocity due to the head of water by between 9 and 10 feet; the difference is due to the effect of the centrifugal force.

The theory of this machine is yet imperfect: but there can be no doubt of its utility in cases where the stream is small, with a considerable fall.

Mr. James Whiteland, a correspondent of the *Franklin Journal*, proposes to make the horizontal arms of the mill of a *curved* form, such that the water will run from the center to the extremity of the arms in a straight line when the machine is working. For the method of constructing the curve, see *Mechanics' Magazine*, No. 499. It is very clear, however, that the additional efficiency of the machine will not be so great, by any means, as the inventor anticipates.

SECT. II. *Air as a Mechanical Agent.*

In the application of wind to mills, whatever varieties there may be in their internal structure, there are certain rules and maxims with

regard to the position, shape, and magnitude of the sails, which will bring them into the best state for the action of the wind, and the production of useful effect. These have been considered much at large by Mr. Smeaton; for this purpose he constructed a machine, of which a particular description is given in the *Philosophical Transactions*, vol. 51. By means of a determinate weight it carried round an axis with an horizontal arm, upon which were four small moveable sails. Thus the sails met with a constant and equable blast of air; and as they moved round, a string with a weight affixed to it was wound about their axis, and thus showed what kind of size or construction of sails answered the purpose best. With this machine a great number of experiments were made: the results of which were as follows:—

(1.) The sails set at the angle with the axis proposed as the best by M. Parent and others, viz. 55°, was found to be the worst proportion of any that was tried.

(2.) When the angle of the sails with the axis was increased from 72° to 75°, the power was augmented in the proportion of 31 to 45; and this is the angle most commonly in use when the sails are planes.

(3.) Were nothing more requisite than to cause the sails to acquire a certain degree of velocity by the wind, the position recommended by M. Parent would be the best. But if the sails are intended, with given dimensions, to produce the greatest effect possible in a given time, we must, if planes are made use of, confine our angle within the limits of 72° and 75°.

(4.) The variation of a degree or two, when the angle is near the best, is but of little consequence.

(5.) When the wind falls upon concave sails it is an advantage to the power of the whole, though each part separately taken should not be disposed of to the best advantage.

(6.) From several experiments on a large scale, Mr. Smeaton has found the following angles to answer as well as any. The radius is supposed to be divided into six parts; and $\frac{1}{6}$th, reckoning from the center, is called 1, the extremity being denoted 6.

No.	Angle with the axis.	Angle with the plane of motion.	
1	72°	18°	
2	71	19	
3	72	18	middle
4	74	16	
5	77½	12½	
6	83	7	extremity.

(7.) Having thus obtained the best method of *weathering* the sails, *i. e.* the most advantageous manner in which they can be placed, our author's next care was to try what advantage could be derived from an increase of surface upon the same radius. The result obtained was, that a broader sail requires a larger angle; and when the sail

ost proper for enlarged sails; it being found in practice that the hould rather be too little than too much exposed to the direct of the wind.

ne have imagined, that the larger the sail the greater would be ıwer of the windmill, and have therefore proposed to fill up the area; and by making each sail a sector of an ellipsis, according Parent's method, to intercept the whole cylinder of wind, in to produce the greatest effect possible. From our author's iments, however, it appeared, that when the surface of all the xceeded seven-eighths of the area, the effect was rather dimi- l than augmented. Hence he concludes, that when the whole er of wind is intercepted, it cannot then produce the greatest for want of proper interstices to escape.

t is certainly desirable," says Mr. Smeaton, "that the sails of nills should be as short as possible; but it is equally desirable, he quantity of cloth should be the least that may be, to avoid ge by sudden squalls of wind. The best structure, therefore, ge mills, is that where the quantity of cloth is the greatest in a circle that can be: on this condition, that the effect holds out portion to the quantity of cloth; for otherwise the effect can gmented in a given degree by a lesser increase of cloth upon a radius than would be required if the cloth was increased upon me radius."

) The ratios between the velocities of windmill sails unloaded, hen loaded to their maximum, turned out very different in dif- experiments; but the most common proportion was as 3 to 2. neral it happened that where the power was greatest, whether enlargement of the surface of the sails, or an increased velocity wind, the second term of the ratio was diminished.

) The ratios between the least load that would stop the sails ıe maximum with which they would turn, were confined be- that of 10 to 8 and 10 to 9; being at a medium about 10 to nd 10 to 9, or about 6 to 5: though on the whole it appeared

sails in ten or fifteen minutes; and, from the length of the arms from tip to tip, has computed, that if an hoop of the same size was to run upon plain ground with an equal velocity, it would go upwards of thirty miles in an hour.

(11.) The load at the maximum is nearly, but somewhat less than, as the square of the velocity of the wind; the shape and position of the sails being the same.

(12.) The effects of the same sails at a maximum are nearly, but somewhat less than, as the cubes of the velocity of the wind.

(13.) The load of the same sails at a maximum is nearly as the squares, and the effect as the cubes of their number of turns in a given time.

(14.) When sails are loaded so as to produce a maximum at given velocity, and the velocity of the wind increases, the load continuing the same; then the increase of effect, when the increase of the velocity of the wind is small, will be nearly as the squares of these velocities: but when the velocity of the wind is double, the effects will be nearly as 10 to $27\frac{1}{2}$; and when the velocities compared are more than double of that where the given load produces a maximum, the effects increase nearly in a simple ratio of the velocity of the wind. Hence our author concludes, that windmills, such as the different species for draining water, &c., lose much of their effect by acting against one invariable opposition.

(15.) In sails of a similar figure and position, the number of turns in a given time will be reciprocally as the radius or length of the sail.

(16.) The load at a maximum that sails of a similar figure and position will overcome, at a given distance from the center of motion, will be as the cube of the radius.

(17.) The effects of sails of similar position and figure are as the square of the radius. Hence augmenting the length of the sail without augmenting the quantity of cloth, does not increase the power; because what is gained by length of the lever is lost by the slowness of the motion. Hence also, if the sails are increased in length, the breadth remaining the same the effect will be as the radius.

(18.) The velocity of the extremities of the Dutch sails, as well as of the enlarged sails, either unloaded or even when loaded to a maximum, is considerably greater than that of the wind itself. This appears plainly from the observations of Mr. Ferguson, already related, concerning the velocity of sails.

(19.) From many observations of the comparative effects of sails of various kinds, Mr. Smeaton concludes, that the enlarged sails are superior to those of the Dutch construction.

(20.) He also makes several just remarks upon those windmills which are acted upon by the direct impulse of the wind against sails fixed to a vertical shaft: his objections have, we believe, been justified in every instance by the inferior efficacy of these horizontal mills.

"The disadvantage of horizontal windmills," he remarks, "does

not consist in this, that each sail, when directly opposed to the wind, is capable of a less power than an oblique one of the same dimensions; but that in an horizontal windmill little more than one sail can be acting at once: whereas in the common windmill, all the four act together; and therefore, supposing each vane of an horizontal windmill to be of the same size with that of a vertical one, it is manifest that the power of a vertical mill will be four times as great as that of an horizontal one, let the number of vanes be what they will. This disadvantage arises from the nature of the thing; but if we consider the further disadvantage that arises from the difficulty of getting the sails back again against the wind, &c., we need not wonder if this kind of mill is in reality found to have not above one-eighth or one-tenth of the power of the common sort; as has appeared in some attempts of this kind."

COULOMB'S EXPERIMENTS.

M. Coulomb, whose experiments have tended to the elucidation of many parts of practical mechanics, devoted some time to the subject of windmills. The results of his labours were published in the Memoirs of the Paris Academy for 1781. The mills to which he directed his attention, were in the vicinity of Lille, and were, in fact, oil mills. From the outer extremity of one sail to the corresponding extremity of the opposite sail, was 70 feet, the breadth of each sail $6\frac{1}{2}$ feet, of which the sail-cloth when extended occupies $5\frac{1}{2}$ feet, being attached on one side to a very light plank; the line of junction of the plank and of the sail-cloth, forms, on the side struck by the wind, an angle sensibly concave at the commencement of the sail, but diminishes gradually all along so as to vanish at the remoter extremity. The angle with the axis, at seven feet from the shaft, is 60°, and it increases continually so as to amount to nearly 84° at the extremity. The shaft upon which the sails turn, is inclined to the horizon, in different angles in different mills, from 8° to 15°.

Coulomb infers from his experiments,

(1.) That the ratio between the space described by the wind in a second, and the number of turns of a sail in a minute, is nearly constant, whatever be the velocity of the wind; the said ratio being about 10 to 6, or 5 to 3.

(2.) That with a wind whose velocity is $21\frac{1}{3}$ feet per second, the quantity of action produced by the impulsion of the wind is equivalent to a weight of 1080 pounds avoirdupois raised 270 feet in a minute; the *useful* effect being equivalent to a weight of 1080 pounds raised 232 feet in the same time: whence it results that the quantity of effect absorbed by the stroke of the stampers, the friction, &c., is nearly a sixth part of the quantity of action.

(3.) Suppose one of these mills to work eight hours in a day, Coulomb regards its daily useful effect as equivalent to that of 11 horses working at a walking-wheel, in a path of the usual radius.

(4.) It is observable, that in most windmills the velocity at the

extremity of the sails is greater than that of the wind. In some cases, indeed, these velocities have been found in about the ratio of 5 to 2. Now, it is evident that the impulsion of a fluid against any surface whatever, can only produce pressure, or mechanical effect, when the velocity of the surface exposed to the impulse is less than that of the fluid; and that the pressure will be nothing when the velocity of the surface is equal to, or greater than that of, the fluid. Indeed, in the latter case, the pressure may operate *against* the motion of the sails, and be injurious. It is desirable, therefore, in order to derive from a windmill all the effect of which it is susceptible, so to adjust the number of the turns that the velocity of the extremity of the sails shall be *less*, or, at most, *equal* to that of the wind.

It would be highly expedient to make comparative experiments on windmills, with a view to the determination of that velocity of the extremity of the sails which corresponds with the maximum of effect.

If v denote the velocity of the wind in feet per second, t the time of one revolution of the sails, A the angle of inclination of the sails to the axis, and D the distance from the shaft or axis of rotation to the point which is not at all acted on by the wind, or beyond which the sail-cloth ought to be folded up; then theoretical considerations supply the following theorem, viz.:—

$$D = \cdot 1092\, t v \tan A.$$

Ex. Suppose $v = 30$ feet per second, $t = 2{\cdot}25$ seconds, and $A = 75°$; then

$$D = \cdot 1092 \times 30 \times 2{\cdot}25 \times 3{\cdot}73205 = 27{\cdot}509 \text{ feet.}$$

This result agrees nearly with one of Coulomb's experiments, in which the velocity of the wind was 30 feet per second, the sails made 17 turns in a minute, and they were obliged to fold up more than 6 feet from the extremity of each sail, which were 34 feet long, to obtain a maximum of effect. The angle A at that distance from the tip of the sail was 75° or 76°.

Sect. III. *Mechanical Agents depending upon Heat:—the Steam Engine.*

In the steam engine the moving power is derived from the vapour produced from water, by the action of heat. Vapour is formed from water under ordinary circumstances at its usual temperature, whatever that may be; but the rapidity with which it is formed, and its elastic force, vary with the temperature and the pressure to which the water is subjected.

In order to a proper understanding of this subject, it will be necessary to explain the exact meaning of a few of the terms em-

inity. The density or specific gravity of steam, is, therefore,
ly as its specific volume. The *pressure*, *tension*, or *elastic*
f the steam, is the force expressed in pounds which it exerts
every square inch of the interior surface of any vessel con-
it.

en water contained in an open vessel is exposed to any source
, its temperature becomes elevated, and a portion of the water
erted into vapour and passes off in that form; this elevation
ies until the temperature (as indicated by the thermometer)
s 212°, when the vapour is formed with such rapidity as to
e a considerable commotion in the water, which is then famili-
id to boil. Notwithstanding, however, that fresh quantities of
e (as before) being poured into the water, no further elevation
emperature above 212° can be produced so long as the vessel
s open. But, if we now close the vessel, so as to prevent the
from escaping as it is formed, the sensible temperature of the
will again begin to rise, and will continue to do so as long as
it is applied; it will also be found that the temperature of the
in the upper part of the vessel will always be the same as that
water: and further, that as its temperature increases, so will
tic force, or the effort which it makes to escape, also increase;
g found that steam contained in a closed vessel in contact
vater and exposed to any given temperature will, under all
stances, be of the same density and have the same pressure.
nowledge of the nature of heat, of the precise mode in which
ates in the production of steam from water, of the species of
ation between the heat and the particles of the water, and of
nge (if any) which the particles of the water undergo in their
sion into steam, is so imperfect, that we are not able, by any
tions or reasoning, to arrive at the law which subsists between
sible temperature and the pressure of steam in contact with
; as, however, the determination of the relation which they
each other is essential to arriving at correct results in our

series, are the following, which have been collected by Pambour.

Range of pressure within which the formulæ are sufficiently exact.	Author.	Value of p = the pressure in lbs. per square inch.	Value of t = the t by Fahren
		(I.)	(II.)
Below 1 atmosphere	Southern	$\cdot04948 + \left(\frac{51\cdot3 + t}{155\cdot7256}\right)^{5\cdot13}$	$155\cdot7256\ (p - \cdot049$
		(III.)	(IV.)
From 1 to 4	De Pambour	$\left(\frac{98\cdot806 + t}{198\cdot562}\right)^{6}$	$198\cdot562\ p^{\frac{1}{6}} -$
		(V.)	(VI.)
From 4 to 50	{ Dulong and Arago }	$(\cdot269704 + \cdot0068031\ t)^{5}$	$146\cdot991\ p^{\frac{1}{5}} -$

As these formulæ are of universal importance in all in nected with steam, whether as applied directly to the st or to other purposes, such as warming buildings, &c., and peculiar form having fractional or very high powers and comes necessary to employ logarithms in their applicatic express them logarithmically; and, for the convenience o are not conversant with algebraical formulæ, we shall reduce the logarithmic formulæ to verbal rules, which we trate by a few examples.

Expressed then, logarithmically, Southern's formulæ be

I. ... $\mathrm{Log}\,(p - \cdot04948) = 5\cdot13 \log (51\cdot3 + t) - 11$

II. ... $\mathrm{Log}\,(t + 51\cdot3) = 2\cdot1923601 + \frac{\log (p - \cdot0}{5\cdot13}$

De Pambour's—

III. ... $\mathrm{Log}\,p = 6 \log (98\cdot806 + t) - 13\cdot7873772$;

IV. ... $\mathrm{Log}\,(t + 98\cdot806) = 2\cdot2978962 + \frac{\log p}{6}$.

And those of Dulong and Arago—

V. ... $\mathrm{Log}\,p = 5 \log (\cdot269704 + \cdot0068031\ t)$;

VI. ... $\mathrm{Log}\,(t + 39\cdot644) = 2\cdot1672906 + \frac{\log p}{5}$.

Or reduced to verbal rules:—

I. *To find the pressure by Southern's formula.*—Add temperature of the steam, multiply the logarithm of 5·13, and from the product subtract 11·2468073, the rer be the logarithm of a number, to which, if we add ·049 will represent the pressure in pounds per square inch.

mple. Required the pressure of steam at the temperature of

Log of (150 + 51·3 = 201·3) =	2·3038438
Multiplying by.....	5·13
	11·818718694
Subtracting.....	11·246807300
Log of 3·73174 =	·571911394

·73174 — ·04948 = 3·78122 = the pressure required.

To find the temperature by Southern's formula.— From the e in pounds subtract ·04948, and divide the logarithm of the ler by 5·13, to the quotient add 2·1923601, and the sum will ogarithm of a number, which, if we subtract 51·3 from it, resent the temperature of the steam.

mple. What is the temperature of steam whose pressure is per square inch?

Log of (10 — ·04948 = 9·95052) =	·9978458
Dividing by.....	5·13
	·1945118
Adding.....	2·1923601
Log of 243·70 =	2·3868719

∴ 243·70 — 51·3 = 192·4, the required temperature.

To find the pressure by De Pambour's formula.—Add 98·806 temperature, and 6 times the logarithm of their sum, minus 3772, will be the logarithm of the required pressure.

mple. What is the pressure of steam at 247°?

Log of (247 + 98·806 = 345·806) =	2·5388325
Multiplying by.....	6
	15·2329950
Subtracting.....	13·7873772
Log of 27·901 (pressure required) =	1·4456178

To find the temperature by De Pambour's formula.—Divide rithm of the pressure by 6, and to the quotient add 72, the sum will be the logarithm of a number, which, if we 98·806 from it, will equal the temperature of the steam.

Example. Required the temperature of steam having a pressure of 35 lbs.

Log of 35 = 1·5440680
Dividing by... 6

·2573447
Adding..... 2·2978962

Log of 359·121 = 2·5552409

∴ 359·121 − 98·806 = 260·215, the required temperature.

V. *To find the pressure by Dulong and Arago's formula.*—Multiply the temperature by ·0068031, and to the product add ·269704; then 5 times the logarithm of their sum will be the logarithm of the pressure.

Example. What is the pressure of steam having a temperature 330°?

Log of (·0068031 × 330 + ·269704 = 2·514727) = ·4004909
Multiply by 5

Log of 100·566 (pressure required) = 2·0024545

VI. *To find the temperature by Dulong and Arago's formula* Divide the logarithm of the pressure by 5, and to the quotient add 2·1672906, and the sum will be the logarithm of a number which we subtract 39·644 from it, will equal the temperature.

Example. What is the temperature of steam at the pressure 120 lbs.?

Log of 120 = 2·0791812
Dividing by... 5

·4158362
Adding..... 2·1672906

Log of 382·937 = 2·5831268

∴ 382·937 − 39·644 = 343·293 = the temperature required

In any boiler, or other evaporating vessel, if the source of heat be quite uniform, so that exactly equal portions of heat are given to the water in equal intervals of time, it will be found that the rate of increase in the sensible temperature of the water will *diminish* as the sensible temperature itself *increases;* that is, that the temperature of the water will rise (for example) from 60° to 70° in less

heat thus required to convert any given weight of water into
diminishes as the temperature of the water increases, and is
that the sum of the sensible temperature and the latent heat is
s constant and equivalent to 1170° of Fahrenheit's thermometer.
this circumstance two important consequences follow, viz., that
n weight of steam, whatever may be its pressure and sensible
rature, will really contain exactly the same amount of heat,
ill therefore require precisely the same quantity of fuel for its
sion from water at 212° to steam at the given density, what-
hat may be; or in other words, that the same weight of fuel is
ed to evaporate a given weight of water, and convert it into
whether it be contained in an open vessel, or closed and sub-
to any pressure, and this result has been amply confirmed by
ment. The second consequence is, that if a quantity of steam
arated from the water with which it was in contact, and then
d to expand into a larger space, or by pressure compressed into
ler, so long as it does not lose any portion of its own heat, or
e any fresh heat from surrounding bodies, its sensible tempera-
ill always be precisely such as it would have been at the same
re when in contact with the water from which it had been
ced. And as in the steam engine, under ordinary circum-
s, and where the usual precautionary means are adopted for
ıting loss of heat by the steam in its passage to the cylinder,
mperature may without any practical error be supposed to suf-
loss from those causes; it results from the property which we
xplained above, that the temperature of the steam in the cylin-
the engine and its pressure will always bear the same constant
n to each other that we have already shown they do while in con-
ith the water in the boiler; and, therefore, the formulæ and
already given may be applied to determine either the tempera-
r pressure of the steam (one being known) in any part of its
ss through the engine.
ıen water contained in a closed vessel is subjected to the con-
l action of heat in the manner we have already supposed, the
rature of the steam, together with that of the water, gradually
ses, and fresh quantities of water are converted into steam;
rom both these causes that the elastic force or pressure of the
increases with the elevation of its sensible temperature. It
esults, that as fresh quantities of water are converted into
, while the space which contains it is but very slightly increased,

the density of the steam must rapidly increase as the temperature rises, and its specific volume being inversely as its density, must become proportionally diminished. As the real quantity of water contained in the steam which passes through the engine, (and which depends upon the specific volume of the steam,) is one of the necessary elements required in the calculation of the power of a steam engine, it becomes of importance to investigate the changes which steam undergoes in its density with any change in its temperature or elastic force.

Let, therefore, V be the specific volume of steam at any temperature t; then, since steam as well as all other aeriform fluids expand $\frac{1}{480}$ th part of their bulk at 32° for every degree that their sensible temperature is raised, putting V_1 for the specific volume of steam at 32°, we have

$$V = V_1 + \frac{V_1 (t - 32)}{480}.$$

This formula supposes that no change has taken place in the pressure of the steam during its change in temperature, but as we have just shown that during all the changes which the steam undergoes in its passage through the engine, there is a constant relation between the sensible temperature of the steam and its pressure; it therefore becomes necessary to determine in what way the specific volume of the steam depends upon its pressure. Now, when steam is separated from the water which produced it, and enclosed *per se* in any vessel, it then follows the same law relative to its density as any other aeriform fluid. This law has been already given at page 279, and is as follows, viz.:—that while the temperature remains the same, the elasticity of the steam, or its pressure against the sides of the vessel containing it, varies inversely as the space which it is made to occupy, or, directly as its density.

Therefore, if P be put for the pressure of steam at 212°, when its specific volume equals V, and p_1 be put for its pressure, supposing its temperature unchanged but that its specific volume has become v, then,

$$P : p_1 :: v : V$$

$$\therefore v = V \frac{P}{p_1}.$$

If, however, t_1 represent the temperature which steam of the pressure p_1 should have, and v_1 represent its specific volume at that temperature, we shall have

$$v : v_1 :: V_1 + \frac{V_1 (212° - 32)}{480} : V_1 + \frac{V_1 (t_1 - 32)}{480};$$

then, substituting the value of v obtained above, and reducing in respect of v_1, we obtain

$$v_1 = V \frac{P}{p_1} \cdot \frac{448 + t_1}{660}.$$

Now, it is found that V, or the specific volume of steam at 212°, is 1700, that is, steam at 212° occupies just 1700 times the space that the water from which it was formed occupied, and in this state its pressure, or P, is equal to one atmosphere, that is, thirty inches of mercury, or 14·706 lbs. on the square inch. Therefore, substituting these values of V and P in the preceding formula, we obtain

$$v_1 = \frac{16969 + 37{\cdot}879\,t_1}{p_1} \quad \ldots\ldots\ldots \text{(VII.)}$$

By this formula it appears to be necessary that both the temperature and pressure of the steam should be known, in order to arrive at its specific volume; but we have already shown that the temperature and pressure of the steam, both in the boiler and in its passage through the engine, preserve an invariable relation to each other, so that one can always be expressed in terms of the other; and we can therefore, by combining this formula with those already given, eliminate t_1 altogether, and thereby obtain the value of v_1 in terms of p_1 only. But although the three sets of formulæ which we have given, taken separately express this relation for a certain portion of the scale of pressures, with a sufficient degree of accuracy for all practical purposes, they will not admit of being correctly employed beyond the limits assigned to each. As, however, in the investigation of the changes which take place in the pressure of the steam in the cylinder of the engine, it becomes essential to have some general formula which shall express with equal exactness this relation between the specific volume and the pressure, for all temperatures and pressures which can occur in the working of the engine, none of the foregoing formulæ are sufficiently comprehensive to serve for this purpose; since, for example, the steam in an expansive engine may pass from a pressure of five or six atmospheres to that of one and a half to two atmospheres, which case would require the employment of both formulæ (VI.) and (IV.), in order to discover the corresponding changes which had taken place in its specific volume.

In order to remove this objection, Navier has proposed the following general formula, which expresses the specific volume of steam, in terms of its pressure only, and which, from the simplicity of its form, is peculiarly well adapted for the purposes of calculation*; it is

$$v = \frac{1}{n + qp} \quad \ldots\ldots\ldots\ldots\ldots \text{(VIII.)}$$

in which v is the specific volume of the steam, having the pressure p in lbs. per *square foot*; n and q are constants to be determined by

* The following formula, derived from an expression for the relation between the pressure and temperature, given by the writer of the article on the Steam Engine, in the *Encyclopædia Metropolitana*, gives the specific volume of steam for all pressures above one atmosphere with singular exactness, but its complicated form prevents its being applied in the subsequent investigation.

$$v = \frac{12664{\cdot}24}{p} + \frac{8086{\cdot}16}{p^{\cdot 8412366}}.$$

experiment, and the values of which, according to De Pambour, should be as follows, viz. :—For condensing engines,

$$n = \cdot 00004227, \text{ and } q = \cdot 000000258;$$

and for non-condensing engines,

$$n = \cdot 0001421, \text{ and } q = \cdot 00000023.$$

The reason of the distinction being made between condensing and non-condensing engines, is that the first values of n and q are found more correct for low temperatures, and the second values for high temperatures.

If we represent by S the space actually occupied by a given weight of steam at the pressure p, and whose relative volume is v, and by S_1 the space which the same weight of steam will occupy at the pressure p_1, and whose relative volume is v_1, we have the following proportion :—

$$v : v_1 :: S : S_1;$$

and substituting for v and v_1 their values in terms of p and p_1, as derived from equation (VIII.) we have

$$\frac{1}{n + q\,p} : \frac{1}{n + q\,p_1} :: S : S_1;$$

and reducing in respect of p, we obtain

$$p = \frac{S_1}{S}\left(\frac{n}{q} + p_1\right) - \frac{n}{q} \quad \ldots\ldots\ldots\ldots \quad \text{(IX.)}$$

Having thus explained the mechanical properties of steam, as far as is requisite for developing the theory of the steam engine, we shall conclude the subject by giving a table of the temperature and specific volume of steam for pressures varying from 5 lbs. on the square inch to 10 atmospheres; the fourth column of which is calculated by Formulæ (II.), (IV.) and (VI.), and the fifth column by Formula (VII.)

TABLE

OF THE PRESSURE, TEMPERATURE, AND SPECIFIC VOLUME OF STEAM, FROM 5 TO 150 POUNDS ON THE SQUARE INCH.

Pressure in atmospheres.	Pressure in pounds per square foot.	Pressure in pounds per square inch.	Temperature of the steam by Fahrenheit.	Specific volume of the steam, that of the water from which it was formed being unity.	Pressure in atmospheres.	Pressure in pounds per square foot.	Pressure in pounds per square inch.	Temperature of the steam by Fahrenheit.	Specific volume of the steam, that of the water from which it was formed being unity.
	720	5	161·4	4617		7056	49	281·0	564
	864	6	169·2	3896		7200	50	282·3	553
	1008	7	176·0	3376		7344	51	283·6	543
	1152	8	182·0	2983		7488	52	284·8	534
	1296	9	187·5	2673		7632	53	286·0	525
	1440	10	192·5	2426		7776	54	287·2	515
	1584	11	197·1	2221		7920	55	288·4	507
	1728	12	201·3	2050		8064	56	289·6	499
	1872	13	205·9	1905		8208	57	290·7	490
	2016	14	209·1	1778		8452	58	291·9	483
1	2160	15	213·0	1669		8596	59	293·0	476
	2304	16	216·4	1573	4	8640	60	293·9	468
	2448	17	219·6	1487		8784	61	294·8	461
	2592	18	222·6	1411		8928	62	295·9	454
	2736	19	225·5	1343		9072	63	297·0	448
	2880	20	228·3	1281		9216	64	298·1	441
	3024	21	231·0	1225		9360	65	299·1	435
	3168	22	233·6	1173		9504	66	300·1	429
	3312	23	236·0	1127		9648	67	301·2	424
	3456	24	238·4	1083		9792	68	302·2	418
	3600	25	240·7	1043		9936	69	303·2	412
	3744	26	243·0	1010		10080	70	304·2	407
	3888	27	245·1	972		10224	71	305·1	402
	4032	28	247·2	940		10368	72	306·1	397
	4176	29	249·2	911		10512	73	307·1	392
2	4320	30	251·2	882		10656	74	308·0	386
	4464	31	253·1	857	5	10800	75	308·9	382
	4608	32	255·0	832		10944	76	309·9	378
	4752	33	256·8	809		11088	77	310·8	373
	4896	34	258·6	787		11232	78	311·7	369
	5040	35	260·3	767		11376	79	312·6	365
	5184	36	262·0	747		11520	80	313·5	361
	5328	37	263·7	729		11664	81	314·4	357
	5472	38	265·3	711		11808	82	315·2	353
	5616	39	266·9	694		11952	83	316·1	349
	5760	40	268·4	678		12096	84	316·9	345
	5904	41	269·9	663		12240	85	317·8	341
	6048	42	271·4	649		12384	86	318·6	338
	6192	43	272·9	635		12528	87	319·4	334
	6336	44	274·3	622		12672	88	320·3	331
3	6480	45	275·7	609		12816	89	321·1	327
	6624	46	277·1	597	6	12960	90	321·9	324
	6768	47	278·4	585		13104	91	322·7	321
	6912	48	279·7	574		13248	92	323·6	318

Pressure in atmospheres.	Pressure in pounds per square foot.	Pressure in pounds per square inch.	Temperature of the steam by Fahrenheit.	Specific volume of the steam, that of the water from which it was formed being unity.	Pressure in atmospheres.	Pressure in pounds per square foot.	Pressure in pounds per square inch.	Temperature of the steam by Fahrenheit.	Specific volume of the steam, that of the water from which it was formed being unity.
	13392	93	324·3	315		15840	110	336·7	271
	13536	94	325·0	312		16560	115	340·1	260
	13680	95	325·8	309	8	17280	120	343·3	250
	13824	96	326·6	306		18000	125	346·4	241
	13968	97	327·3	303		18720	130	349·5	233
	14112	98	328·1	300	9	19440	135	352·4	225
	14256	99	328·8	297		20160	140	355·3	217
	14400	100	329·6	295		20880	145	358·1	211
7	15120	105	333·2	282	10	21600	150	360·8	204

GENERAL DESCRIPTION OF THE MODE OF ACTION OF THE STEAM ENGINE.

Before proceeding to develope the general theory of the steam engine, or to investigate the mode of action of the steam in the cylinder, it will be well to enter into a general description of its construction and mode of action. In this preliminary description we shall only notice two forms of engine, namely, the non-condensing expansive engine, and the condensing engine without expansion.

Plate VII. exhibits a sectional elevation of a condensing steam-engine of eight horse power, with its boiler attached. The steam being generated in the boiler A, is conveyed by the steam-pipe C into the valve-box or chamber I, which is shown upon a larger scale in fig. 1, plate IX.; from this chamber there are three passages, one (*a*) communicating with the top of the cylinder, the other (*b*) with the bottom of the cylinder, and the third (*c*) with a vessel to be afterwards described, called the condenser. These passages are covered by a sliding valve, *d*, so formed that when in the position shown in fig. 1, all three of the passages are closed, but if it be moved downwards, and brought into the position shown in fig. 2, the upper passage *a* is made to communicate with the steam chamber I, while *b* and *c* are made to communicate with each other; but if it be moved upwards, and brought into the position shown in fig. 3, then the lower passage *c* communicates with I, and *a* and *b* with each other. The action of this valve being understood, if we suppose it to be in the position shown in fig. 2, we must immediately perceive that the steam, which has been conveyed into the chamber I by the pipe C, will pass by means of the passage *a* into the top of the steam cylinder D, and by its pressure on the piston E give motion to it, and cause it to descend. This piston is connected by a rod *e* with a beam capable of turning about the center F, and the other end of which is connected by the rod *g* with a crank *h*, upon the shaft of which is fixed the fly-wheel G. It is therefore evident, that any

motion of the piston E will by means of the beam and connecting rods be immediately communiated to the crank shaft *h*, and cause it to revolve. Upon this shaft a contrivance termed an excentric (H) is fixed; this is nothing more than a circular disc, which has the shaft or axis passing on one side of its center, the effect of which arrangement is to cause any point in its circumference to move nearer to or further from its center as it is made to revolve, by which means a reciprocating motion is given to the connecting rod *k*, which communicates through the beam *l*, and rods *m* and *n*, to the slide valve *d* already described. Now, when the piston E, by the continued pressure of the steam upon its upper surface, has been forced to the bottom of the cylinder D, this valve *d* will, by means of the excentric, have been moved upwards, and brought into the position shown in figure 1, in which the communication between the boiler and the cylinder is closed, so that no more steam can be admitted to press upon the piston E. The downward motion, however, of the piston having been communicated, in the manner already described, to the fly-wheel G, the momentum which it has thereby acquired causes it to continue its motion, and as it carries round the crank *h*, and the excentric H will produce the following twofold effect; first, it will, through the instrumentality of the beam F, and connecting rods, *e* and *g*, reverse the motion of the piston, and cause it to commence its ascent in the cylinder; secondly, it will, by means of the excentric and system of rods attached to it, cause the valve *d* to slide upwards, towards the position shown in fig. 3. As soon as this motion of the slide valve commences, the communication between the passage *c* and the box I being opened, the steam will now pass to the bottom of the cylinder, and there pressing on the lower side of the piston, will cause its reascent.

We must now, however, notice what becomes of the steam which already occupies the upper portion of the cylinder. By reference to fig. 3, (which it must be recollected represents the position in which the slide valve is now supposed to stand,) the upper part of the cylinder will be seen to be in immediate communication, by means of the passages *a* and *b*, and the pipe O, with the condenser M, which is a cylindrical vessel entirely surrounded with cold water, and which also has a small jet of cold water constantly playing into it. If we now suppose the whole of the air to have been previously expelled from the vessel M, (the manner of doing which will be presently described,) so that nearly a perfect vacuum is formed within it, it will readily be understood that the moment the communication is opened between it and the upper part of the cylinder, the steam contained in the latter will rush into this vacuum with a very great velocity, and being there immediately condensed by the cold water playing into and also surrounding the vessel M, the vacuum will be preserved, and thus no resistance will be offered to the motion of the piston by the pressure of the steam upon its upper surface. As, however, the injected water, together with that resulting from the condensed steam, would in time fill the vessel M, an air pump, L, is

made to communicate with it, which being worked by a rod from the beam of the engine, always keeps the condenser empty. And further, as the heat lost by the steam would in a short time so far elevate the temperature of the water surrounding the condenser as to render it incapable of continuing properly to condense it, it becomes necessary continually to change the water, with which object a pump, P, worked by a rod from the engine beam, raises cold water from a well or other source, and pours a continued stream into the vessel surrounding the condenser, while an equal quantity of the warmer water is allowed to run off.

Before the engine is set to work, the cylinder D, the condenser M, and the passages between them, are filled with common air, which it is necessary to extract. To effect this, by opening the valves a communication is made between the steam-pipe C, the space below the piston in the cylinder D, the eduction-pipe O, and the condenser M. The steam will not at first enter the cylinder D, or will only enter it a little way, because it is resisted by the air; but the air in the eduction-pipe O, and the condenser M, is forcibly driven before it, and this part of the air makes its exit through the valve N, called the snift valve, and which is kept covered with water. The steam-admission valve is now closed, and the steam already admitted is converted into water, in the manner already described, by the coldness of the condenser M, and by the jet of cold water which enters it from the well S S, in which the condenser is immersed. When this steam is condensed, all the space it occupied would be a vacuum, did not the air in the cylinder D expand, and fill all the space that the original quantity of it filled; but by the repetition of the means for extracting a part of the air, the remainder is blown out, and the cylinder becomes filled with steam alone.

In order that the connecting rod *e* may work freely, and yet possess the desirable property of being steam-tight, it passes through what is called a *stuffing* or *packing box*. This stuffing consists of some material which the steam will rather adapt to its office than injure; leather, which answers well for the stuffing or collars of machines never to be subjected to heat, will not answer here; hempen yarn is the material usually employed. The rod of the piston *l* passes through a stuffing box of the same kind as that of the piston E; and the pistons themselves are surrounded with stuffing.

The cylinder D is surrounded by a case, to keep it from being cooled by contact with the external air. The extremity, or any given point removed from the center of the great beam, can describe only the arc of a circle; but it is necessary that the piston rod *e* should rise and fall vertically. An apparatus is therefore used, called the parallel joint, which is easily understood by inspection. By this means the rod *e* not only rises and falls perpendicularly, but is perfectly rigid, and communicates all its motion to the great beam in each direction of its motion. The connecting rod *g* does not require the same contrivance, because it does not rise and fall perpendicularly; its lower end, with the outer end of the crank, describ-

ing a circle: it has therefore only a simple joint, admitting of this deviation.

In order to communicate a rotatory motion to the fly-wheel, instead of the crank may be used a contrivance giving twice the rapidity to the fly. For this purpose, on the outside of the axis of the fly, where the crank is shown in the plate, a small toothed wheel is fixed, and can only be moved with the fly: at the extremity of the rod *g*, and on that side of it which is next the fly-wheel, another toothed wheel is fixed, in such a manner that it cannot turn round on its axis, but must rise and fall with the rod to which it is attached. These two wheels work in each other, and that attached to the connecting rod cannot leave its fellow, because their centers are connected by a strap or bar of iron. When, therefore, the connecting rod rises, the wheel upon it moves round the circumference of the wheel upon the axis of the fly. By this means the fly makes an entire revolution for every stroke of the piston, and some mechanics are apt to think that they are great gainers by such an arrangement: the contrivance is certainly elegant, but with respect to utility, the fact is, that a crank is preferable; for it is more simple, cheaper, and less likely to be out of order, while, if the fly be large enough to receive, with less velocity, all the momentum that can be communicated to it, the effect will certainly not be inferior.

We now pass on to describe the non-condensing expansive engine shown in plate VIII., in which fig. 1 is a side elevation; fig. 2 an end elevation; fig. 3 a plan; and fig. 4 plate IX., a section of the cylinder, showing the steam passages and valves. By a glance at this plate, the means by which the motion of the piston is transmitted to the crank shaft will be seen to be similar to that just described; the real difference between them consists in there being no condenser and air pump in the engine which we are now describing, and in its having a second excentric by which the admission of steam to the chamber I from the boiler can be cut off at any desired portion of the stroke. The mode of admitting the steam to the top and bottom of the cylinder, alternately, by means of the slide valve *d*, is precisely similar to that already described; but after the steam has done its duty in the cylinder, instead of passing into the condenser it escapes by the passage *b*, directly into the atmosphere. We have yet, however, to explain the use of the second excentric, and to show in what manner the steam is caused to expand in the cylinder, by which a saving is effected in the quantity of steam used by the engine. In the engine which we have just described, the steam is supposed to remain on during nearly the whole stroke of the engine, and to have therefore nearly the same pressure throughout the stroke; but in that which we are now describing, the communication between the boiler and the cylinder can be closed at any period of the stroke that is desired, so that after the steam has been cut off, it then expands in the cylinder as the piston moves before it, and therefore presses upon the piston at each instant with a pressure due to its specific volume at that instant.

The method of adjusting the excentric rod, so as to cut off the

steam from the cylinder at any desired period of the stroke, is shown in fig. 5, plate IX., and is as follows:—$a\,b$ is a bell-crank lever, by means of which the motion of the excentric is communicated to the valve, the arc $c\,d$ being the distance (termed its effective stroke) through which the valve has to be moved to cut off the steam entirely from the cylinder; then, if we make the angle $g\,k\,h$ equal to the angle $c\,k\,d$, and taking the diameter of the excentric's path, or double the distance between the center of the excentric and the center of the shaft in the compasses, apply them along the two dotted lines, the point marked θ where they fall upon both, or where the angular distance is equal to the diameter of the excentric's path, will be the point at which the excentric rod must be fixed to the arm b of the lever, in order that the steam may remain on during the entire stroke. Then, fig. 6, representing the excentric's path, if it be desired to cut off the steam at any portion of the stroke, as at $\frac{1}{4}$, $\frac{1}{2}$, or $\frac{3}{4}$, we have only to measure such a proportion of the semi-circumference of the excentric's path, and taking the length of its chord in the compasses, apply them in the manner already described, to the angle $g\,k\,h$, and the distances from k, at which they severally fall upon both lines, will be the distances at which the excentric rod should be fixed when it is desired to cut the steam off from the cylinder at any such portions of the stroke.

Or, if we put l for the length of the arm a of the lever, s for the effective stroke of the valve or the distance $c\,d$, δ for the diameter of the excentric's path, d for the distance from k to the point at which the excentric rod should be attached to the arm b of the lever, and n the fraction of the stroke at which it is desired to cut the steam off from the cylinder, then

$$d = \frac{l\,\delta \sin (90\,n)}{s}.$$

For example, if l equal 6 inches, s equal 1·5 inch, and δ equal 3 inches, and we give to n the successive values ·25, ·5, ·75, or suppose the steam to be cut off at $\frac{1}{4}$, $\frac{1}{2}$ and $\frac{3}{4}$ of the stroke, we shall have for the corresponding values of d,

$$\text{at } \tfrac{1}{4} \text{ stroke } d = \frac{6 \times 3 \times \cdot 383}{1 \cdot 5} = 4 \cdot 6 \text{ inches,}$$

$$\text{at } \tfrac{1}{2} \text{ stroke } d = \frac{6 \times 3 \times \cdot 707}{1 \cdot 5} = 8 \cdot 5 \quad \text{,,}$$

$$\text{and at } \tfrac{3}{4} \text{ stroke } d = \frac{6 \times 3 \times \cdot 924}{1 \cdot 5} = 11 \cdot 1 \quad \text{,,}$$

THEORY OF THE STEAM ENGINE.

Before entering upon the general theory of the steam engine, it is but right to mention that the manner in which we purpose considering the subject is similar to that first laid down by the Comte De Pambour, in his very able work* upon this subject; and at the

* The Theory of the Steam Engine, by Comte De Pambour. Published by John Weale, 1839.

same time to acknowledge the great services which he has rendered by his investigations, which have led to the adoption of a theory alike applicable to every form of engine, and working under all circumstances. The propositions upon which his theory is founded, are thus stated by him at page 25 of his work:—

"From what has been stated, it plainly appears that we ground all our theory on these two incontestable facts: 1st, that the engine having attained uniform motion, there is necessarily equilibrium between the power and the resistance; that is, between the pressure of the steam *in the cylinder*, and the resistance against the piston, which furnishes the first relation *,

$$P_1 = R.$$

And 2dly, that there is also a necessary equality between the production of the steam and its expenditure, which furnishes the second relation †,

$$v = \frac{m S}{a} \cdot \frac{P}{P_1}.$$

And these two equations suffice for the solution of all the problems."

The manner in which Tredgold, and other writers upon this subject, had determined the power of an engine, was by measuring the area of its piston, and then, assuming that the engine would move at a given velocity, and that the pressure in the cylinder would be identical with that in the boiler, they thought that the continual product of these quantities would give them the load which the engine ought to raise in a given time, and to a given height, providing no loss had arisen from the friction of the machine, and other causes; and further, supposing that the portion thus lost always bore the same constant ratio to the whole, in the same class of engines, they determined arbitrary coefficients, by which they multiplied the above product, and they conceived that the quantity thus obtained represented the actual effective power of the engine with sufficient accuracy for all practical purposes, although no notice whatever was taken in the calculation of the evaporative power of the boiler.

A little consideration will, however, show the inaccuracy of this method of calculation. The engine itself is not the moving power by which we produce the desired effects; the boiler is the part in which the power resides, and the engine is only the instrument or machine through which this power is transmitted, and by which it can be applied precisely at the point, in the mode, and with the

* In which P_1 represents the pressure of the steam against the piston, per unit of surface, and R represents the resistance of the load against the piston, divided in like manner per unit of surface.

† In which v is the velocity of the piston under the resistance R, m the specific volume of the steam in the boiler under the pressure P, or the ratio of its volume to that of the water which produced it, s the volume of water evaporated by the boiler in a unit of time, and a the area of the cylinder.

velocity which we desire. A steam engine may, in this respect, compared with a crane, by means of which a number of men enabled to raise a considerable weight; but here no power resides the crane, it is merely the instrument through which the men enabled to apply simultaneously their united energies, and to prod certain effects; but those men could produce an equal amount effect *without* the crane, although not with so much ease or conve ence. And, in like manner, the power which can be transmitted the engine can only equal that which resides in the boiler. For, in the crane, the power exerted through it must depend on t number of men by whom it is moved, so also in the steam engine must the power which it can exert, or the amount of work which can perform, depend *only* upon the power of the boiler. Not however, that we mean to assert, that any amount of power which it is possible to derive from a boiler could be transmitted and applied by the engine, for, as the crane has been designed and proportioned only for the production of a certain effect, and the application of a greater power than that required for its production might strain and injure the machine, so it is with the steam engine, every engine is constructed only to transmit safely a certain amount of power from the boiler to the working point, and the attempt to produce a greater effect might be attended by the derangement of the machinery.

If a equal the area of the piston in square feet, and p_1 the pressure of the steam in the cylinder, per square foot of surface, the product, or ap_1 will obviously represent the resistance which the engine is capable of overcoming; and if this be multiplied by the length of the piston's stroke $= l_1$, we shall have the whole amount of work which the engine will perform in each stroke, or represented this amount by w_1, we shall have

$$w_1 = a l_1 p_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(X.)}$$

This equation, however, supposes the steam to act with the pressure p_1 during the whole stroke of the engine, and therefore only applies to engines in which the steam is not cut off until the end of the stroke, and which therefore do not work expansively.

In order then to obtain a general expression which will include the expansive engine also, let us suppose that l_1 represents only the portion of the stroke during which the steam is not cut off, in which case the above formula will still represent the work done by the engine during such portion of the stroke. Now, if λ equals the length of a portion of the cylinder, equal in content to the steam passages and the space left at each end of the cylinder for the clearance of the piston, we shall have $a(l_1 + \lambda)$ for the actual space occupied by the steam in the cylinder (having a pressure of p_1) at the moment when the steam is cut off; then, if l represent the length of the stroke from its commencement, at any moment after the expansion of the steam has commenced, and p the pressure of the steam due to its altered volume, we shall have from formula (IX.)

$$p = \frac{l_1 + \lambda}{l + \lambda}\left(\frac{n}{q} + p_1\right) - \frac{n}{q}.$$

Then, since the work done by the steam in the cylinder is equal to the pressure upon the piston multiplied by its area and the length of the stroke, or the distance that the piston has moved under that pressure, if Δl represent the distance moved over by the piston after the steam has been cut off, we shall have for the work performed during such motion,

$$a p \Delta l,$$

in which, substituting for p its value derived above, we have

$$a(l_1 + \lambda)\frac{\Delta l}{l + \lambda}\left(\frac{n}{q} + p_1\right) - \frac{n}{q} a \Delta l;$$

therefore, the whole amount of work done by the steam during its expansion in the cylinder will be represented by

$$a(l_1 + \lambda)\Sigma\frac{\Delta l}{l + \lambda}\left(\frac{n}{q} + p_1\right) - \frac{n}{q} a \Sigma \Delta l;$$

or putting w_2 for this amount of work, and l_2 for the total length of the stroke from its commencement,

$$w_2 = a(l_1 + \lambda)\int_{l_2}^{l_1}\frac{dl}{l + \lambda}\left(\frac{n}{q} + p_1\right) - \frac{n}{q} a \int_{l_2}^{l_1} dl;$$

and integrating between the limits of l_1 and l_2, we have

$$w_2 = a(l_1 + \lambda)\,\text{hyper. log}\frac{l_2 + \lambda}{l_1 + \lambda}\left(\frac{n}{q} + p_1\right) - \frac{n}{q} a (l_2 - l_1) \ldots \text{(XI.)}$$

Now, if we add together the work done before expansion, $(= w_1)$ and the work done after expansion, $(= w_2)$ and represent the whole work performed by the engine each stroke by W, we shall have

$$W = w_1 + w_2;$$

in which, substituting for w_1 and w_2 their values derived from equations (X.) and (XI.), and reducing, we obtain

$$W = a\left(\frac{n}{q} + p_1\right)\left\{l_1 + (l_1 + \lambda)\,\text{hyper. log}\frac{l_2 + \lambda}{l_1 + \lambda}\right\} - \frac{n}{q} a l_2 \ldots \text{(XI.*)}$$

Then, if P represent the pressure or resistance which the work W exerts against every square foot of the surface of the piston, $W = P a l_2$, and therefore

$$P l_2 = \left(\frac{n}{q} + p_1\right)\left\{l_1 + (l_1 + \lambda)\,\text{hyper. log}\frac{l_2 + \lambda}{l_1 + \lambda}\right\} - \frac{n}{q} l_2 \ldots \text{(XII.)}$$

which expression becomes

$$P = p_1 \ldots \text{(XIII.)}$$

when $l_2 = l_1$, or when there is no expansion.

These formulæ, then, enable us to determine the effects which would be produced by any engine working under given circumstances; but it becomes a matter of considerable practical importance to

determine those values of the several quantities which shall produce the greatest mechanical effect with the least expenditure of steam; or, in other words, the quantity of water evaporated by the boiler remaining constant, shall give to P its maximum value. In the high-pressure engine, working without expansion, this takes place when the pressure in the cylinder is as nearly as possible equal to that in the boiler. In the expansive engine, the same law holds with regard to the pressure at which the steam should be admitted to the cylinder to produce the greatest mechanical effect with any given expansion. But there is in every engine working by expansion a certain proportion of the stroke through which the expansion will produce a greater effect than through any other proportion, and this will be easily found after we have obtained expressions for Q, or the number of cubic feet of water which the boiler is capable of evaporating per minute.

If v_1 equals the specific volume of the steam at the pressure p_1, at which it is admitted to the cylinder, we shall have for the number of cubic feet of steam of that pressure, generated per minute, $Q v_1$; and if the engine makes ν strokes per minute, each equal l_1 feet in length, $\nu a (l_1 + \lambda)$ will equal the number of cubic feet of steam at the pressure p_1 used by the engine per minute, and therefore from the second principle laid down by De Pambour*, we have

$$Q v_1 = a \nu (l_1 + \lambda);$$

then, if V equal the velocity of the piston in feet per minute, we have $V = \nu l_1$, or $\nu = \frac{V}{l_1}$; and also from equation (VIII.) we have $v_1 = \frac{1}{n + q p_1}$; therefore, substituting these values above, we obtain

$$\frac{Q}{n + q p_1} = \frac{a V (l_1 + \lambda)}{l_1},$$

whence we obtain for the value of p_1,

$$p_1 = \frac{l_1 Q}{a q V (l_1 + \lambda)} - \frac{n}{q} \quad \ldots\ldots\ldots\ldots\ldots \text{(XIV.)}$$

We have also seen by equation (XIII.) that in an unexpansive engine $p_1 = P$, P being the resistance against each square foot of the piston's surface, arising from its load, from the friction of the engine, and from the pressure of the atmosphere, or imperfectly condensed steam on the other side of the piston. Now, taking the most simple case, or that of the stationary engine, if we put ϱ for the resistance occasioned by the load upon each square foot of the piston's surface, f for the resistance arising from the friction of the engine when unloaded, and $\varrho \varphi$ for the additional friction produced

* See page 317.

by the load ϱ upon every square foot of the piston's surface, and also π to represent the pressure of the atmosphere or uncondensed steam upon each square foot, we have

$$P = \varrho(1 + \varphi) + f + \pi = p_1,$$

and substituting this value of p_1 in equation (XIV.) above,

$$\varrho(1 + \varphi) + f + \pi = \frac{l_1 Q}{a q V (l_1 + \lambda)} - \frac{n}{q} \quad \text{..... (XV.)}$$

which equation expresses the relation between all the several quantities for *unexpansive* engines.

In *expansive* engines l_1 represents only that portion of the stroke which is performed before the steam is cut off, the whole stroke being equal to l_2; therefore in this case $r = \frac{V}{l_2}$, and equation (XIV.) becomes

$$p_1 = \frac{l_2 Q}{a q V (l_1 + \lambda)} - \frac{n}{q} \quad \text{..... (XVI.)}$$

Now, from equation (XII.) we have for expansive engines,

$$p_1 = \frac{l_2 \left(P + \frac{n}{q}\right)}{l_1 + (l_1 + \lambda) \text{ hyper. log } \frac{l_2 + \lambda}{l_1 + \lambda}} - \frac{n}{q},$$

and substituting for P its value determined above, reducing, and solving in respect of ϱ, we have

$$\varrho = \frac{Q\left(\frac{l_1}{l_1 + \lambda} + \text{hyper. log } \frac{l_2 + \lambda}{l_1 + \lambda}\right)}{a q V (1 + \varphi)} - \frac{f + \pi + \frac{n}{q}}{1 + \varphi} \quad \text{... (XVII.)}$$

Then, since the actual useful mechanical effect produced by the engine per minute equals $\varrho a V$, we have for the power of the engine

$$\varrho a V = \frac{Q\left(\frac{l_1}{l_1 + \lambda} + \text{hyper. log } \frac{l_2 + \lambda}{l_1 + \lambda}\right)}{q(1 + \varphi)} - \frac{a V \left(f + \pi + \frac{n}{q}\right)}{1 + \varphi}.$$

Now, in order to arrive at that value of l_1 which will give the maximum value to $\varrho a V$, or, in other words, to ascertain at what portion of the stroke the steam should be cut off in order to produce the greatest effect, let us substitute for $a V$ its value derived from equation (XIV.), whence by reduction we obtain

$$\frac{Q}{q(1 + \varphi)}\left(\frac{l_1}{l_1 + \lambda} + \text{hyper. log } \frac{l_2 + \lambda}{l_1 + \lambda} - \frac{l_2}{l_1 + \lambda} \cdot \frac{n + q(f + \pi)}{n + q p_1}\right)$$

= a maximum;

then, differentiating in respect of l_1 and remembering that since the above expression is a maximum its differential will $= 0$, we have

$$\frac{\lambda\, dl_1}{(l_1+\lambda)^2} - \frac{dl_1}{l_1+\lambda} + \frac{l_2 dl_1}{(l_1+\lambda)^2}\cdot\frac{n+q(f+\pi)}{n+qp_1} = 0,$$

whence, reducing and solving in respect of $\frac{l_1}{l_2}$, we obtain

$$\frac{l_1}{l_2} = \frac{n+q(f+\pi)}{n+qp_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(XVIII.)}$$

equal that portion of the stroke (the whole stroke being unity) at which the steam should be cut off to produce the greatest effect, with the least expenditure of steam.

DESCRIPTION OF THE VARIOUS KINDS OF ENGINES, AND THE FORMULÆ FOR CALCULATING THEIR POWER.

Although almost every manufacturer of steam engines adopts his own peculiar form of construction, by which a great diversity in their external appearance is occasioned, the principles upon which they act are similar in all engines of the same class; and, therefore, in attempting a classification of the different kinds of engines, we shall pay no attention to the details of their construction, but only regard the mode of employing the steam in the cylinders for the purpose of producing any particular species of motion. Proceeding, then, upon these principles, the following table represents at one view the various forms under which steam is at present employed as a moving power.

- Engines working *without* expansion
  - Non-condensing
    - Stationary.
    - Locomotive.
  - Condensing.
- Engines working *with* expansion
  - Double acting
    - Non-condensing
      - Stationary.
      - Locomotive.
    - Condensing
      - One cylinder.
      - Two cylinders.
  - Single acting
    - Steam pressing only on the *upper* surface of the piston.
    - Steam pressing only on the *lower* surface of the piston.

Stationary non-condensing Engines without expansion.

The mode of action of the steam in the cylinders of this class of engines is precisely similar to that already described at page 31[illegible] and a view of which is given in Plate VIII., with the exception that the steam is never cut off from the valve-box I, and therefore presses with an equal pressure upon the piston during nearly the whole extent of the stroke. With regard to the mechanical construction of this kind of engine, it depends in a great measure upon the fancy of the maker, or upon the purposes to which the engine is to be applied; there is perhaps a greater diversity in the form of this class of engine than of any other. For particular examples of the method

of constructing all kinds of engines, we must refer to the last edition of Tredgold's elaborate work upon the steam engine, which, with its appendices, forms one of the most magnificent works ever published.

The formulæ for calculating the power of engines of this class may be immediately derived from equation (XV.) already given, by solving in respect of any of the quantities which are wanted. Thus, we obtain for the values of V, Q, and ϱ,

$$V = \frac{Q l_1}{(l_1 + \lambda) a \{n + q(\varrho[1 + \varphi] + f + \pi)\}},$$

$$Q = \frac{V}{l_1}(l_1 + \lambda) a \{n + q(\varrho[1 + \varphi] + f + \pi)\},$$

$$\varrho = \frac{1}{1 + \varphi}\left\{\frac{Q l_1}{V a q (l_1 + \lambda)} - \left(f + \pi + \frac{n}{q}\right)\right\},$$

or, multiplying ϱ by Va, which then gives the actual work performed by the engine in each minute, we have

$$\varrho a V = \frac{1}{1 + \varphi}\left\{\frac{Q l_1}{q(l_1 + \lambda)} - aV\left(f + \pi + \frac{n}{q}\right)\right\}.$$

We may reduce these formulæ to a more practical form by inserting the numerical values of those quantities which are either constant, or sufficiently so for all practical purposes. Thus, π being in this case the pressure in pounds of the atmosphere upon each square foot of the piston's surface, equals 2118 lbs.; for n and q, we have the values already given for non-condensing engines at page 310, and assuming that $\lambda = \cdot 05\ l_2$, $\varphi = \cdot 14$, and $f = 144$, as adopted by De Pambour, the above formulæ become

$$V = \frac{10000\, Q}{a(6 \cdot 95536 + \cdot 0027531\, \varrho)},$$

$$Q = \frac{V a (6 \cdot 95536 + \cdot 0027531\, \varrho)}{10000},$$

$$\varrho a V = 3632268\, Q - 2526 \cdot 37\, a V.$$

Locomotive Engine working without expansion.

The mode of action of the steam in the cylinder of a locomotive engine, does not in any way differ from that in the engine already described; the only essential differences of which we have to take notice being in the details of its construction, and in the nature of the work which it has to perform.

Plate X. represents the longitudinal, and Plate XI. two transverse sections of one of the engines employed on the Great Western Railway. The principal peculiarity in the construction of a locomotive consists in the form of the boiler, which, in order to lessen the weight to be moved, is much smaller than the ordinary boiler for other

engines of the same power, and which, therefore, requires a very intense heat in the furnace, and a large evaporating surface: the first of these is effected by employing coke instead of coal, and producing rapid combustion by a strong draught; and the second by causing the water entirely to surround the fire-box, or furnace, and by carrying the smoke and heated air from the same through a number of small tubes which are likewise surrounded with water. Thus, it will be seen by an inspection of Plate X., and also of fig. 2, Plate XI., (which latter is a transverse section through the fire-box,) that the water not only covers its upper surface, but that it surrounds it on every side, leaving only a small space on one side for the opening, I, by which the stoker regulates the fire. The tubes *a a*, to which we have already alluded, for conveying the smoke from the fire to the chimney, are ninety-one in number, and expose a very large heating surface to the water; and in addition there is a hollow bridge, E, also filled with water passing through the center of the fire. The bars, X X, of the furnace, are so arranged, that when it is desired to put out the fire they can all be simultaneously lowered, allowing the burning contents of the furnace to fall upon the ground, or into a vessel placed to receive them underneath the engine. The sides of the fire-box, being flat, would be liable to be forced out by the pressure of the water and steam, to obviate which the sides are held together by short rivets, and the top is strengthened by a number of short girders *g*, bolted to its upper side. The flat ends of the boiler are prevented from being forced out by the fire tubes *a a*, which are secured at each end, and by tie rods which run from one end to the other; the other parts of the boiler derive sufficient strength from its cylindrical form. There are two safety-valves, one *b*, under the immediate inspection of the engine-driver, and which by means of a spring manometer Q, always indicates to him the pressure existing in the boiler; the other L, is placed (as an additional precaution) beyond his reach, so that it cannot be meddled with, or have its pressure altered. A glass tube W, fig. 1, Plate XI., made to communicate with the boiler at both its upper and lower extremities, so that the water in it always stands at the same level as that in the boiler, serves to indicate to the engine-driver when the quantity of water falls short. An opening K, called a man-hole, is left in the upper part of the boiler, being covered with a screwed lid, in order to enable the interior of the boiler to be examined; *c* is a whistle, so contrived, that the steam being allowed to rush forcibly through it produces a very shrill whistle, which is employed as a very general and useful means of signaling on railways. It will be observed that the boiler is surrounded by an exterior case, a small space being left between and filled with air, which prevents in a considerable degree the loss of heat from radiation, that would otherwise take place.

In consequence of the steam space in a locomotive boiler being very limited, the steam is formed almost as it is used, and immediately the communication with the cylinders is opened and the steam allowed to flow into them, a violent commotion in the water in the

boiler is occasioned by the sudden production of steam to supply its place, the result of which is, that a large quantity of water, in a state of minute mechanical division, is carried over with the steam into the cylinders, and there gradually accumulating, would at length materially interfere with the working of the engine. This effect, which is called *priming*, is to a certain extent obviated by a dome and steam-chest (H) being formed at the top of the boiler, and by the entrance of the steam pipe G by which the steam is conveyed to the cylinders being carried to the upper part of this chest, and having its mouth turned upwards, by which the space for steam is somewhat increased, and more time is given for the separation of the water from the steam. The admission of steam to the cylinders is regulated by the valve F, which the engine-driver can open or close by means of the handle *d*. The steam pipe has an expansive joint Z, which allows of a slight elongation or contraction of the pipe, in consequence of its varying temperature. The steam pipe passes straight through the boiler into the chamber in the fore part of the locomotive, termed the smoke-box, it there divides as shown in fig. 1, Plate XI., into two branches *ff*, each leading to one of the cylinders; the manner in which the admission of the steam alternately to the top and bottom of the cylinder is effected by the slide valve T, is precisely similar to that already described at page 312. But in a locomotive it is necessary that the engine-driver should possess the means of making the engine revolve in either direction, in order to effect which there are two excentrics to each slide valve, fixed upon the crank shaft, opposite to each other, and so arranged that, by means of a lever within reach of the engine-driver, he can connect either excentric with the slide valve, and by reversing its motion also reverse that of the engine itself: only one of these excentrics (C) is seen in the plate. The motion of the piston *n*, is transmitted to the crank *us*, by the piston rod *o*, and connecting rod *r*; the piston rod being made to move straight by the cross head B, moving in guides. After the steam has performed its office in the cylinders it is expelled through the pipe M, into the lower part of the chimney, where by its partial condensation it produces a powerful current of air through the fire and tubes *aa*. The water formed in the cylinder by condensation, or that which is brought over by priming, is allowed to escape by the cock P. The supply of water to the boiler is maintained by a small force-pump D, the plunger of which is attached to the cross head B of the piston rod, and which draws the water from the tender through the pipe O, and forces it into the boiler. The cylinders, being entirely within the smoke-box, are always maintained at a considerable temperature, by which means very little loss results from condensation of steam in the steam passages or cylinders.

The nature of the resistances to which a locomotive engine is subjected are somewhat different to those of a stationary engine, arising from the two following causes, viz., the resistance occasioned against the piston, by the ejected steam being blown forcibly through the contracted pipe M into the chimney, for the purpose of

producing a draught in the fire, as already described; and the resistance which the air occasions to the progressive motion of the train. The former of these, the Comte de Pambour, by his experiments, has ascertained to increase directly as the velocity of the piston, and to amount upon an average to about 1·75 lbs. per square inch of the piston's surface, when moving with a velocity of 150 feet per minute, or putting $\pi_1 V$ for this pressure per square foot, at the velocity V, we have

$$\pi_1 : 1{\cdot}75 \times 144 :: V : 150;$$
$$\therefore \pi_1 = 1{\cdot}6848.$$

The second resistance, both theory and experiments show to increase as the square of the velocity, and from the results of De Pambour's investigations it appears that for a train of average length, moving at a velocity of ten miles per hour, this resistance amounts to about 33 lbs., which, assuming the engine to be of the most usual dimensions, is equal to ·0055136 lbs. for every square foot of the piston, when moving with a velocity of one foot per minute; or, putting r for this resistance, we have rV^2 when the velocity is V. As the value of r depends not only on the number and description of the carriages composing the train, but also on the size of the driving wheels, the diameter of the piston, the length of its stroke, the gauge of the railway, and even on the direction and force of the wind, it will be perceived that the value which we have assumed above is very general indeed, and that, wherever great accuracy is required, these circumstances must be taken into consideration, and the more exact value of r deduced therefrom.

We have therefore, in this case, for the value of P, or the sum of the resistances against the piston,

$$(\varrho + rV^2)(1 + \varphi) + f + \pi + \pi_1 V,$$

which being substituted in equation (XV.) for $\varrho(1 + \varphi) + f + \pi$, and reducing, we have for the values of V, Q, and ϱa V, as follows:—

$$V = \frac{l_1 Q}{a(l_1 + \lambda)\{n + q[(\varrho + rV^2)(1 + \varphi) + f + \pi + \pi_1 V]\}};$$

$$Q = \frac{V}{l_1}(l_1 + \lambda)a\{n + q[(\varrho + rV^2)(1 + \varphi) + f + \pi + \pi_1 V]\};$$

$$\varrho a V = \frac{1}{1 + \varphi}\left\{\frac{Q l_1}{q(l_1 + \lambda)} - aV\left(f + \pi + \pi_1 V + \frac{n}{q}\right)\right\} - arV^3.$$

Or, substituting for n, q, λ, π and φ, the same numerical values as before, and making $f = 3{\cdot}0125$ lbs. per square foot of the piston surface, when moving with a velocity of one foot per minute, which would be its average value, these formulæ become,

$$V = \frac{10000\,Q}{a(6{\cdot}6143 + {\cdot}002753\varrho + {\cdot}00001518\,V^2 + {\cdot}0040688\,V)};$$

$$Q = \frac{a\,V}{10000}(6{\cdot}6143 + {\cdot}002753\,\varrho + {\cdot}00001518\,V^2 + {\cdot}0040688\,V);$$

$$\varrho\,a\,V = 3632400\,Q - a\,({\cdot}005514\,V^3 + 1{\cdot}4779\,V^2 + 2402{\cdot}49\,V).$$

Condensing Engines working without expansion.

One form of this class of engine is shown in Plate VII., and its general principles and mode of action have already been described at page 312. This class includes the greater number of engines employed for propelling steam-boats, which, although different in form, are identical in principle with the engine shown in Plate VII.

The general formulæ for calculating the values of V, Q and $\varrho\,a\,V$, are precisely the same as those already given at page 323, for non-condensing engines, and therefore need not be recapitulated here; but the practical formulæ differ in consequence of the coefficients and constants having different numerical values. The value of π, or the resistance on the under-side of the piston, is in this case much smaller than in the preceding, and may be taken on the average as only equal to 576 lbs. on each square foot; the value of n and q for condensing engines will be found at page 310, and those of λ, φ, and f, may be taken the same as those already given at page 323, for stationary non-condensing engines. Substituting these values in the general formulæ, and reducing, we obtain

$$V = \frac{10000\,Q}{a\,(2{\cdot}3943 + {\cdot}0030883\,\varrho)};$$

$$Q = \frac{V\,a\,(2{\cdot}3943 + {\cdot}0030883\,\varrho)}{10000};$$

$$\varrho\,a\,V = 3237280\,Q - 775{\cdot}281\,a\,V.$$

Stationary non-condensing Engines, working expansively.

This engine, which is shown in Plates VIII. and IX., has been already described at page 315, and it therefore only remains to deduce the formulæ by which the values of V, Q and $\varrho\,a\,V$ may be obtained.

Now we have, at page 321,

$$\varrho\,a\,V = \frac{Q\left(\frac{l_1}{l_1+\lambda} + \text{hyper. log}\,\frac{l_2+\lambda}{l_1+\lambda}\right)}{q\,(1+\varphi)} - \frac{a\,V\left(f + \pi + \frac{n}{q}\right)}{1+\varphi};$$

and from formula (XVII.), by reduction, we immediately obtain

$$V = \frac{Q\left(\frac{l_1}{l_1+\lambda} + \text{hyper. log}\,\frac{l_2+\lambda}{l_1+\lambda}\right)}{a\,\{n + q\,[\varrho\,(1+\varphi) + f + \pi]\}};$$

$$Q = \frac{V a \{n + q[\varrho(1+\varphi) + f + \pi]\}}{\frac{l_1}{l_1+\lambda} + \text{hyper. log}\frac{l_2+\lambda}{l_1+\lambda}}.$$

And substituting in these formulæ the numerical values of the constant quantities, viz., π equal the pressure of the atmosphere, equal 2118 lbs., n and q as given for non-condensing engines, at page 310, and λ and φ as before; also, giving to f the value assigned to it by Comte de Pambour for this class of engine, viz., 864 lbs. per square foot of the piston's surface, we have

$$\varrho a V = 3813883\, Q \left\{\frac{l_1}{l_1+\lambda} + \text{hyper. log}\frac{l_2+\lambda}{l_1+\lambda}\right\} - 3157{\cdot}73\, a V;$$

$$V = \frac{10000\, Q \left\{\frac{l_1}{l_1+\lambda} + \text{hyper. log}\frac{l_2+\lambda}{l_1+\lambda}\right\}}{a(8{\cdot}2796 + {\cdot}002622\,\varrho)};$$

$$Q = \frac{V a (8{\cdot}2796 + {\cdot}002622\,\varrho)}{\frac{l_1}{l_1+\lambda} + \text{hyper. log}\frac{l_2+\lambda}{l_1+\lambda}}.$$

In order to simplify the calculation of these quantities, and for the benefit of those who are not conversant with the use of logarithms, we subjoin a table of the values of the expression $\frac{l_1}{l_1+\lambda} + \text{hyper. log}\frac{l_2+\lambda}{l_1+\lambda}$, for values of $\frac{l_1}{l_2}$, (or the fraction expressing that portion of the stroke which is performed before the steam is cut off,) varying from one-tenth to nine-tenths of the whole stroke.

Value of $\frac{l_1}{l_2}$.	Value of $\frac{l_1}{l_1+\lambda} + \text{hyper. log}\frac{l_2+\lambda}{l_1+\lambda}$	Value of $\frac{l_1}{l_2}$.	Value of $\frac{l_1}{l_1+\lambda} + \text{hyper. log}\frac{l_2+\lambda}{l_1+\lambda}$
·10	2·61258	·50	1·55558
·15	2·40823	·55	1·47628
·20	2·23507	·60	1·40266
·25	2·08610	·66	1·31213
·30	1·95576	·70	1·26981
·33	1·87721	·75	1·20943
·35	1·84008	·80	1·15248
·40	1·73619	·85	1·09862
·45	1·64194	·90	1·04739

Locomotive Engine working expansively.

Lately the principle of working locomotive engines expansive has been successfully adopted, and we therefore give the formu

necessary for calculating the effects which they will then produce. In this case we have

$$V = \frac{Q\left(\frac{l_1}{l_1+\lambda} + \text{hyper. log}\,\frac{l_2+\lambda}{l_1+\lambda}\right)}{a\{n + q\,[(\varrho + r\,V^2)(1+\phi) + f + \varpi + \varpi_1 V]\}};$$

$$Q = \frac{V\,a\{n + q\,[(\varrho + r\,V^2)(1+\phi) + f + \varpi + \varpi_1 V]\}}{\frac{l_1}{l_1+\lambda} + \text{hyper. log}\,\frac{l_2+\lambda}{l_1+\lambda}};$$

$$\varrho\,a\,V = \frac{Q\left(\frac{l_1}{l_1+\lambda} + \text{hyper. log}\,\frac{l_2+\lambda}{l_1+\lambda}\right)}{q(1+\phi)} - \frac{a\,V\left(f + \varpi + \varpi_1 V + \frac{n}{q}\right)}{1+\phi} - a\,r\,V^3.$$

And substituting for the constant quantities the same numerical values as those already employed at page 326, for locomotives working without expansion, we have

$$V = \frac{10000\,Q\left(\frac{l_1}{l_1+\lambda} + \text{hyper. log}\,\frac{l_2+\lambda}{l_1+\lambda}\right)}{a(6{\cdot}2993 + {\cdot}002622\,\varrho + {\cdot}00001446\,V^2 + {\cdot}003875\,V)};$$

$$Q = \frac{V\,a(6{\cdot}2993 + {\cdot}002622\,\varrho + {\cdot}00001446\,V^2 + {\cdot}003875\,V)}{10000\left(\frac{l_1}{l_1+\lambda} + \text{hyper. log}\,\frac{l_2+\lambda}{l_1+\lambda}\right)};$$

$$\varrho\,a\,V = 3813882\,Q\left(\frac{l_1}{l_1+\lambda} + \text{hyper. log}\,\frac{l_2+\lambda}{l_1+\lambda}\right) - a\,({\cdot}005514\,V^3 + 1{\cdot}4779\,V^2 + 2402{\cdot}49\,V).$$

The apparent complication of these formulæ will disappear when we substitute for $\left(\frac{l_1}{l_1+\lambda} + \text{hyper. log}\,\frac{l_2+\lambda}{l_1+\lambda}\right)$ its value from the above table, which cannot be done until the proportion of the stroke at which the steam is cut off is known.

Stationary Condensing Engines, with one Cylinder working expansively.

This engine only differs from that already described at page 312, in its working expansively, which is effected by the agency of a second excentric, in the manner explained at page 315.

The general formulæ for the values of V, Q, and $\varrho\,a\,V$, are identical with those already given at pages 327 and 328, for non-condensing engines working expansively; but the practical formulæ differ in

consequence of the constants π, n, q, λ, φ, and f, having in this case the value assigned to them at page 327, for condensing engines working without expansion; these being substituted in the general formulæ, they become as follows:—

$$V = \frac{10000\,Q\left(\frac{l_1}{l_1+\lambda} + \text{hyper. log}\frac{l_2+\lambda}{l_1+\lambda}\right)}{a\,(2{\cdot}0953 + {\cdot}0029412\,\varrho)};$$

$$Q = \frac{V\,a\,(2{\cdot}0953 + {\cdot}0029412\,\varrho)}{10000\left(\frac{l_1}{l_1+\lambda} + \text{hyper. log}\frac{l_2+\lambda}{l_1+\lambda}\right)};$$

$$\varrho\,a\,V = 3399972\,Q\left(\frac{l_1}{l_1+\lambda} + \text{hyper. log}\frac{l_2+\lambda}{l_1+\lambda}\right) - 712{\cdot}123\,a\,V.$$

Condensing Engines, with two cylinders, working expansively.

This form of engine was invented by Mr. Arthur Woolfe, and patented by him in the year 1804. The peculiarity in its mode of action consists in employing two cylinders, through both of which the same steam is made successively to pass. The details of its construction will be better understood by reference to Plate XII., in which fig. 1 is a section of the two cylinders and slide valves. Supposing the various parts to be in the positions shown in this figure, the steam from the boiler enters by the aperture A, and passes through the passage B into the top of the smaller cylinder C, and there, pressing on the upper surface of the piston, produces a downward motion; when the piston has performed a certain portion of its stroke the communication with the boiler is closed, so that the steam acts upon the small piston only by its expansion during the remaining portion of its stroke. The rod *a* of the small piston, and that *b* of the large one, being both connected to the same beam, descend together, during which the steam which occupied the space below the small piston passes through the passages D and E into the space above the larger piston; but, since the content of the cylinder F is larger than that of C, in doing so it expands and produces a pressure on the upper surface of the large piston, the under-side of which is in immediate communication with the condenser by means of the passage G and aperture H. As the pistons descend, the slide valves, moved by an excentric in the ordinary manner, change their position, and during the up-stroke of the pistons are in the situation shown in fig. 2, in which it will be seen that the steam from the boiler now presses directly on the *upper* side of the smaller piston, while the *top* of the small cylinder now communicates with the *bottom* of the larger one, by which the effort of the steam, in expanding from one cylinder to the other, is exerted on the under surface of the large piston, the *upper* side of the same being now in direct communication with the condenser by means of the passage E and aperture I. The mode of action of the

steam in the cylinders being then correctly understood, it is only necessary to state, that the manner of transmitting the motion of the pistons to the working point, and the arrangement of the condenser and air pump, may be the same as in other steam engines, and similar to that which we have already described.

In order to deduce formulæ for calculating the effects of this class of engine, let a_1 represent the area of the small piston, and a_2 that of the larger piston, both in square feet; let l_1 equal that portion of the small piston's stroke which is performed before the steam is cut off, l_2 the whole length of the small piston's stroke, and l_3 the length of the larger piston's stroke; also, let p_1 be the pressure of the steam in the small cylinder before the expansion commences, p_2 the pressure of the same in the large cylinder before expansion, and λ_1 and λ_2 the length of a portion of each cylinder, equivalent in content to the space left for clearance and the steam passages. Then we have from equation (XI.*), page 319, for the work down in the small cylinder,

$$a_1\left(\frac{n}{q}+p_1\right)\left\{l_1+(l_1+\lambda_1)\text{ hyper. log }\frac{l_2+\lambda_1}{l_1+\lambda_1}\right\}-\frac{n}{q}a_1 l_2.$$

Now, in order to obtain the work done by the expansion of the steam in the large cylinder, put $s_1 = a_1 l_2 =$ the space moved through by the small piston, $s_2 = a_2 l_3 =$ the space moved through by the large piston, and $s = a_2 l =$ the space moved through by the same at any portion (l) of its stroke. Then, the space occupied by the steam (having a pressure equal to p_2) before the commencement of the stroke, equals $s_1 + a_1\lambda_1 + a_2\lambda_2$, and that which it occupies at any portion (l) of the stroke equals $s + a_1\lambda_1 + a_2\lambda_2$; therefore, putting $\beta = a_1\lambda_1 + a_2\lambda_2$, we have from formula (IX.), for the pressure of the steam due to its altered volume,

$$p=\frac{s_1+\beta}{s+\beta}\left(\frac{n}{q}+p_2\right)-\frac{n}{q}.$$

Then, since Δs represents the elementary space moved through by the larger piston, we have for the work developed during the motion, $p\Delta s$, which, by substituting the above value of p, becomes

$$(s_1+\beta)\frac{\Delta s}{s+\beta}\left(\frac{n}{q}+p_2\right)-\frac{n}{q}\Delta s;$$

therefore, the whole amount of work developed is represented by

$$(s_1+\beta)\Sigma\frac{\Delta s}{s+\beta}\left(\frac{n}{q}+p_2\right)-\frac{n}{q}\Sigma\Delta s;$$

which being taken between the proper limits of $s = s_1$, and $s = s_2$, equals

$$(s_1+\beta)\int_{s_2}^{s_1}\frac{ds}{s+\beta}\left(\frac{n}{q}+p_2\right)-\frac{n}{q}\int_{s_2}^{s_1}ds;$$

whence, by integrating, we obtain

$$(s_1 + \beta)\,\text{hyper. log}\,\frac{s_2 + \beta}{s_1 + \beta}\left(\frac{n}{q} + p_2\right) - \frac{n}{q}(s_2 - s_1).$$

Then, in order to eliminate p_2, we can derive its value from equation (IX.) in terms of p_1, viz.,

$$p_2 = \frac{a_1(l_1 + \lambda_1)}{s_1 + \beta}\left(\frac{n}{q} + p_1\right) - \frac{n}{q},$$

which being substituted for p_2 in the expression above, it becomes

$$a_1(l_1 + \lambda_1)\left(\frac{n}{q} + p_1\right)\text{hyper. log}\,\frac{s_2 + \beta}{s_1 + \beta} - \frac{n}{q}(s_2 - s_1);$$

and again, substituting for s_1, s_2, and β, their several values above, and reducing, we have

$$a_1(l_1 + \lambda_1)\left(\frac{n}{q} + p_1\right)\text{hyper. log}\,\frac{a_2(l_3 + \lambda_2) + a_1\lambda_1}{a_1(l_2 + \lambda_1) + a_2\lambda_2} - \frac{n}{q}(a_2 l_3 - a_1 l_2),$$

for the whole amount of the work done by the expansion of the steam in the large cylinder, minus the resistance produced by the reaction of the steam in the small cylinder; whence, adding this to the expression already obtained for the work done in the small cylinder, and reducing, we obtain

$$a_1\left(\frac{n}{q} + p_1\right)\left\{l_1 + \left[(l_1 + \lambda_1)\text{hyper. log}\,\frac{l_2 + \lambda_1}{l_1 + \lambda_1} + \text{hyper. log}\,\frac{a_2(l_3 + \lambda_2) + a_1\lambda_1}{a_1(l_2 + \lambda_1) + a_2\lambda_2}\right]\right\} - \frac{n}{q}a_2 l_3,$$

which equals the actual work performed by Woolfe's engine, during one stroke.

Then, in order to obtain the value of P, in this case the resistance against each square foot of the *larger* piston's surface, arising from the load and friction of the engine, and from the imperfect vacuum in the condenser; let π, f, and φ, represent the same quantities as before, only taken per unit of the large piston's surface; then we have

$$P = \varrho(1 + \varphi) + f + \pi,$$

which being multiplied by $a_2 l_3$, equals the whole amount of resistance developed during one stroke of the engine; equal

$$a_2 l_3\{\varrho(1 + \varphi) + f + \pi\}.$$

Therefore, since where the engine has attained uniform motion the resistance must equal the work performed, we have

$$a_1\left(\frac{n}{q}+p_1\right)\left\{l_1+\left[(l_1+\lambda_1)\,\text{hyper. log}\,\frac{l_2+\lambda_1}{l_1+\lambda_1}+\text{hyper. log}\,\frac{a_2(l_3+\lambda_2)+a_1\lambda_1}{a_1(l_2+\lambda_1)+a_2\lambda_2}\right]\right\}-\frac{n}{q}a_2 l_3$$
$$=a_2 l_3\{\varrho(1+\varphi)+f+\varpi\}.$$

Then, if V equals the velocity of the larger piston, and ν the number of strokes which it makes per minute, we have $\nu=\frac{V}{l_3}$, whence we obtain, for the value of p_1, from equation (XIV.)

$$p_1=\frac{l_3 Q}{a_1 q V(l_1+\lambda_1)}-\frac{n}{q}.$$

Then, substituting this value of p_1 in the above expression, reducing, and solving in respect of Q, V, and $\varrho a_2 V$, we have

$$Q=\frac{a_2 V\{n+q[\varrho(1+\varphi)+f+\varpi]\}}{\frac{l_1}{l_1+\lambda_1}+\text{hyper. log}\,\frac{l_2+\lambda_1}{l_1+\lambda_1}+\text{hyper. log}\,\frac{a_2(l_3+\lambda_2)+a_1\lambda_1}{a_1(l_2+\lambda_1)+a_2\lambda_2}};$$

$$V=\frac{Q\left\{\frac{l_1}{l_1+\lambda_1}+\text{hyper. log}\,\frac{l_2+\lambda_1}{l_1+\lambda_1}+\text{hyper. log}\,\frac{a_2(l_3+\lambda_2)+a_1\lambda_1}{a_1(l_2+\lambda_1)+a_2\lambda_2}\right\}}{a_2\{n+q[\varrho(1+\varphi)+f+\varpi]\}};$$

$$\varrho a_2 V=\frac{Q\left\{\frac{l_1}{l_1+\lambda_1}+\text{hyper. log}\,\frac{l_2+\lambda_1}{l_1+\lambda_1}+\text{hyper. log}\,\frac{a_2(l_3+\lambda_2)+a_1\lambda_1}{a_1(l_2+\lambda_1)+a_2\lambda_2}\right\}}{(1+\varphi)q}$$
$$-a_2 V\frac{f+\varpi+\frac{n}{q}}{1+\varphi}.$$

In order to put these formulæ under a more practical form, we must substitute for the constants and coefficients their average values, viz., for n, q, and ϖ, as before, for condensing engines, $f=125$ lbs., φ as before, $\lambda_1=\cdot05 l_2$, and $\lambda_2=\cdot05 l_3$, then we obtain

$$Q=\frac{a_2 V(2\cdot2313+\cdot0029412\varrho)}{9524+10000\left\{\text{hyper. log}\,\frac{l_2+\lambda_1}{l_1+\lambda_1}+\text{hyper. log}\,\frac{a_2(l_3+\lambda_2)+a_1\lambda_1}{a_1(l_2+\lambda_1)+a_2\lambda_2}\right\}};$$

$$V=\frac{10000\,Q\left\{\cdot9524+\text{hyper. log}\,\frac{l_2+\lambda_1}{l_1+\lambda_1}+\text{hyper. log}\,\frac{a_2(l_3+\lambda_2)+a_1\lambda_1}{a_1+(l_2\lambda_1)+a_2\lambda_2}\right\}}{a_2(2\cdot2313+\cdot0029412\varrho)};$$

$$\varrho a_2 V=3399972\,Q\left\{\cdot9524+\text{hyper. log}\,\frac{l_2+\lambda_1}{l_1+\lambda_1}+\text{hyper. log}\,\frac{a_2(l_3+\lambda_2)+a_1\lambda_1}{a_1(l_2+\lambda_1)+a_2\lambda_2}\right\}-758\cdot61\,a_2 V.$$

The following table, giving the numerical value of the expression $\left(\text{hyper. log}\,\frac{l_2+\lambda_1}{l_1+\lambda_1}+\text{hyper. log}\,\frac{a_2(l_2+\lambda_2)+a_1\lambda_1}{a_1(l_2+\lambda_1)-a_2\lambda_2}\right)$ for values of $\frac{l_1}{l_2}$ varying from $\frac{1}{10}$ to $\frac{1}{2}$, and for the three cases where $\frac{a_2}{a_1}$ equals 4, 3 and 2, may sometimes be found useful: in this table the two strokes are assumed to be of the same length, that is, $l_2 = l_3$.

Value of $\frac{l_1}{l_2}$.	Value of $\left(\text{hyper. log}\,\frac{l_2+\lambda_1}{l_1+\lambda_1}+\text{hyper. log}\,\frac{a_2(l_2+\lambda_2)+a_1\lambda_1}{a_1(l_2+\lambda_1)+a_2\lambda_2}\right)$		
	when $a_2 = 4a_1$.	when $a_2 = 3a_1$.	when $a_2 = 2a_1$.
·10	3·16969	2·92674	2·57162
·15	2·88200	2·63906	2·28393
·20	2·65886	2·41591	2·06079
·25	2·47654	2·23359	1·87847
·30	2·32239	2·07944	1·72432
·33	2·23901	1·99606	1·64094
·35	2·18885	1·94591	1·59078
·40	2·07107	1·82813	1·47300
·45	1·96561	1·72277	1·36764
·50	1·87088	1·62793	1·27281

Single-acting Engine, in which the Steam acts only upon the upper surface of the Piston.

This class of steam engine is that usually known by the appellation of the Cornish pumping engine, for which purpose it is more peculiarly adapted; and from the surprising amount of work which has been realized by means of it, with the consumption of a given weight of fuel, has attracted a great deal of attention, and has become almost universally adopted in mining districts.

As it is not our province here to enter into a detail of the mechanical construction of these engines, we have only shown in plate XII, figure 3, a section of the cylinder and valves; it may, however, be remarked, that these engines exhibit the greatest mechanical skill in their design and construction, and doubtless one cause of the high duty which they perform results from the accuracy and precision with which the various parts are formed, and the precautions adopted for preventing loss of heat by the steam in passing through the engine.

In the plate, A is the cylinder, which is surrounded by an outer casing sufficiently large to leave a space of about an inch round the whole of the exterior of the cylinder, which space (termed the steam jacket) is always kept filled with steam of the same pressure as that in the boiler, by means of the pipe B, which is in direct communica-

tion with the boiler; a similar space is also left below the cylinder, and kept constantly filled with steam. The object of thus surrounding the cylinder with hot steam, is to prevent its interior surface from being cooled during the up-stroke; and for the same purpose, the top of the cylinder, as also the upper part of the valve box, and the entire steam pipe from the boiler to the cylinder, are enveloped in an external case, the space between being filled with sawdust or some other non-conducting substance.

The piston being at the top of the cylinder, the valve C is opened, allowing the steam to pass from the boiler into the cylinder, where, pressing on the upper surface of the piston, it causes its descent, raising in so doing the pump rods and counterweight which are attached to the other end of the engine beam. After the engine has performed a portion of its stroke (varying from $\frac{1}{10}$ to $\frac{1}{3}$ in different engines), the valve C is closed, and the remainder of the downward, or, as it is termed in Cornwall, the *in-door* stroke, is performed by the expansion of the steam: the under-side of the piston is in communication with the condenser by means of the pipe D, which is closed by the valve E towards the conclusion of the down-stroke. There being in the Cornish engine no crank, or other means by which the precise length of the stroke is determined, it becomes necessary to adopt some means of preventing the piston descending too far, and by doing so injuring the bottom of the cylinder; the way in which this is effected, is by so regulating the quantity of steam admitted through the valve C at the commencement of the stroke, that it shall be just sufficient to carry the piston to the proper distance, and no further; but since any variation in the pressure of the steam in the boiler, or in the quantity of water raised by the pump, would immediately destroy this adjustment, and cause the piston either to fall short of or to exceed its proper stroke, springs are placed so as to receive and stop the beam of the engine when it descends too low, and at the same time a bell is rung, which warns the engine-driver to lessen the quantity of steam admitted by the valve C. The piston having been brought to a state of rest at the bottom of the cylinder, the valve F is closed by which the communication with the condenser is cut off, and the valve G, termed the *equilibrium valve*, is opened, by which a free communication is effected between the top and bottom of the cylinder, and an equilibrium established between the pressure on the upper and under side of the piston, which, therefore, having now nothing to oppose its motion but the friction of the engine, is drawn to the top of the cylinder by the counterweight which it had raised during its down-stroke, and which is sufficiently heavy to raise the columns of water in the various pump mains, and to overcome the friction of the engine. As, however, the piston would continue its motion until it came into contact with the cylinder cover, were it not checked, to do which the equilibrium valve G is closed before the piston has completed its up-stroke, after which the further motion of the piston, by compressing the steam on its upper side and attenuating that below it, occasions a

sufficient preponderance of pressure on its upper surface to bring it gradually to a state of rest. The piston is now ready to resume its downward stroke as soon as the valve C shall be opened, and the steam from the boiler admitted to the top of the cylinder; this is effected at the proper time by means of a contrivance termed the cataract, by which the time which elapses between the up and down strokes can be regulated at the pleasure of the person who has charge of the engine. The various contrivances by which the valves are opened are not shown in the plate, as it would require too long a description to render their use clearly intelligible; it is sufficient to remark, that the valve C can be closed at any portion of the stroke which is desired, by means of a simple adjustment; that the exhaustion valve F is so arranged as to close somewhat before the equilibrium valve G is opened; that the injection cock of the condenser is only open during the same time that the exhaustion valve F is open; and that the valve F is opened somewhat before the valve C, in order that the steam beneath the piston may be perfectly condensed as soon as the down-stroke of the engine commences.

We will now proceed to deduce the formulæ for calculating the effects of these engines, but since the nature of the work performed by the Cornish engine during its in-door and out-door strokes are so essentially different, it will be necessary to investigate each separately.

First, then, during the in-door or down stroke, we have the pressure of the steam direct from the boiler upon the upper surface of the piston during the first portion of its stroke, which pressure will be so little less than that in the boiler itself, that without any appreciable error it may be assumed to be the same, for the piston, engine beam, pump rods, and counterweight, comprising a joint mass weighing several tons, require considerable force to put them in motion, and only acquire that motion very gradually, so that the steam has time to attain in the cylinder the same density and pressure as it had in the boiler; and during the after part of the stroke, we have the force due to the expansion of the steam above the piston. Now, if we put a for the area of the piston, λ for the length of the cylinder equivalent to the clearance and steam passages, p_1 for the pressure in the boiler, l_1 for the portion of the stroke performed before expansion, and l_2 for the whole length of the stroke from its commencement, we have from formula (XI.*)

$$a\left(\frac{n}{q}+p_1\right)\left\{l_1+(l_1+\lambda)\,\text{hyper. log}\,\frac{l_2+\lambda}{l_1+\lambda}\right\}-\frac{n}{q}a\,l_2,$$

for the whole amount of work performed during the in-door stroke of the engine; or, indeed, during both the in-door and out-door strokes, for the latter is, as already explained, performed by the descent of a counterweight which was raised during the in-door stroke; and since the work expended in raising it is precisely equivalent to that which it performs during the out-door stroke, the amount which represents it would have to appear first on one side of

the equation as a resistance, and afterwards upon the other as work performed, we shall therefore simplify the calculation by omitting it altogether.

Now, the actual resistances to which the Cornish engine is exposed are as follows; viz., during the in-door stroke, the pressure on the under side of the piston arising from the imperfect vacuum, which we will as before designate by π; the friction proper to the motion of the engine when unloaded, equal f; and the additional friction caused by the load of the engine, represented by $\varrho\varphi_1$ during the down stroke, and $\varrho\varphi_2$ during the up stroke, φ having a different value in consequence of the different load upon the engine, and ϱ being the effective load measured per square foot of the piston's surface: there is also a slight resistance occasioned by the steam pressing with rather more force upon the upper surface of the piston than the lower, in consequence of its having to pass through the passage and equilibrium valve, but this difference of pressure is too small to require being taken notice of; the compression of the steam, however, in the upper part of the cylinder at the conclusion of the out-door stroke, and after the equilibrium valve has been closed, is of more consequence, and requires to be considered. If, then, we put p_2 for the pressure of the steam on each side of the piston at the moment that the equilibrium valve is closed, p_3 for the pressure of the steam above the piston at the conclusion of the out-door stroke, and l_3 the length of the stroke performed after the valve is closed, we have from equation (IX.),

$$p_2 = \frac{l_1 + \lambda}{l_2 + 2\lambda}\left(\frac{n}{q} + p_1\right) - \frac{n}{q},$$

$$\text{and } p_3 = \frac{l_3 + \lambda}{\lambda}\left(\frac{n}{q} + p_2\right) - \frac{n}{q};$$

or, substituting for p_2, in the last equation, its value from the first,

$$p_3 = \frac{l_3 + \lambda}{\lambda}\left\{\frac{l_1 + \lambda}{l_2 + 2\lambda}\left(\frac{n}{q} + p_1\right)\right\} - \frac{n}{q}.$$

Then, if l represent the length of the piston's out-door stroke remaining to be performed at any moment, and p the pressure of the steam above the piston at the same moment, we have from equation (IX.),

$$p = \frac{\lambda}{l + \lambda}\left(\frac{n}{q} + p_3\right) - \frac{n}{q};$$

and for the work performed in the elementary space Δl, we have

$$ap\Delta l,$$

in which, substituting the above value of p, and taking the integral between the limits of $l = 0$, and $l = l_3$, we have

$$a\lambda\int_{l_3}^{0}\frac{dl}{l+\lambda}\left(\frac{n}{q}+p_3\right)-a\frac{n}{q}\int_{l_3}^{0}dl$$
$$=a\lambda\left(\frac{n}{q}+p_3\right)\text{ hyper. log }\frac{l_3+\lambda}{\lambda}-\frac{n}{q}al_3,$$

which becomes, by substituting for p_3 its value above in terms of p_1,

$$a(l_3+\lambda)\frac{l_1+\lambda}{l_2+2\lambda}\left(\frac{n}{q}+p_1\right)\text{ hyper. log }\frac{l_3+\lambda}{\lambda}-\frac{n}{q}al_3,$$

which expression equals the work required to be done by the piston in compressing the steam in the upper part of the cylinder, after the equilibrium valve is closed. But a portion of this work is performed by the expansion of the steam on the under side of the piston, and therefore we must deduct this from the above. Now, the pressure of the steam on the under side of the piston at any moment being represented by p, and the length of the piston's stroke from its commencement at the same moment being represented by l, we have

$$p=\frac{l_2-l_3+\lambda}{l+\lambda}\left(\frac{n}{q}+p_2\right)-\frac{n}{q};$$

and for the work performed in the elementary portion of the stroke represented by Δl, we have $ap\Delta l$, in which, substituting the above value of p, and integrating between the limits of $l=l_2$, and $l=l_2-l_3$, we obtain

$$a(l_2-l_3+\lambda)\left(\frac{n}{q}+p_2\right)\int_{l_2-l_3}^{l_2}\frac{dl}{l+\lambda}-a\frac{n}{q}\int_{l_2-l_3}^{l_2}dl$$
$$=a(l_2-l_3+\lambda)\left(\frac{n}{q}+p_2\right)\text{ hyper. log }\frac{l_2+\lambda}{l_2-l_3+\lambda}-\frac{n}{q}al_3,$$

r substituting for p_2 its value above,

$$a(l_2-l_3+\lambda)\frac{l_1+\lambda}{l_2+2\lambda}\left(\frac{n}{q}+p_1\right)\text{ hyper. log }\frac{l_2+\lambda}{l_2-l_3+\lambda}+\frac{n}{q}al_3$$

equal the work which the steam below the piston performs by its expansion after the equilibrium valve is closed; and therefore the resistance actually to be overcome by the engine equals

$$a(l_3+\lambda)\frac{l_1+\lambda}{l_2+2\lambda}\left(\frac{n}{q}+p_1\right)\text{ hyper. log }\frac{l_3+\lambda}{\lambda}-\frac{n}{q}al_3$$
$$-a(l_2-l_3+\lambda)\frac{l_1+\lambda}{l_2+2\lambda}\left(\frac{n}{q}+p_1\right)\text{hyper. log}\frac{l_2+\lambda}{l_2-l_3+\lambda}+\frac{n}{q}l_3$$
$$=a\frac{l_1+\lambda}{l_2+2\lambda}\left(\frac{n}{q}+p_1\right)\left\{(l_3+\lambda)\text{ hyper. log }\frac{l_3+\lambda}{\lambda}\right.$$
$$\left.-(l_2-l_3+\lambda)\text{ hyper. log }\frac{l_2+\lambda}{l_2-l_3+\lambda}\right\}.$$

Therefore, the total resistance during both strokes of the piston, equals

$$2\,\mathrm{P}al_2 = al_2\{\varrho(1+\varphi_1+\varphi_2)+\pi+2f)$$
$$+a\frac{l_1+\lambda}{l_2+2\lambda}\left(\frac{n}{q}+p_1\right)\Big\{(l_3+\lambda)\text{ hyper. log}\frac{l_3+\lambda}{\lambda}$$
$$-(l_2-l_3+\lambda)\text{ hyper. log}\frac{l_2+\lambda}{l_2-l_3+\lambda}\Big\};$$

and therefore, from the necessary equality between the work and the resistance, we have

$$a\left(\frac{n}{q}+p_1\right)\Big\{l_1+(l_1+\lambda)\text{ hyper. log}\frac{l_2+\lambda}{l_1+\lambda}\Big\}-\frac{n}{q}al_2$$
$$=al_2\{\varrho(1+\varphi_1+\varphi_2)+\pi+2f\}$$
$$+a\frac{l_1+\lambda}{l_2+2\lambda}\left(\frac{n}{q}+p_1\right)\Big\{(l_3+\lambda)\text{ hyper. log}\frac{l_3+\lambda}{\lambda}$$
$$-(l_2-l_3+\lambda)\text{ hyper. log}\frac{l_2+\lambda}{l_2-l_3+\lambda}\Big\}.$$

Now, since the Cornish engine only performs work in pumping during its in-door stroke, it is usual to take its effective velocity as $\nu l_2 = \mathrm{V}$, ν being the number of double strokes which the engine makes per minute. Then, the quantity of steam actually consumed by the engine each double stroke, is only that which remains below the piston when the equilibrium valve is closed, and which equals $a(l_2-l_3+\lambda)$; and further, since the quantity of steam generated equals that consumed, we have

$$\mathrm{Q}v_1 = a\nu(l_2-l_3+\lambda),$$

in which $v_1 = \dfrac{1}{n+qp_2}$, and $\nu = \dfrac{\mathrm{V}}{l_2}$; making which substitutions and solving in respect of p_2, we obtain

$$\frac{l_2\mathrm{Q}}{aq\mathrm{V}(l_2-l_3+\lambda)}-\frac{n}{q}=p_2=\frac{l_1+\lambda}{l_2+2\lambda}\left(\frac{n}{q}+p_1\right)-\frac{n}{q},$$

which latter expression is the value of p_2, already given at page 337. Then reducing, and solving in respect of p_1, we have

$$p_1=\frac{l_2\mathrm{Q}}{aq\mathrm{V}(l_2-l_3+\lambda)}\cdot\frac{l_2+2\lambda}{l_1+\lambda}-\frac{n}{q}.$$

Then, substituting this value of p_1 in the general equation between the work and resistance given above, reducing, and solving in respect of V, Q, and $\varrho a V$, we obtain

$$Q = \frac{aqV(l_2 - l_3 + \lambda)\left\{\varrho(1 + \varphi_1 + \varphi_2) + \pi + 2f + \frac{n}{q}\right\}}{\frac{l_2 + 2\lambda}{l_1 + \lambda}\left\{l_1 + (l_1 + \lambda)\,\text{hyper. log}\,\frac{l_2 + \lambda}{l_1 + \lambda}\right\} - \left\{(l_3 + \lambda)\,\text{hyper. log}\,\frac{l_3 + \lambda}{\lambda} - (l_2 - l_3 + \lambda)\,\text{hyper. log}\,\frac{l_2 + \lambda}{l_2 - l_3 + \lambda}\right\}};$$

$$V = Q\,\frac{\frac{l_2 + 2\lambda}{l_1 + \lambda}\left\{l_1 + (l_1 + \lambda)\,\text{hyper. log}\,\frac{l_2 + \lambda}{l_1 + \lambda}\right\} - \left\{(l_3 + \lambda)\,\text{hyper. log}\,\frac{l_3 + \lambda}{\lambda} - (l_2 - l_3 + \lambda)\,\text{hyper. log}\,\frac{l_2 + \lambda}{l_2 - l_3 + \lambda}\right\}}{aq(l_2 - l_3 + \lambda)\left\{\varrho(1 + \varphi_1 + \varphi_2) + \pi + 2f + \frac{n}{q}\right\}};$$

$$\varrho a V = Q\,\frac{\frac{l_2 + 2\lambda}{l_1 + \lambda}\left\{l_1 + (l_1 + \lambda)\,\text{hyper. log}\,\frac{l_2 + \lambda}{l_1 + \lambda}\right\} - \left\{(l_3 + \lambda)\,\text{hyper. log}\,\frac{l_3 + \lambda}{\lambda} - (l_2 - l_3 + \lambda)\,\text{hyper. log}\,\frac{l_2 + \lambda}{l_2 - l_3 + \lambda}\right\}}{q(l_2 - l_3 + \lambda)(1 + \varphi_1 + \varphi_2)}$$

$$- aV\,\frac{\pi + 2f + \frac{n}{q}}{1 + \varphi_1 + \varphi_2}.$$

The above are the general expressions for Q, V, and $\varrho a V$, and in this case we cannot reduce them to a more practical form, for want of experiments from which to deduce the values of the several resistances.

Single-acting Engine, in which the steam presses only upon the lower surface of the piston.

The engines belonging to this class are better known under the title of *atmospheric* engines, from the down stroke being performed entirely by the pressure of the atmosphere upon the upper surface of the piston.

Plate XII, figure 4, is a section of the working cylinder of an atmospheric engine; the steam at a low pressure enters the bottom of the cylinder A, by the valve B, and assists the counterweight at the other end of the beam in raising the piston. When a certain portion of the stroke has been completed, the valve B is closed, and the counterweight being insufficient of itself to overcome all the resistances of the engine, its velocity becomes gradually diminished, and the piston at last comes to rest at the top of the cylinder. The valve C is then opened, by which a communication is effected with the condenser; and a vacuum being formed under the piston, the pressure of the atmosphere upon its upper surface causes it to descend. In order to prevent the piston making too long a down stroke and coming in contact with the bottom of the cylinder, the valve C is closed some time before the conclusion of the stroke, and the uncondensed steam then remaining in the cylinder being compressed below the piston, gradually brings it to a state of rest.

The work performed by the steam during the up stroke will be represented by formula (XI.*) and if we put Π for the pressure of the atmosphere, that performed during the down stroke will be $\Pi a l_2$, therefore we have

$$a\left(\frac{n}{q} + p_1\right)\left\{l_1 + (l_1 + \lambda)\text{ hyper. log }\frac{l_2 + \lambda}{l_1 + \lambda}\right\} - \frac{n}{q} a l_2 + \Pi a l_2,$$

for the whole work performed during both strokes of the engine.

Then we have for the resistance resulting from the friction of the engine when moving without any load, $2l_2 f$, and for the additional friction caused by the load, together with the resistance of the load itself, $l_2 \varrho (1 + \varphi_1 + \varphi_2)$; and, finally, for the resistance caused by the imperfect vacuum, if we put l_3 for the length traversed by the piston after the valve C is closed, we have for the resistance before the closing of the valve, $a\pi(l_2 - l_3)$; then if we put p for the pressure of the uncondensed steam, at any moment when the piston is at a distance equal to l from the end of its stroke, we have from equation (IX.),

$$p = \frac{l_3 + \lambda}{l + \lambda}\left(\frac{n}{q} + \pi\right) - \frac{n}{q};$$

whence the work to be performed will be represented by

$$a(l_3 + \lambda)\int_{l_2}^{0} \frac{dl}{l + \lambda}\left(\frac{n}{q} + \pi\right) - a\frac{n}{q}\int_{l_2}^{0} dl$$

$$= a(l_3 + \lambda)\left(\frac{n}{q} + \pi\right)\text{ hyper. log }\frac{l_3 + \lambda}{\lambda} - \frac{n}{q} a l_3.$$

Therefore, collecting these several resistances, we have

$$a\,(l_3+\lambda)\left(\frac{n}{q}+\pi\right)\text{ hyper. log }\frac{l_3+\lambda}{\lambda}$$

$$-\frac{n}{q}a l_3 + a l_2\{\varepsilon(1+\varphi_1+\varphi_2)$$

$$+ a\pi(l_2-l_3) = a\left(\frac{n}{q}+p_1\right)\left\{l_1+(l_1+\lambda)\text{ hyper. log }\frac{l_2+}{l_1+}\right.$$

$$-a l_2\left(\frac{n}{q}\right.$$

Now, in order to find the quantity of steam consumed engine, we must deduct the quantity left below the piston w valve C is closed, from the quantity admitted through the during the early portion of the stroke; the latter equals $a($ with the pressure p_1, and the former $a\,(l_3+\lambda)$ with the pres then, in order to find the space (equal S) which it would oc tha pressure p_1, we have from equation (IX.)

$$p_1 = \frac{a\,(l_3+\lambda)}{\mathrm{S}}\left(\frac{n}{q}+\pi\right)-\frac{n}{q};$$

and solving in respect of S, we have

$$\mathrm{S} = a\,(l_3+\lambda)\,\frac{n+q\pi}{n+q p_1}.$$

And therefore the quantity consumed per minute equals

$$\nu a\left\{(l_1+\lambda)-(l_3+\lambda)\,\frac{n+q\pi}{n+q p_1}\right\} = \mathrm{Q}\,v_1;$$

then, substituting for ν and v_1 their values as before, reduci solving in respect of p_1, we have

$$p_1 = \frac{\mathrm{Q}\,l_2}{a\,q\,\mathrm{V}\,(l_1+\lambda)} + \frac{l_3+\lambda}{l_1+\lambda}\left(\frac{n}{q}+\pi\right)-\frac{n}{q}.$$

Then, substituting this value of p_1 in the preceding e reducing, and solving in respect of V, Q, and εa V, we obtain

$$\mathrm{Q} = a q \mathrm{V}\,\frac{\left(\frac{n}{q}+\pi\right)\left\{1-\frac{l_2}{l_3}+\frac{l_2+\lambda}{l_3}\text{ hyper. log }\frac{l_2+\lambda}{\lambda}\right\}+\varepsilon(1+\varphi_1+\varphi_2)+}{\frac{l_1}{l_1+\lambda}+\text{hyper. log }\frac{l_2+\lambda}{l_1+\lambda}}$$

$$-a\,\mathrm{V}\,\frac{l_2+\lambda}{l_2}(n+q\pi);$$

$$V = \frac{Q\left\{\frac{l_1}{l_1+\lambda} + \text{hyper. log}\,\frac{l_2+\lambda}{l_1+\lambda}\right\}}{a q\left\{\left(\frac{n}{q}+\varpi\right)\left[1-\frac{l_2}{l_2}+\frac{l_2+\lambda}{l_2}\,\text{hyper.log}\,\frac{l_2+\lambda}{\lambda}\right]+\varrho(1+\varphi_1+\varphi_2)+2f-\Pi\right\}}$$

$$\cdot\frac{1}{1-\frac{l_2+\lambda}{l_2}\left(\frac{n}{q}+\varpi\right)\left\{\frac{l_1}{l_1+\lambda}+\text{hyper. log}\,\frac{l_2+\lambda}{l_1+\lambda}\right\}};$$

$$\varrho a V = \frac{Q\left\{\frac{l_1}{l_1+\lambda} + \text{hyper. log}\,\frac{l_2+\lambda}{l_1+\lambda}\right\}}{q(1+\varphi_1+\varphi_2)\left\{1-\frac{l_2+\lambda}{l_2}\left(\frac{n}{q}+\varpi\right)\left[\frac{l_1}{l_1+\lambda}+\text{hyper. log}\,\frac{l_2+\lambda}{l_1+\lambda}\right]\right\}}$$

$$-aV\,\frac{\left(\frac{n}{q}+\varpi\right)\left\{1-\frac{l_2}{l_2}+\frac{l_2+\lambda}{l_2}\,\text{hyper. log}\,\frac{l_2+\lambda}{\lambda}\right\}-2f+\Pi}{1+\varphi_1+\varphi_2}.$$

The remark made above with regard to the Cornish engine, of the want of experiments from which to deduce the values of the several resistances, will equally apply to the atmospheric engine, and we are therefore obliged to leave the above formulæ in their general form.

PRACTICAL APPLICATION OF THE FOREGOING FORMULÆ.

In the preceding investigation we have deduced for each engine three forms of expression, giving the velocity of the piston in feet per minute or V, the quantity of water in cubic feet evaporated by the boiler per minute or Q, and the effective useful work performed by the engine per minute expressed in pounds raised through a height of one foot or $\varrho a V$; it may, however, be desirable, before concluding the subject, to deduce from these, expressions for the horse power, and the quantity of coals consumed.

In estimating the horse power of a steam engine, it is usual to assume the power of a horse to be equivalent to raising 33,000 lbs. through a height of one foot in a minute; and, therefore, to derive the horse power of an engine whose effective useful work per minute equals $\varrho a V$, we have

$$\frac{\varrho a V}{33000} = \left\{\begin{array}{l}\text{the number of horses' power}\\ \text{of the engine.}\end{array}\right.$$

Then, if we put F for the weight in pounds of coal, which is required to evaporate a cubic foot of water, we have

$$QF = \left\{\begin{array}{l}\text{the number of pounds of coal consumed}\\ \text{by the engine per minute,}\end{array}\right.$$

$$\text{and } 33000\,\frac{QF}{\varrho a V} = \left\{\begin{array}{l}\text{the number of pounds of coal consumed}\\ \text{per horse power.}\end{array}\right.$$

The following table contains the value of F for four different varieties of coal, compared together by Mr. Wicksteed.

Description of Coal.	No. of lbs. of Coal required to evaporate one cubic foot of water = F.
Welsh coal, used in Cornwall . .	6·7414
Best ditto, used at Old Ford . . .	6·5838
Anthracite, ditto . . .	6·9337
Best Newcastle small, ditto . . .	7·3322
Derbyshire coals, ditto . . .	9·2292

TABLE I.—*Quantity of Coals equivalent to the horse power of* 33,000 *lbs. raised one foot per minute in high pressure steam engines, when the greatest possible effect is obtained* *.

Temperature of steam.	Total force in inches of mercury.	Force in lbs. per square inch above the atmosphere.	Quantity of coal equivalent to one horse power.		Pounds raised one foot high equivalent to the immediate power of the steam produced by 84 lbs. of coal.	
			When working at full pressure.	When acting expansively.	When working at full pressure.	When working expansively.
°	inches.	lbs.	lbs.	lbs.	lbs.	lbs.
234·5	45	7·4	480		2,780,000	
251·0	60	14·8	163	143	8,200,000	9,300,000
275·0	90	29·7	98	77	13,700,000	17,700,000
292·8	120	44·5	82	59	16,600,000	22,700,000
307·7	150	59·3	74	51	18,000,000	26,200,000
320·2	180	74·2	70	48	19,200,000	28,700,000
343·6	240	104·0	65	41½	20,300,000	32,200,000

TABLE II.—*Quantity of Coals equivalent to the horse power of* 33,000 *lbs. raised one foot per minute in condensing steam engines, when the greatest possible effect is obtained.*

Temperature of steam.	Total force in inches of mercury.	Force in lbs. per square inch above the atmosphere.	Quantity of coal equivalent to one horse power.		Pounds raised one foot high equivalent to the immediate power of the steam produced by 84 lbs. of coal.	
			When working at full pressure.	When acting expansively.	When working at full pressure.	When working expansively.
°	inches.	lbs.	lbs.	lbs.	lbs.	lbs.
220·0	35	2·5	63½	40½	21,000,000	33,100,000
234·5	45	7·4	62	38½	21,400,000	35,200,000
251·0	60	14·8	60	35½	22,400,000	37,500,000
275·0	90	29·7		33½		40,000,000
292·8	120	44·5		32½		41,000,000
307·7	150	59·3		32		42,400,000
320·2	180	74·2		31½		42,700,000
343·6	240	104·0		31		43,500,000

* The tables here given, marked I. II. III., were extracted, with the author's permission, from Mr. Tredgold's work on Railroads.

Remarks on ***Tables*** I. and II.—The columns showing the pounds an engine ought to raise one foot high, by the heat of one bushel of coals, are added chiefly for the purpose of comparison with actual practice. Now, it is stated, that after the most impartial examination for several years in succession, it is found that Woolf's engine, at Wheal Abraham Mine, raised 44,000,000 pounds of water, one foot high, with a bushel of coals. And, "the burning of one bushel of good Newcastle or Swansea coals in Mr. Watt's reciprocating engines, working more or less expansively, was found, by the accounts kept at the Cornish mines, to raise from 24,000,000 to 32,000,000 pounds of water one foot high; the greater or less effect depending upon the state of the engine, its size, and rate of working, and the quality of the coal."

We shall further add the results of half a year's reports taken, without selection, from *Lean's* Monthly Reports on the work performed by the steam engines in Cornwall, with each bushel of coals. The numbers show the pounds of water raised one foot high with each bushel, from January to June, 1818.

Name of Engine.	Pounds raised one foot by a bushel of coals.					
	January.	February.	March.	April.	May.	June.
22 to 26 Common Engines average	22,188,000	22,494,000	21,898,000	22,982,000	23,608,000	23,836,000
Wheal Vor (Woolf's Engine)	30,834,000	26,158,000	29,511,000	26,064,000	29,032,000	30,336,000
Wheal Abraham (ditto)	41,847,000	35,364,000	30,445,000	32,723,000	31,520,000	34,352,000
Ditto (ditto)	27,942,000	28,000,000	26,978,000	23,636,000	29,702,000	34,846,000
Wheal Unity (ditto)	31,900,000	32,306,000				
Dolcouth Engine	42,622,000	41,354,000	40,499,000	41,888,000	38,233,000	38,143,000
Wheal Abraham Engine	32,239,000	36,180,000	35,715,000	33,934,000	33,714,000	34,291,000
United Mines Engine	26,396,000	31,830,000	31,427,000	33,564,000	33,967,000	30,105,000
Treskirby Engine	38,733,000	39,375,000	41,867,000	41,823,000	40,615,000	42,098,000
Wheal Chance Engine	28,496,000	32,319,000	33,594,000	33,932,000	—	35,797,000

These numbers are less than the immediate power of the engines, by the friction and loss of effect in working the pumps; hence, in comparing them with Mr. *Tredgold's* table, it may be inferred that he made his calculations from such data as can be realised in practice. It is known from experience, that a cubic foot of water can be converted into steam equal in force to the atmosphere, with 7 lbs. of Newcastle coals; but we also know the attention necessary to produce that effect, and therefore have assumed that $8\frac{4}{10}$ lbs. will be required for that purpose.

According to Mr. *Lean's* Monthly Report, for January, 1833, the following engines raised more than 50,000,000 pounds, one foot high, by consuming a bushel of coals:—

Name of Mine.	Diameter of cylinder.	Pounds raised by a bushel of coals.	Strokes per minute.
Cardrew Downs . . .	66 inches single	51,831,751	7·34
Binner Downs	70 ,,	57,942,435	10·13
Ditto	65 ,,	55,931,852	8·50
Consolidated Mines . .	90 ,,	51,713,913	4·86
Ditto	70 ,,	61,846,133	5·93
Ditto	65 ,,	54,726,957	3·60
Ditto	90 ,,	59,978,983	4·69
Ditto	90 ,,	52,040,672	6·50
Ditto	65 ,,	65,617,011	5·28
Polgooth	66 ,,	70,240,452	8·00
Pembroke	40 ,,	61,170,237	6·67
East Crinnis	76 ,,	62,097,533	7·04
Wheal Leisure	66 ,,	53,506,372	3·58
Wheal Vor	80 ,,	88,504,900	6·22
Ditto	80 ,,	65,471,147	6·07
Ditto	53 ,,	53,938,177	6·37
Poladras Downs . . .	70 ,,	56,766,668	8·78
Great Work	60 ,,	65,460,248	6·08
Wheal Towan	80 ,,	73,159,628	4·90
Ditto	80 ,,	68,782,390	6·98
Wheal Falmouth Consols	70 ,,	54,334,137	6·22
Wheal Darlington . . .	80 ,,	66,058,518	8·47

Of the above, the engine of greatest operation, the first at Wheal Vor, raises the water 190 fathoms, at seven lifts, drawing perpendicularly 160 fathoms, and the remainder diagonally. Main beam over the cylinder; stroke in the cylinder ten feet; one balance-bob at the surface, and three under ground.

The following extract from *Lean's* Monthly Report, for June 1841, will show the advance which had been then made.

Name of Mine and Engine.	Diameter of cylinder, in inches.	Pounds lifted one foot by a bushel of coals.	No. of strokes per minute.
Wheal Darlington, Eastern . .	80, single	81,581,775	9·76
Great Wheal Fortune, Wheal Prosper	80 ,,	54,159,465	6·45
Duffield	80 ,,	54,438,341	6·58
Wheal Julia	80 ,,	59,965,983	6·77
Godolphin, Sims'	80 ,,	50,166,186	6·30
Great Work, Leeds'	60 ,,	66,776,208	6·70
Wheal Vor, Borlase's	80 ,,	74,927,175	6·90
Ditto, Trelawny's . . .	80 ,,	60,534,127	6·00
North Roskear, New	70 ,,	57,354,238	4·76
Wheal Unity Wood, Williams's	80 ,,	61,168,549	5·54
Hallen-Beagle	70 ,,	62,314,755	6·50

Name of Mine and Engine.	Diameter of cylinder, in inches.	Pounds lifted one foot by a bushel of coals.	No. of strokes per minute.
Wheal Unity	70, single	68,462,346	6·70
Carn Brea	76 ,,	70,635,787	3·50
Ditto, Sims'	50, combined	88,096,178	4·30
United Mines, Taylors' . . .	85, single	98,559,570	4·60
Ditto, Cardozo's . . .	90 ,,	55,219,358	5·40
Ditto, Eldon's . . .	30 ,,	61,652,553	10·30
Ditto, Loam's	85 ,,	67,044,127	7·07
Ditto, Hocking's . .	85 ,,	69,131,761	6·60
Bissoe Bridge	50 ,,	54,477,451	3·65
United Hills, Williams's . . .	80 ,,	59,267,244	4·36
Fowey Consols, Austin's . . .	80 ,,	77,854,927	5·30
Polgooth	66 ,,	71,144,002	5·90

TABLE III.—*Showing the effects of a force of traction of* 100 *lbs. at different velocities, on Canals, Railroads, and Turnpike-roads* *. (*From Tredgold.*)

Velocity of Motion.		Load moved by a power of 100 lbs.					
		On a Canal.		On a level Railway.		On a level Turnpike Road.	
Miles per hour.	Feet per second.	Total mass moved.	Useful effect.	Total mass moved.	Useful effect.	Total mass moved.	Useful effect.
		lbs.	lbs.	lbs.	lbs.	lbs.	lbs.
$2\frac{1}{2}$	3·66	55,500	39,400	14,400	10,800	1,800	1,350
3	4·40	38,542	27,361	14,400	10,800	1,800	1,350
$3\frac{1}{2}$	5·13	28,316	20,100	14,400	10,800	1,800	1,350
4	5·86	21,680	15,390	14,400	10,800	1,800	1,350
5	7·33	13,875	9,850	14,400	10,800	1,800	1,350
6	8·80	9,635	6,840	14,400	10,800	1,800	1,350
7	10·26	7,080	5,026	14,400	10,800	1,800	1,350
8	11·73	5,420	3,848	14,400	10,800	1,800	1,350
9	13·20	4,282	3,040	14,400	10,800	1,800	1,350
10	14·66	3,468	2,462	14,400	10,800	1,800	1,350
13·5	19·9	1,900	1,350	14,400	10,800	1,800	1,350

This table is intended to exhibit the work that may be performed by the same mechanical power, at different velocities, on canals, railroads, and turnpike-roads. Ascending and descending by locks or canals, may be considered equivalent to the ascent and descent of inclinations on railroads and turnpike-roads. The load carried, added to the weight of the vessel or carriage which contains it, forms the total

* Though the force of traction on a canal varies as the square of the velocity, the mechanical power necessary to move the boat is usually reckoned to increase as the cube of the velocity. On a railroad, or turnpike, the force of traction is constant; but the mechanical power necessary to move the carriage increases as the velocity.

mass moved; and the useful effect is the load. To find the effect on canals at different velocities, the effect of the given power at one velocity being known, it will be as $3^2 : 2{\cdot}5^2 :: 55{,}500 : 38{,}542$. The mass moved being very nearly inversely as the square of the velocity; at least, within certain limits.

This table shows, that when the velocity is five miles per hour, it requires less power to obtain the same effect on a railway than on a canal; and the lower range of figures is added to show the velocity at which the effect on a canal is only equal to that on a turnpike-road. By comparing the power and tonnage of steam vessels, it will be found that the rate of decrease of power by increase of velocity, is not very distant from the truth; but we know that in a narrow canal the resistance increases in a more rapid ratio than as the square of the velocity*, that is, within certain limits; beyond them, there is a remarkable change in the circumstances of resistance.

SECT. IV. *Animal Strength as a Mechanical Agent.*

1. The force obtained through the medium of animal agency, evidently varies, not only in different species of animals, but also in different individuals. And this variation depends, first, on the particular constitution of the individual, and upon the complication of causes which may influence it; secondly, upon the particular dexterity acquired by habit. It is plain, that such a variation cannot be subjected to any law, and that there is no expedient to which we can have recourse but that of seeking mean results.

Secondly, the force varies according to the nature of the labour. Different muscles are brought into action in different gestures and positions of an animal which labours; the weight itself of the animal machine is an aid in some kinds of labour, and a disadvantage in others; whence it is not surprising that the force exerted is different, in different kinds of work. Thus the force exerted by a man is

* According to the interesting researches of Du Buat, the resistance to the motion of boats, even in canals, may be regarded as proportional to the square of the velocity, or R as V^2 nearly, provided R be made to depend upon the transverse sections of the vessel and the canal in which it moves. If c be the vertical section of the canal, and b the vertical section of the immersed portion of the boat, or barge; then

$$R = \frac{\frac{c}{b} + 2}{K}.$$

The mean of Du Buat's experiments gives, $K = 8{\cdot}46$, or

$$R = \frac{\frac{c}{b} + 2}{8{\cdot}46};$$

but these experiments were not so numerous and varied as might be wished. See *Principes d'Hydraulique*, tom ii. pp. 340, 342, &c.

different, in carrying a weight, in drawing or pushing it horizontally, and in drawing or pushing it vertically.

Thirdly, the force varies according to the duration of the labour. The force, for example, which a man can exert in an effort of a few instants, is different from that which he can maintain equably in a course of action continued, or interrupted only by short intervals, for a whole day of labour, without inducing excessive fatigue. The former of these may be called *Absolute Force*, the latter *Permanent Force*. It is of use to become acquainted with them both, as it is often advantageous to avail ourselves sometimes of the one, sometimes of the other.

Lastly, the force varies according to the different degrees of velocity with which the animal, in the act of labouring, moves either its whole body, or that part of it which operates. The force of the animal is the greatest when it stands still; and becomes weaker as it moves forward, in proportion to its speed; the animal acquiring, at last, such a degree of velocity as renders it incapable of exerting any force.

2. Let φ be a weight equivalent to the force which a man can exert, standing still: and let V be the velocity with which, if he proceeds, he is no longer capable of exerting any force: also, let F be a weight equivalent to the force which he exerts, when he proceeds, equably, with a velocity v.

Then F will be a function of v, such that, 1st, it decreases whilst v increases; 2ndly, when $v = 0$, then $F = \varphi$; 3rdly, when $v = V$, $F = 0$.

3. Upon the nature of this function, we have the three following suppositions.

1. $F = \varphi \left(1 - \frac{v}{V}\right)$. (Bouguer, *Man. des Vais.*)

2. $F = \varphi \left(1 - \frac{v^2}{V^2}\right)$. (Euler, *Nov. Comm. Pet.* tom. III.)

3. $F = \varphi \left(1 - \frac{v}{V}\right)^2$. (Ib. tom. VIII; and *Act of Rowers.*)

4. *Coroll.* 1. The effect of the permanent force being measured by the product $F\,v$, the expression for the effect will be one of the three following, accordingly as one or other of the suppositions is adopted.

1. $F\,V \left(1 - \frac{F}{\varphi}\right)$, or $\varphi v \left(1 - \frac{v}{V}\right)$.

2. $F\,V \sqrt{\left(1 - \frac{F}{\varphi}\right)}$, or $\varphi v \left(1 - \frac{v^2}{V^2}\right)$.

3. $F\,V \left(1 - \sqrt{\frac{F}{\varphi}}\right)$, or $\varphi v \left(1 - \frac{v}{V}\right)^2$.

5. *Coroll.* 2. To know the weight with which a man should be loaded, or the velocity with which he ought to move, in order to produce the greatest effect, we must make $d \cdot Fv = 0$.

Whence we shall have

1. $F = \frac{1}{2}\varphi$; and $v = \frac{1}{2}V$.

2. $F = \frac{2}{3}\varphi$; and $v = \frac{1}{\sqrt{3}}V = 0{\cdot}5773\,V$.

3. $F = \frac{4}{9}\varphi$; and $v = \frac{1}{3}V$.

6. *Coroll.* 3. And the value of the greatest effect, according to the several hypotheses, will be,

1. $\frac{1}{4}\varphi V$:

2. $\frac{2}{3\sqrt{3}}\varphi V = 0{\cdot}3836\,\varphi V$:

3. $\frac{4}{27}\varphi V$.

But which of the three suppositions ought we to prefer? And are we certain that any of them approximates to the true law of nature?

Mr. Schulze made a series of experiments with a view to the determination of this point*, and with regard to men decided in favour of the last of Euler's formulæ: viz.

$$F = \varphi\left(1 - \frac{v}{V}\right)^2.$$

As the experiments of this philosopher are very little known in England, I shall here present his brief account of them.

7. To make the experiments on human strength, he took at random 20 men of different sizes and constitutions, whom he measured and weighed. The result is exhibited in the following table.

Order.	Height.	Weight.	Order.	Height.	Weight.
1	5′ 3″ 4‴	122	11	5′ 9″ 7‴	132
2	5 2 3	134	12	5 1 4	157
3	5 7 2	165	13	5 3 2	175
4	5 5 0	131	14	5 4 1	117
5	5 11 2	177	15	5 10 8	192
6	6 0 4	158	16	5 0 3	133
7	5 8 3	180	17	4 11 2	147
8	5 2 1	117	18	5 3 9	124
9	5 4 8	140	19	5 6 0	163
10	5 0 4	126	20	5 10 1	181

* Mem. Acad. Scienc. Berlin, for 1783.

e the heights are expressed in feet (marked '), inches (''), is ('''), the feet being those of Rhinland, each 12·35 English
The weights are in pounds, which are to our avoirdupois 30 to 29.

ind the strength that each of these men might exert to raise a vertically, Mr. Schulze made the following experiments:—
took various weights increasing by 10 lbs. from 150 lbs. up to s.; all these weights were of lead, having circular and equal
To use them with success in the proposed experiments, he the same time a kind of bench made, in the middle of which hole of the same size as the base of the weights: this hole was y a circular cover when pressed against the bench; at other t was kept at about the distance of a foot and a half above the by means of a spring and some iron bars. To prevent the with which this cover was loaded during the experiment from down the cover, lower than the level of the surface of the he had several grooves made in the four iron bars, which sus- the cover, and which at the same time served to hold up the at any height where it might arrive by the pressure of the as soon as the pressure of the weight ceased.

r having laid the 150 lbs. weight on the cover, and the other s in succession increasing by 10 lbs. up to 250 lbs., he made llowing experiments with the men whose size and weight are bove, by making them lift up the weights as vertically as pos- ll at once, and by observing the height to which they were lift them. The annexed table gives the heights observed for ferent weights marked at its head.

50	160	170	180	190	200	210	220	230	240	250
'''	'' '''	'' '''	'' '''	'' '''	'' '''	'' '''	'' '''	'' '''	'' '''	
9	6 4	4 11	4 4	3 8	2 8	1 1				
10	6 6	5 7	4 7	3 11	2 5	0 5	1 7	0 3		
9	7 3	6 5	5 9	4 11	4 0	3 0	3 8	3 1	1 4	
3	7 6	7 2	5 10	5 3	4 7	4 0	4 7	3 2	1 3	
4	11 1	9 7	8 5	7 10	7 1	5 10	6 6	4 1	0 1	
5	14 0	13 5	12 8	11 5	10 1	8 6	3 8	1 11	0 2	
11	11 3	10 5	9 3	8 1	6 9	5 3	5 1	3 2	1 0	
9	10 2	9 4	8 11	8 1	6 11	5 10				
5	8 3	7 1	5 6	4 1	2 9	1 3				
1	6 5	4 7	3 9	2 5	1 7	0 4				

s table proves that the size of the men employed to raise the s vertically has considerable influence on the height to which rought the same weight. We find also that the height dimi- in a much more considerable ratio than the weight increases; e may therefore conclude, that it is advantageous to employ nen when it becomes necessary to draw vertically from below ds: and on the contrary, it is more advantageous to employ of a considerable *weight*, when it is required to lift up loads by

means of a pulley about which a cord passes, that the workmen may draw in a vertical direction, from above downwards. To find the absolute strength of these men in a horizontal direction, Mr. Schulze proceeded thus:—

Having fixed over an open pit a brass pulley, extremely well made, of 15 inches diameter, whose axis, made of well polished steel to diminish the friction, was $\frac{1}{4}$ of an inch in diameter, he passed over this pulley a silk cord, worked with care, to give it both the necessary strength and flexibility. One of the ends of this cord carried a hook to hang a weight to it which hung vertically in the pit, whilst the other end was held by one of the 20 men, who in the first order of the following experiments made it pass above his shoulders; instead of which, in the second, he simply held it by his hands.

Mr. Schulze had taken the precaution to construct this in such a manner that the pulley might be raised or lowered at pleasure, in order to keep the end of the cord held by the man always in a horizontal direction, according as the man was tall or short, and exerted his strength in any given direction.

He had made the necessary arrangements so as to be able to load successively the basin of a balance which was attached to the hook at the end of the cord which descended into the pit, whilst the man who held the other end of this cord employed all his strength without advancing or receding a single inch.

The following tables give the weights placed in the basin when the workmen were obliged to give up, having no longer sufficient strength to sustain the pressure occasioned by the weight. To proceed with certainty, Mr. Schulze increased the weight each time by five pounds, beginning from 60, and took the precaution to make this augmentation in equal intervals of time; having always precisely a space of 10 seconds between them. The result of these observations repeated several days in succession, is contained in the following tables.

I. When the cord passed over the shoulders of the workmen:

Order.	lbs.	Order.	lbs.	Order.	lbs.	Order.	lbs.
1	95	6	100	11	95	16	95
2	105	7	115	12	100	17	100
3	110	8	105	13	110	18	90
4	100	9	95	14	90	19	100
5	105	10	90	15	110	20	100

II. When the cord was simply held before the man:

Order.	lbs.	Order.	lbs.	Order.	lbs.	Order.	lbs.
1	90	6	100	11	90	16	90
2	105	7	110	12	90	17	90
3	105	8	100	13	100	18	85
4	90	9	90	14	85	19	100
5	95	10	85	15	105	20	100

These two tables show that men have less power in drawing a cord before them than when they make it pass over their shoulders; they show, also, that the largest men have not always the greatest strength to hold, or to draw in a horizontal direction, by means of a cord. To obtain the absolute velocity of these twenty men, Mr. Schulze proceeded as follows: —

Having measured very exactly a distance of 12,000 Rhinland feet, in a plane nearly level, he caused these twenty men to march with a fair pace, but without running, and so as to continue during the period of four or five hours; the following is the time employed in describing this space, with the velocity resulting for each of them.

Order.	Time.	Veloc.	Order.	Time.	Veloc.	Order.	Time.	Veloc.
1	40·18	4·94	8	40·09	4·99	15	36·17	5·51
2	41·12	4·85	9	40·20	4·96	16	41·28	4·82
3	39·08	5·55	10	40·51	4·90	17	42·25	4·71
4	39·40	5·04	11	36·17	5·51	18	40·19	4·98
5	34·19	5·83	12	38·11	5·24	19	39·57	5·01
6	35·11	5·68	13	38·05	5·25	20	37·51	5·29
7	38·07	5·25	14	37·01	5·40			

It is necessary to mention, with regard to these experiments, that Mr. Schulze took care to place at certain distances persons in whom he could place confidence, in order to observe whether these men marched uniformly and sufficiently quick without running.

Having thus obtained not only the absolute force, but the absolute velocity also, of several men, he took the following method to determine their relative force.

He made use of a machine composed of two large cylinders of very hard marble, which turned round a vertical cylinder of wood, and moved by a horse, which described in his march a circle of 10 Rhinland feet. This machine appeared the most proper to make the subsequent experiments, which serve to determine the relative strength that the men had employed to move this machine, and which is used hereafter to determine which of Euler's two formulæ ought to be preferred.

To obtain this relative force, he took here the same pulley which

served in the preceding experiments, by applying a cord to the vertical cylinder of wood, and attaching to the other end of this cord, which entered into an open pit, a sufficient weight to give sucessively to the machine different velocities.

Having applied in this manner a weight of 215 lbs., the machine acquired a motion, which after being reduced to a uniform velocity, taking into account the acceleration of the weight, of the friction, and of the stiffness of the cord, gave 2·41 feet velocity; and having applied in the same manner a weight of 220 lbs., the resulting uniform motion gave a velocity of 2·47 feet. These two limits are mentioned because they serve as a comparison with what immediately follows: Mr. Schulze began these experiments with a weight of 100 lbs., and increased it by five every time from that number up to 400 lbs.

He made this machine move by the first seven of his workmen, placing them in such a way that their direction remained almost always perpendicular to the arm on which was attached the cord which passed over their shoulders in an almost horizontal direction.

Thus situated, they made 281 turns with this machine in two hours, which gave for their relative velocity $v = 2{\cdot}45$ feet per second. We have also the absolute force, or φ, from these seven men, by the above table, = 730 lbs.: and their absolute velocity or $V = 5{\cdot}30$ feet.

Therefore, by substituting these values in the first formula, we find the relative force $F = 205$ lbs., which agrees very well with what we have just found above.

If instead of this first formula the second be taken, it gives $F = 153$ lbs., which is far too little.

By this it is evident, that the last of Euler's two formulæ is to be preferred in all respects. Mr. Schulze made a great number of combinations, and almost always found the same effect.

Dividing the 205 lbs. which we have just found by seven, the number of workmen, we get 29 lbs. for the relative force with 2·45 feet relative velocity for each man, which is rather more than the values commonly adopted in the computation of machinery. A number of other observations on different machines have given the same result; that is to say, we must value the mean human strength at 29 or 30 lbs., with a velocity of $2\frac{1}{2}$ feet per second.

To obtain the ratio of the strength of a horse to that of a man, Mr. Schulze proceeded in a similar manner; but his results, in reference to that inquiry, are neither so correct nor so interesting.

8. In the first volume of my *Mechanics*, I stated the average force of a man at rest to be 70 lbs., and his utmost walking velocity when unloaded, to be about six feet per second; and thence inferred that a man would produce the greatest momentum when drawing $31\frac{1}{3}$ lbs. along a horizontal plane with a velocity of two feet per second. But that is not the most advantageous way of applying human strength.

9. Dr. Desaguliers asserts, that a man can raise of water or any other

weight about 550 lbs., or one hogshead (weight of the vessel included), ten feet high in a minute; this statement, though he says it will hold good for six hours, appears from his own facts to be too high, and is certainly such as could not be continued one day after another. Mr. Smeaton considers this work as the effect of haste or distress; and reports, that six good English labourers will be required to raise 21141 cubic feet of sea water to the height of four feet in four hours: in this case, the men will raise a very little more than six cubic feet of fresh water each to the height of 10 feet in a minute. Now, the hogshead containing about $8\frac{1}{2}$ cubic feet, Smeaton's allowance of work proves less than that of Desaguliers in the ratio of 6 to $8\frac{1}{2}$ or 3 to $4\frac{1}{4}$. And as his good English labourers who can work at this rate are estimated by him to be equal to a double set of common men picked up at random, it seems proper to state that, with the probabilities of voluntary interruption, and other incidents, a man's work for several successive days ought not to be valued at more than half a hogshead raised 10 feet high in a minute. Smeaton likewise states, that two ordinary horses will do the work in three hours and twenty minutes, which amounts to little more than two hogsheads and a half raised 10 feet high in a minute. So that, if these statements be accurate, one horse will do the work of five men.

Mr. Emerson affirms, that a man of ordinary strength, turning a roller by the handle, can act for a whole day against a resistance equal to 30 lbs. weight; and if he works 10 hours a day, he will raise a weight of 30 lbs. through $3\frac{1}{2}$ feet in a second of time; or, if the weight be greater, he will raise it to a proportionally less height. If two men work at a windlass, or roller, they can more easily draw up 70 lbs. than one man can 30 lbs., provided the elbow of one of the handles be at right angles to that of the other. Men used to bear loads, such as porters, will carry from 150 lbs. to 200 or 250 lbs., according to their strength. A man cannot well draw more than 70 lbs. or 80 lbs. horizontally: and he cannot thrust with a greater force acting horizontally at the height of his shoulders, than 27 or 30 lbs. But one of the most advantageous ways in which a man can exert his force is to sit and pull towards him nearly horizontally, as in the action of rowing.

M. Coulomb communicated to the French National Institute the results of various experiments on the quantity of action which men can afford by their daily work, according to the different manners in which they employ their strength. In the first place he examined the quantity of action which men can produce when, during a day, they mount a set of steps or stairs, either with or without a burden. He found that the quantity of action of a man who mounts without a burden, having only his own body to raise, is double that of a man loaded with a weight of 68 kilogrammes, or 150 lbs. avoirdupois, both continuing at work for a day. Hence it appears how much, with equal fatigue and time, the total or absolute effort may obtain different values by varying the combinations of effort and velocity.

But the word *effect* here denotes the total quantity of labour

employed to raise, not only the burden, but the man himself; and, as Coulomb observes, what is of the greatest importance to consider is the *useful effect*, that is to say, the total effect, deducting the value which represents the transference of the weight of the man's body. This total effect is the greatest possible when the man ascends without a burden; but the *useful* effect is then nothing: it is also nothing if the man be so much loaded as to be scarcely capable of moving, and consequently there exists between these two limits a value of the load such that the useful effect is a maximum. M. Coulomb supposes that the loss of quantity of action is proportional to the load (an hypothesis which experience confirms), whence he obtains an equation which, treated according to the rules of maxima and minima, gives 53 kilogrammes (117 lbs. avoirdupois) for the weight with which the man ought to be loaded, in order to produce during one day, by ascending stairs, the greatest useful effect: the quantity of action which results from this determination has for its value 56 kilogrammes (123$\frac{1}{2}$ lbs. avoirdupois) raised through one kilometre, or nearly 1094 yards. But this method of working is attended with a loss of three-fourths of the total action of the man, and consequently costs four times as much as work, in which, after having mounted a set of steps without any burden, the man should suffer himself to fall by any means, so as to raise a weight nearly equal to that of his own body.

From an examination of the work of men walking on a horizontal path, with or without a load, M. Coulomb concludes that the greatest quantity of action takes place when the men walk being loaded; and is to that of men walking under a load of 58 kilogrammes (128 lbs. avoirdupois) nearly as 7 to 4. The weight which a man ought to carry in order to produce the greatest *useful effect*, namely, that effect in which the quantity of action relative to the carrying his own weight is deducted from the total effect, is 50·4 kilogrammes, or 111·18 lbs. avoirdupois.

There is a particular case which always obtains with respect to burdens carried in towns, viz., that in which the men, after having carried their load, return unloaded for a new burden. The weight they should carry in this case, to produce the greatest effect, is 61·25 kilogrammes (135$\frac{1}{8}$ lbs. avoirdupois). The quantity of useful action in this case, compared with that of a man who walks freely and without a load, is nearly as 1 to 5, or, in other words, he employs to pure loss $\frac{4}{5}$ths of his power. By causing a man to mount a set of steps freely and without burden, his quantity of action is at least double of what he affords in any other method of employing his strength.

When men labour in cultivating the ground, the whole quantity afforded by one man during a day amounts to 100 kilogrammes elevated to 1 kilometre, that is, 220·6 lbs. raised 1094 yards. M. Coulomb, comparing this work with that of men employed to carry burdens up an ascent of steps, or at the pile-engine, finds a loss of about $\frac{1}{10}$th part only of the quantity of action, which may be neglected in researches of this kind.

In estimating mean results we should not determine from experiments of short duration, nor should we make any deductions from the exertions of men of more than ordinary strength. The mean results have likewise a relation to climate. "I have caused," says M. Coulomb, "extensive works to be executed by the troops at Martinico, where the thermometer (of Réaumur) is seldom lower than 20° (or 77° of Fahrenheit). I have executed works of the same kind by the troops in France, and I can affirm, that under the fourteenth degree of latitude, where men are almost always covered with perspiration, they are not capable of performing half the work they could perform in our climate."

10. Entirely according with these are the experiments of Regnier, by means of a dynamometer, the results of which not only established the superiority of civilized men over savages, but that of the Englishman over the Frenchman. The following is reduced from one of Regnier's tables of mean results.

Strength.	With the hands.		With the reins.	
	lbs.	oz.	lbs.	oz.
Savages, of Van Diemen's Land . .	30	6	0	0
———— New Holland	51	8	14	8
———— Timor	58	7	16	2
Civilized men: French	69	2	22	1
———— English	71	4	23	8

11. A porter in London is accustomed to carry a burden of 200 lbs. at the rate of three miles an hour; and a couple of chairmen continue at the rate of four miles an hour, under a load of 300 lbs. Yet these exertions, Professor Leslie remarks, are greatly inferior to the subsultory labour performed by porters in Turkey, the Levant, and generally on the shores of the Mediterranean. At Constantinople, an Albanian porter will carry 800 or 900 lbs. on his back, stooping forward, and assisting his steps by a short staff. Such loads, however, are carried for very short intervals. At Marseilles it is affirmed that four porters carry the immense load of nearly two tons, by means of soft hods passing over their heads, and resting on their shoulders, with the ends of poles from which the goods are suspended.

12. With regard to the magnitude of the comparative efforts of man in different employments, the late Mr. Robertson Buchanan ascertained, that in working a pump, in turning a winch, in ringing a bell, and rowing a boat, the dynamic results are as the numbers 100, 167, 227, and 248.

According to the interesting experiments described in M. Hachette's *Traité des Machines*, the dynamic unity being *the weight of a cubic metre of water raised to the height of one metre* [that is, 2208 lbs. avoirdupois, or 4 hogsheads raised to the height of 3·281 feet, or

1·3124 hogsheads to the height of 10 feet], we have the follow measures, at a medium, of the daily actions of men.

	Dyn. U
1. A man marching 7½ hours on a slope of 7 degrees with a load of from 15 to 18 lbs.	=
2. Marching in a mountainous country without load	=
3. Carrier of wood up a ladder, his weight 123 lbs., his load 117 lbs.	=
4. Carrier of peat, up steps, his own weight comprised, 112 to	=
5. Man working at the cord of a pulley to raise the ram of a pile engine: three examples	=
6. A man drawing water from a well by means of a cord	=
7. Man working at a capstan	=
8. Man working at a capstan to raise water, mean of 24 observations	=

The unit of transport being the weight of a cubic metre of w carried a metre (or 2208 lbs. carried 3281 feet) upon a horizo road, we have for the daily action,

	Dyn. U
1. A man travelling without load on a flat road, his weight 154 lbs. his journey 31¼ miles	= 3
2. A soldier, carrying from 44 to 55 lbs., travelling 12½ miles, 1800 to	= 1
3. Ditto, a forced march of 25 miles	= 2
4. A French porter, weight of the man not included, 792 to	=
5. Porter with wheel-barrow, weight of the man not included	= 1
6. Porters with a sledge	=
7. A man drawing a boat on a canal; 110310 lbs. conveyed 6⅛ miles	= 550

14. Mr. B. Bevan, an able engineer, has made experiments on application of human energy to the use of augurs, gimlets, scr drivers, &c. He has presented to the public the following list, specimen; premising that many ordinary operations are performe a short space of time, and may therefore be done by greater exer than if a longer time was necessary. Thus a person, for a s time, is able to use a tool or instrument called

	lbs.
A drawing-knife, with a force of	100
An augur, with two hands	100
A screw-driver, one hand	84
A common bench vice handle	72
A chisel and awl, vertical pressure	72
A windlass, handle revolving	60

	lbs.
Pincers and pliers, compression	60
A hand-plane, horizontally	50
A hand or thumb-vice	45
A hand-saw	36
A stock-bit, revolving	16
Small screw-drivers, or twisting by the thumb and fore-finger only	14

15. M. Morisot informs us that the time employed by a French stone-mason's sawyer, to make a section of a square toise (40·89 square feet English) in different stones, is as below: viz.

	hours.
Calcareous stone, equal grain, spec. grav. 2000	45
——— hard, spec. grav. 2300	62
Liais, ditto hard, fine grain, spec. grav. 2400	67
Pyrénéan alabaster, the softest of the marbles	56
Normandy granite	504
Granite from Vosges	700
Red and green porphyry	1177

The workmen ordinarily made 50 oscillations in a minute; each stroke about 15¾ inches.

16. Hassenfratz assigns 13 kilogrammes as the mean effort of such a man; but M. Navier, in his new edition of *Belidor, Architecture Hydraulique*, regards this estimate as too high. If Hassenfratz were correct, the daily quantity of action of the sawyer would be equivalent to 376 kilogrammes elevated to a kilometre (or 818 lbs. raised ⅛ of a mile), a quantity more than triple that of a man working at a winch. M. Navier gives, as a more correct measure of this labour for 12 hours, 188 kilogrammes raised a kilometre: half the former measure. But all this is probably very vague.

17. Among quadrupeds, those which are employed to produce a mechanical effect are the dog, the ass, the mule, the ox, the camel, and the horse. Of these the *horse* is the only one, so far as we are aware, whose animal energy has been subjected to cautious experiments; and, even with regard to this noble animal, opinions as to actual results are very much afloat. The dynamic effort of the horse is, however, probably about 6 times that of a strong and active labourer. Desaguliers states the proportion as 5 to 1, coinciding with the deductions of Smeaton. The French authors usually regard seven men as equivalent to one horse. As a fair mean between these, I assumed in vol. i. of my *Mechanics* the proportion of 6 to 1, and stated the strength of a horse as equivalent to 420 lbs. at a dead pull. But the proportion must not be regarded as constant, but obviously varies much according to the breed and training of the animal, as well as according to the nature of the work about which he is employed. Thus the worst way, as De la Hire observed, of

applying the strength of a horse is to make him carry a weight up a steep hill; while the organization of a man fits him very well for that kind of labour: hence *three* men, climbing up such a hill with a weight of 100 lbs. each, will proceed faster than a horse with a load of 300 lbs.

18. In the memoirs of the French Academy for 1703 are inserted the comparative observations of M. Amontons, on the velocity of men and of horses; in which he states the velocity of a horse loaded with a man and walking to be rather more than $5\frac{1}{4}$ feet per second, or $3\frac{1}{2}$ miles per hour, and when going a moderate trot with the same weight to be about $8\frac{1}{2}$ feet per second, or about 6 miles per hour. These velocities, however, are somewhat less than what might have been taken for the mean velocities.

19. But the best way of applying the strength of horses is to make them draw weights in carriages, &c. To this kind of labour, therefore, the inquiries of experimentalists should be directed. A horse put into harness, and making an effort to draw, bends himself forward, inclines his legs, and brings his breast nearer to the earth; and this so much the more as the effort is the more considerable. So that, when a horse is employed in drawing, his effort will depend, in some measure, both upon his own weight and that which he carries on his back.

Indeed it is highly useful to load the back of a drawing horse to a certain extent; though this, on a slight consideration, might be thought to augment unnecessarily the fatigue of the animal: but it must be recollected that the mass with which the horse is charged vertically is added in part to the effort which he makes in the direction of traction, and thus dispenses with the necessity of his inclining so much forward as he must otherwise do: and may, therefore, under this point of view, relieve the draught more than to compensate for the additional fatigue occasioned by the vertical pressure. Carmen, and waggoners in general, are well aware of this, and are commonly very careful to dispose of the load in such a manner that the shafts shall throw a due proportion of the weight on the back of the shaft horse. This is most efficaciously accomplished at Yarmouth, in Norfolk, where a number of narrow streets connecting the market-place with the quay, have led to the invention and use of the low, strong, narrow carts, thence denominated *Yarmouth carts*, drawn by one horse; and on which the loads are frequently shifted, especially when the vehicles pass over the bridge, in order to give the animals better foot-hold, and consequently a greater dynamic effort.

20. The best disposition of the traces during the time a horse is drawing is perpendicular to the position of the collar upon his breast and shoulders: when the horse stands at ease, this position of the traces is rather inclined upwards from the direction of the road; but when he leans forward to draw the load, the traces should then become nearly parallel to the plane over which the carriage is to be drawn; or, if he be employed in drawing a sledge, or any thing without wheels, the inclination of the traces to the road should (from the

table at page 220) be about 18½°, when the friction is one-third of the pressure. If the relation of the friction to the pressure be different from this, the same table will exhibit the angle which the traces must make with the road.

21. When a horse is made to move in a circular path, as is often practised in mills and other machines moved by horses, it will be necessary to give the circles which the animal has to walk round the greatest diameter that will comport with the local and other conditions to which the motion must be subjected. It is obvious, indeed, that, since a rectilinear motion is the most easy for the horse, the less the line in which he moves is curved, with the greater facility he will walk over it, and the less he need recline from a vertical position: and besides this, with equal velocity the centrifugal force will be less in the greatest circle, which will proportionally diminish the friction of the cylindrical part of the trunnions, and the labour of moving the machine. And, further, the greater the diameter of the horse-walk, the nearer the chord of the circle in which the horse draws is to coincidence with the tangent, which is the most advantageous position of the line of traction. On these accounts it is that, although a horse may draw in a circular walk of 18 feet diameter, yet in general it is advisable that the diameter of such a walk should not be less than 25 or 30 feet; and in many instances 40 feet would be preferable to either.

22. It has been stated by Desaguliers and some others, that a horse employed daily in drawing nearly horizontally can move, during eight hours in the day, about 200 lbs. at the rate of 2½ miles per hour, or 3⅔ feet per second. If the weight be augmented to about 240 or 250 lbs., the horse cannot work more than six hours a day, and that with a less velocity. And, in both cases, if he carry some weight, he will draw better than if he carried none. M. Sauveur estimates the mean effort of a horse at 175 French, or 189 avoirdupois pounds, with a velocity of rather more than three feet per second. But all these are probably too high to be continued for eight hours, day after day. In another place Desaguliers states the mean work of a horse as equivalent to the raising a hogshead full of water (or 550 lbs.) 50 feet high in a minute. But Mr. Smeaton, to whose authority much is due, asserts, from a number of experiments, that the greatest effect is the raising 550 lbs. forty feet high in a minute. And, from some experiments made by the Society for the Encouragement of Arts, under the direction of Mr. Samuel Moore, it was concluded, that a horse moving at the rate of three miles an hour can exert a force of 80 lbs. Unluckily, we are not sufficiently acquainted with the nature of the experiments and observations from which these deductions were made to institute an accurate comparison of their results. Neither of them ought to express what a horse can draw upon a carriage; because in that case friction only is to be overcome (after the load is once put into motion); so that a middling horse, well applied to a cart, will often draw much more than 1000 lbs. The proper estimate would be that which measures the weight that a

horse would draw up out of a well; the animal acting by a horizontal line of traction turned into the vertical direction by a simple pulley, or roller, whose friction should be reduced as much as possible.

23. Mr. Tredgold, in his valuable publication on Railroads, has directed his attention to the subject of "*horse power*." The following is his expression for the power of a horse, $250\ v\left(1-\frac{v}{V}\right)$; and $\frac{250\ d\ v\left(1-\frac{v}{V}\right)}{1+n}$ for the day's work in lbs. raised one mile; d being the hours which the horse works in a day, and the weight of the carriage to that of the load as $n:1$. He also gives $\frac{14{\cdot}7}{\sqrt{d}}$ for the greatest speed in miles per hour, when the horse is unloaded. These expressions must, at present, be regarded as tentative. The following is his table of the comparison of the duration of a horse's daily labour and maximum of velocity, unloaded.

Duration of labour. *Hours.*	Max. velocity unloaded in miles per hour.
1	14·7
2	10·4
3	8·5
4	7·3
5	6·6
6	6·0
7	5·5
8	5·2
9	4·9
10	4·6

Taking the hours of labour at 6 per diem, the utmost that Mr. *Tredgold* would recommend, the maximum of useful effect he assigns at 125 lbs. moving at the rate of three miles per hour, and regarding the expense of carriage, in that case, as unity; then—

Miles per hour.	Proportional expense.	Moving force.
2	$1\frac{1}{8}$ or 1·125	166 lbs.
3	1	125
$3\frac{1}{2}$	$1\frac{1}{35}$ or 1·0285	104
4	$1\frac{1}{8}$ or 1·125	83
$4\frac{1}{2}$	$1\frac{1}{3}$ or 1·333	$62\frac{1}{2}$
5	$1\frac{4}{5}$ or 1·8	$41\frac{2}{3}$
$5\frac{1}{8}$	2	$36\frac{1}{2}$

That is, the expense of conveying goods at 3 miles per hour being 1; the expense of $4\frac{1}{2}$ miles per hour will be $1\frac{1}{3}$; and so on, the expense being doubled when the speed is $5\frac{1}{8}$ miles per hour.

24. Thus, according to Mr. *Tredgold*, we have for the day of

6 hours 2250 lbs. raised one mile. And Mr. *Bevan*, who has made many experiments on the force of traction to move canal boats on the Grand Junction Canal, found the force of traction 80 lbs., and the space travelled in a day 26 miles; hence, it is only equivalent to $26 \times 80 = 2080$ lbs. raised one mile for the day's work; the rate of travelling being 2·45 miles per hour; and the result a little less than Mr. *Tredgold's*, the difference probably arising from the deviation of the angle of the catenary formed by the rope from the horizon.

25. The following experimental data from Mr. *Bevan* also deserve attention.

"In the period from 1803 to 1809, I had the opportunity of ascertaining correctly the mean force exerted by good horses in drawing a plough; having had the superintendence of the experiments on that head at the various ploughing matches both at Woburn and Ashridge, under the patronage of the Duke of Bedford and the Earl of Bridgewater. I find among my memoranda the result of eight ploughing matches, at which there were seldom fewer than seven teams as competitors for the various prizes.

	lbs.
The first result is from the mean force of each horse in six teams, of two horses each team, upon light sandy soil	= 156
The second result is from seven teams of two horses each team, upon loamy ground, near Great Berkhampstead...	= 154
The third result is from six teams of four horses each team, with old Hertfordshire ploughs.....	= 127
The fourth result is from seven teams of four horses each team, upon strong stony land (improved ploughs).....	= 167
The fifth result is from seven teams of four horses each team, upon strong stony land (old Hertfordshire ploughs)	= 193
The sixth result is from seven teams of two horses each team, upon light loam.....	= 177
The seventh result is from five teams of two horses each, upon light, sandy land.....	= 170
The eighth result is from seven teams of two horses each team, upon sandy land.....	= 160

"The mean force exerted by each horse from fifty-two teams, or one hundred and forty-four horses, equals 163 pounds each horse, and although the speed was not particularly entered, it could not be less than the rate of two miles and a half an hour.

"As these experiments were fairly made, and by horses of the common breed used by farmers, and upon ploughs of various counties, these numbers may be considered as a pretty accurate measure of the force actually exerted by horses at plough, and which they are able to do without injury for many weeks; but it should be remembered, that if these horses had been put out of their usual walking pace, the result would have been very different. The mean power of the draught-horse, deduced from the above-mentioned experiments, exceeds the calculated power from the highest formula of

Mr. *Leslie*, which is as follows: $(15 - v)^2$ = lbs. avoirdupois for the traction of a strong horse, and $(12 - v)^2$ = lbs. traction of the ordinary horse, v = velocity in miles per hour."

TABLE I.—*Showing the maximum quantity of labour a Horse of average strength is capable of performing, at different velocities, on Canals, Railways, and Turnpike-roads.* (*From Tredgold.*)

Velocity in miles per hour.	Duration of the day's work at the preceding velocity.	Force of traction in lbs.	Useful effect of one horse working one day, in tons drawn one mile.		
			On a canal.	On a level railway.	On a good level turnpike-road.
miles.	hours.	lbs.	tons.	tons.	tons.
$2\frac{1}{2}$	$11\frac{1}{2}$	$83\frac{1}{3}$	520	115	14
3	8	$83\frac{1}{3}$	243	92	12
$3\frac{1}{2}$	$5\frac{9}{10}$	$83\frac{1}{3}$	153	82	10
4	$4\frac{1}{2}$	$83\frac{1}{3}$	102	72	9
5	$2\frac{9}{10}$	$83\frac{1}{3}$	52	57	7·2
6	2	$83\frac{1}{3}$	30	48	6·0
7	$1\frac{1}{2}$	$83\frac{1}{3}$	19	41	5·1
8	$1\frac{1}{8}$	$83\frac{1}{3}$	12·8	36	4·5
9	$0\frac{9}{10}$	$83\frac{1}{3}$	9·0	32	4·0
10	$0\frac{3}{4}$	$83\frac{1}{3}$	6·6	28·8	3·6

Where horse power is employed for the higher velocities, the animals ought to be allowed to acquire the speed as gradually as possible at the first starting. This simple expedient will save the proprietors of horses much more than they are aware of; and it deserves their attention to consider the best mode of feeding and training horses for performing the work with the least injury to their animal powers.

To compare the preceding table with practice at the higher velocities, it will be necessary to have the total mass moved, which is one-third more than the useful effect in this table. Now, the actual rate at which some of the quick coaches travel, is 10 miles an hour; the stages average about 9 miles; and a coach with its load of luggage and passengers amounts to about 3 tons; therefore the average day's work of 4 coach horses is 27 tons drawn one mile, or $6\frac{3}{4}$ tons drawn one mile by one horse. The table gives 3·6 tons, added $\frac{1}{3}$ of 3·6 = 4·8 tons drawn one mile for the extreme quantity of labour for a horse at that speed, upon a good level road; from which should be deducted the loss of effect in ascending hills, heavy roads, &c., which will make the actual labour performed by a coach-horse average about double the maximum given by the table. The consequences are well known.

According to Mr. *Bevan's* observations, the horses on the Grand

Junction Canal draw 617 tons one mile, at the velocity of 2·45 miles per hour.

According to Mr. *Tredgold*, if V be the maximum velocity of a horse, and v any other velocity, the immediate power of a horse is $250\,v\left(1 - \frac{v}{V}\right)$; and, when the weight of the vessel or carriage is to the weight of the load, as n: 1, we have $\frac{250\,v\left(1 - \frac{v}{V}\right)}{1+n} =$ the effective power; and d being the hours the horse works in one day, the day's work will be $\frac{250\,d\,v\left(1 - \frac{v}{V}\right)}{1+n}$ in lbs. raised 1 mile, and $250\left(1 - \frac{v}{V}\right) =$ the force of traction in lbs. But if the force were immediately applied, the value of V would be $\frac{14{\cdot}7}{\sqrt{d}}$; and to find the value when the waggons alone are moved, we have $1 : \frac{1}{\sqrt{1+n}} :: \frac{14{\cdot}7}{\sqrt{d}} : \frac{14{\cdot}7}{\sqrt{d\,(1+n)}} = V$; whence the day's work is $\frac{250\,d\,v}{1+n}\left(1 - \frac{v\sqrt{d\,(1+n)}}{14{\cdot}7}\right)$; which is a maximum when $\frac{96}{v^2\,(1+n)} = d$. Consequently, when the velocity is given, we have $\frac{96}{v^2(1+n)}$ equal the duration of the day's work in hours; and $\frac{8000}{v\,(1+n)^2} =$ the effective day's work; and $250\left(1 - \frac{9{\cdot}8}{14{\cdot}7}\right) = 83\frac{1}{3}$ lbs. But we may assume n to be always so near $\frac{1}{3}$, as not to affect the result; and then, $\frac{72}{v^2} = d$, and $\frac{4500}{v} =$ the day's work in lbs. or very nearly $\frac{2}{v}$ tons raised one mile. This, being combined with the numbers of the preceding table, gives the effect of a horse on canals, railroads, and turnpike-roads.

It must, however, be here added, that although the deductions from Mr. *Tredgold's* valuable tables, as to the effects on *canals*, are tolerably accurate up to rates of 4 or 5 miles per hour, yet, when boats are moved on canals at rates of from 9 to 12 or 14 miles per hour, the circumstances of the resistances undergo an essential change. The resistance, in fact, becomes so small, that passage-

boats now travel at these high velocities; and it is hence probable that railroads and canals will admit of a competition such as the supporters of railroads never anticipated.

I shall here briefly detail some of the facts, as they have been given in a letter widely circulated by Mr. *W. Grahame* of Glasgow, in the *Nautical Magazine*, and other places.

From the traffic by canal boats, which has been actually going on during the last two years and a half, on the Paisley canal, we learn this remarkable fact, that, while a speed of ten miles per hour has been maintained by the canal boats, the banks have sustained no injury whatever. The cause of injury, in truth, has been entirely suppressed by the velocity of the boat, which passes along the water without raising a ripple.

About two years ago, measures were adopted for increasing the speed of the boats on the Paisley or Ardrossan canal. This canal is by no means favourable to such experiments, being both serpentine in its course, and narrow: it connects the town of Paisley with the city of Glasgow, and the village of Johnstone; the distance being about twelve miles. The boats employed on this canal are 70 feet in length, and 5·6 broad, and carry, if necessary, upwards of 120 passengers. They are formed of light iron plates, and ribs covered with wood, and light oiled cloth, at a whole cost of about 125*l.* They perform stages of four miles in an interval of time varying from 22 to 25 minutes, including all stoppages, and the horses run three or four of these stages alternately every day. The passengers are under cover, or not, as they please, no difference being made in this particular; and the fare is one penny per mile in the first, and three farthings per mile in the second cabin.

The horses drawing the canal boat are guided by a boy, who rides one of them; and, in passing under bridges at night, a light is shown in the bow of the boat, by which he sees his way, and which light is closed when the bridge is passed. Intermediate passengers are also accommodated to distances even as small as a mile; and the facility with which the boat stops, when relieved from the drawing force, is such as avoids all danger whatever. The expense of conveying a load of eight tons *at a rate of nine or ten miles per hour*, including all outlay, interest, and replacement of capital, is not more than 11*d.* per mile. It is also ascertained that one ton weight may be carried on a canal at nearly the same speed as on the railway, at about 1¼*d.* per mile, including an allowance for interest and replacement of capital.

It is also believed, that if the breadth and curvature of the Paisley canal admitted boats of 90 feet length, instead of 70, they would carry more passengers by one-half without an additional expense, and a decrease of labour to the horses.

The foregoing has been deduced from calculations founded on the observation of facts relating to the wear and tear of boats and horses, and the absolute resistance which these boats meet with in passing through the water. On this subject it has been observed,

that, in addition to the common resistance of the water to the motion of the boat, a wave, or body of water, is also raised before it, varying in its height according to the velocity of the boat, and constantly presenting an obstacle to her progress, providing that she only moves through the water at a certain slow rate. The height of this wave will then amount to nearly two feet, often overflowing the banks of the canal, and, from the obstruction it occasions, eventually obliging the boat to be stopped. Now, if, instead of stopping the boat when this wave is raised, her velocity be increased beyond what it had then been, she advances and passes over it, and leaves it to subside in her wake, which it does, and the water becomes perfectly still. The same horses, drawing the boat at this increased speed, are found to perform their work better, the resistance to their progress having become less; and the more the velocity of the boat is thus increased, the less resistance she meets with, merely having to cut the still water instead of the wave. It is a curious fact, that the wave produced by the approach of a slow canal boat is often observed at the distance of a mile, and upwards, along the canal, before the arrival there of the boat. But, in the case of the high wave being raised by the Paisley canal boat, it is customary to stop the boat, and after it has subsided to start again at a greater velocity. When the boat is to be stopped for any purpose, as her speed decreases the wave rises in proportion, and washes over the banks, until the motion of the boat becomes so small as to produce none. The discovery is doubtless a very important one, and, if turned to account, is likely to produce a material alteration in the rate of transport on canals. It was not known until these experiments were made, that if a boat, from a state of rest, was dragged along a canal, in proportion as her speed increased to a certain limit, that the power required was greater: but that, if she were started at, and preserved a speed exceeding the same limit, the power required would be less, and would decrease as her velocity increased. In fact, from a certain velocity there seems to be no limit to the rate at which a boat, as far as animal power can be applied, may thus pass through the water; and as the rate increases the power required decreases. On this principle it is that the boats on the Paisley canal, with ninety passengers in them, are drawn by horses at a speed of *ten miles an hour;* while it would kill them to draw the same boat along the canal at *six miles an hour*. A boat might indeed travel fifteen or twenty miles an hour easier than at six miles. The former of these velocities has already been attained by Mr. *Grahame*, along a distance of two miles, and is considered by him safer both for the boat and the canal.

As a proof of what may be done by this method of carriage, Mr. *Grahame* states that he has performed a voyage of fifty-six miles along two canals in six hours and thirty-eight minutes, which included the descent of five, and the ascent of eleven locks, the passage of eighteen drawbridges where the tracking-line was thrown off, and sixty common bridges, besides a tunnel half a mile long; all of course producing some delay. The boat which performed this

was sixty-nine feet long, and nine broad, drawn by two horses, and carried thirty-three passengers, with their luggage and attendants.

These facts furnish great encouragement to canal companies, to improve the construction of their boats and the speed on their canals; and thus, probably, in some situations, supersede the necessity of railroads.

Mr. Macneill, the assistant engineer upon the Holyhead road, under Mr. Telford, in the course of his examination before a committee of the House of Commons, on steam-carriages, railroads, &c. gave the subjoined curious information.

Well-made roads, formed of clean hard broken stone, placed on a solid foundation, are little affected by changes of atmosphere; but weak roads, or such as are imperfectly formed with gravel, flint, or round pebbles, without a bottoming of stone, pavement, or concrete, are much affected.

On the generality of roads, the proportional injury from the weather and traffic is nearly as follows. When travelled by fast coaches: from atmospheric changes 20; coach-wheels 20; horses' feet 60 = 100. When travelled by waggons: atmospheric changes 20; waggon-wheels 35·5; horses' feet 44·5 = 100. Has ascertained, from a number of observations, that the wear of the iron tire of fast-going coach-wheels is, compared with that of the shoes of the horses which draw them, as 326·8 to 1000, or as 1 to 3-4ths nearly; and infers that the comparative injury done by them to roads is nearly in the same proportion. In the case of slow-going carriages and horses the proportion is as 309 to 360, or as 1 to 1·16, or nearly 1 to $1\frac{1}{4}$. The tire of the wheels of the fast-going coaches last from two to three months, according to the weather, the workmanship, and quality of iron; about 20 years ago, it did not last seven days on an average. Coach-horse shoes remain in use about thirty days; waggon-horse shoes about five weeks on an average. Where roads are weak, and yield under pressure, the injury caused by heavy wheels is far greater than on solid firm roads.

It was found, in one instance, that the wear of hard stone, placed on a wet clay bottom, was four inches, while it was not more than half an inch when placed on a solid dry foundation. On the Highgate archway road, the annual wear is not more than half an inch in depth. To the same gentleman we owe the following useful table.

TABLE II.—*The general Result of Experiments made with a Stage Coach, weighing, exclusive of seven passengers,* 18 *cwt., on the same piece of road, on different inclinations, and at different rates of velocity, furnish the following statement.*

Rate of Inclination.	Rates of Travelling.	Force required.
1 in 20	6 miles per hour	268 lbs.
1 in 26	6	213
1 in 30	6	165

Rate of Inclination.	Rates of Travelling.	Force required.
1 in 40	6 miles per hour	160 lbs.
1 in 600	6	111
1 in 20	8	296
1 in 26	8	219
1 in 30	8	196
1 in 40	8	166
1 in 600	8	120
1 in 20	10	318
1 in 26	10	225
1 in 30	10	200
1 in 40	10	172
1 in 600	10	128

CHAP. VIII.

STRENGTH OF MATERIALS.

SECT. I. *Results of Experiments, and Principles upon which they should be practically applied.*

BY most writers on the strength of materials, it has been customary to start with the assumption of the three under-mentioned principles, and upon them to construct a theory by means of which they afterwards deduced from experiments on rectangular bars the strength of beams of the same material, but of various forms of transverse section. Upon, however, testing these deductions by experiment, they are found in many cases to be entirely erroneous, and to such an extent as to render their practical application not only uncertain, but frequently highly dangerous. Such being the result of experience, it becomes important to examine these principles more closely, in order to discover wherein the cause of this erroneous result consists.

The principles which we allude to are (in the language of Tredgold*) as follows:—

"The first is, that the strength of a bar or rod to resist a given strain, when drawn in the direction of its length, is directly proportional to the area of its cross section; *while its elastic power remains perfect*, and the direction of the force coincides with the axis.

"The second is, that the extension of a bar or rod by a force acting in the direction of its length is directly proportional to the straining force, when the area of the section is the same; *while the strain does not exceed the elastic power*.

"The third is, that *while the force is within the elastic power of the material*, bodies resist extension and compression with equal forces."

It should be remarked, that each of the foregoing propositions is only asserted to hold good "while the strain does not exceed the elastic power of the material." By which is meant that the force applied shall not be sufficient to cause any permanent displacement of the particles, and that when the straining force is removed, the body will not have acquired any set, but will reassume its former shape

* Practical Treatise on the strength of cast iron. Fourth edition, p. 122.

ıs. Tredgold and most other writers upon this subject
d that no permanent displacement or set takes place
ining force amounts to about one-third of the ultimate
he material, and have consequently assumed that these
ght be applied without error to determine the strength
ıs of material, of any form, and subject to any kind

more careful experiments have, however, shown conclu-
permanent displacement or set takes place with a much
ı than one-third of the ultimate strength, and that in
able set is produced by a straining force very much less
which the material will be likely to be exposed in
ch being the case, it is obvious that these principles (as
ove) are inapplicable to any practical case. As, how-
ıve been disposed to imagine that such a strict limitation
ication is not necessary, and that they practically hold
hen the force applied *exceeds* the elastic power of the
ill be desirable to show that not only is such a supposi-
ırroneous, but that the principles themselves, even with
ı, so far from being demonstrable, are, on the contrary,
loubtful.

ıertion, that the strength of a rod or bar subjected to
n is directly proportional to the area of its cross section,
irst sight it may appear obvious enough, nevertheless is
practice to be strictly correct; for, if the material is of
is found that the strength of the exterior hard crust is
hat of the interior substance, and therefore that the
hich the perimeter of the section bears to its area is a
requiring to be taken into consideration; not to men-
ırrangement of the particles during the cooling of the
ıpon which arrangement the strength of the substance so
ɔends, is greatly influenced by the external form of its
l on the other hand, in the case of a fibrous material, as
ds or timber, in consequence of the fibres not laying
llel to each other, in cutting the bar to the required
ıany of the exterior fibres will be cut transversely, and
ıfore be capable of affording so great a proportionate
ıe similar fibres within the more central portion of the
ich it follows, that the tensile strength of a bar of any
t directly proportional to its sectional area, although the
ı trifling as not to require attention in most of the cases
; themselves in practice.

l law, that the extension of a bar or rod exposed to a
is directly proportional to that strain *so long as the area remains unaltered*, is from the very circumstances of
ıible, since it is not capable for a bar to become elon-
having its transverse section proportionately diminished,
we cannot conceive the bar in two successive degrees
having in both the same sectional area. And although

the correct law subsisting between the elongation and the force producing it, has not yet been satisfactorily determined, there is every reason to believe that they are not directly proportional to each other.

With regard to the third law, which may be more definitely stated as follows, viz. :—that within the elastic power of the material, bodies require as great a force to alter their dimensions to any given extent by compression as by extension; we may remark, that any property depending upon the straining force being within the elastic power of the material is incapable of any practical application, and therefore that this law, as limited above, even assuming it to be correct, does not require to be included in any practical inquiry into the laws which regulate the strength of materials. And without such limitation the law is at utter variance with the results of experiment, since no fact is now better ascertained than that cast iron requires a much greater force to compress it to any given extent, than to extend it to the same extent.

Such then being the result of our investigation into the accordance of these principles with practical experience, we can no longer be surprised that the deductions drawn from them by Tredgold, and others, should be equally at variance with the same experience. Indeed the errors into which they have fallen appear to have been caused by reasoning upon hypothetical principles—attempting to form a complete theory of the strength of materials upon data neither sufficient in quantity nor certainty. And even now, although much has been done, since the time when Tredgold wrote, by Hodgkinson, Fairbairn, Thomas Cubitt, Barlow, and some others, this insufficiency has not yet been supplied, and we are still in ignorance upon many points, without a knowledge of which it would be impossible to form any complete theory of the strength of materials sufficient in its extent for all practical purposes, and of whose accuracy we could be so far sure as to allow of its application without fear of erroneous results.

In order to show in what points the deficiency in our knowledge upon this subject consists, and to explain the general principles upon which the strength of materials actually depends, we will briefly investigate the subject.

If A B C D, fig. 230, be a rectangular beam, subjected to three pressures P_1, P_2, and P_3, the two former of which are equal to each other, and to half P_3, and if P_1 and P_2 be applied at the two extremities of the beam, and P_3 at its center in an opposite direction, the beam will be deflected, the upper surface A C becoming convex, and the lower surface B D concave. Then, if the material composing the beam be supposed capable both of extension and compression, the fibres on the convex side will be extended, and those on the concave side compressed; there will, however, be a certain curved superficies E F within the beam (called the *neutral* superficies), the fibres situate in which will not be either extended or compressed.

Now, if tangents E G, F H, to this curved neutral superficies at the

extremities of the beam be drawn, and from them perpendiculars K L, M N, passing through the point I, in which the neutral superficies intersects the plane of rupture O Q; then will the triangle N I L be equal to the sum of the extensions of the whole length of all the extended fibres; and the similar triangle K I M will be equal to the sum of the compressions of the whole length of all the compressed fibres; and if any superficies be taken between A C or B D and the neutral superficies, the portions of such superficies, as $\iota_1 \iota_2$, $\gamma_1 \gamma_2$, contained between K L, and M N, will be equal to the extension or compression of the whole length of the fibres situate in that superficies: from whence it follows that the amount of extension or compression of any fibre is directly proportional to its distance from I, the neutral axis.

If now any number of points be taken in the depth of the beam, as ι_1, ι_2, ι_3, γ_1, γ_2, γ_3, &c. (fig. 231), and lines be drawn through them perpendicular to O Q, each equal in length to the force required to extend or compress the fibres to the extent to which such fibres are actually extended or compressed, and which extent will be proportional to their distances from the neutral axis, and a curved superficies $a_1 b_1 I_1 c_1 d_1 d_2 c_2 I_2 b_2 a_2$ be drawn through the ends of all these lines, then will the solid contained between the plane of rupture $O O_1 Q_1 Q$ and the curved superficies, be equal to the resistance of all the extended and compressed fibres, or to the transverse strength of the beam.

Now, if E equals the solidity of $a_1 b_1 I_1 I_2 b_2 a_2$, and C equals the solidity of $I_1 c_1 d_1 d_2 c_2 I_2$, also δ and Δ equal the perpendicular distances of their center of gravity from the neutral superficies, and $\frac{1}{2}$ L the distance of the points where the pressures P_1 and P_2 are applied, from the neutral axis; we have from the principle of the equality of moments,

$$\tfrac{1}{2} L P_1 = E \delta + C \Delta;$$

or, since E δ must be equal to C Δ, and P_1 equals $\frac{1}{2} P_3$, we have

$$\frac{1}{8} L P_3 = E \delta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(I.)}$$

and this formula will apply, whatever form the transverse section of the beam may be, whether rectangular as fig. 231, cylindrical as fig. 232, or ⊥ shaped as fig. 233.

It is further evident that, when the beam is upon the point of breaking, the upper fibres at O must be strained to the utmost extent to which they can be, before yielding, and that as this extent does not depend upon the form of the beam, but only upon the elastic properties of the material, for the same material the length $a_1 O$ will remain constant in all cases. From this it follows that the lines ι_1, ι_2, ι_3, γ_1, γ_2, γ_3, &c. (fig. 231), which are the ordinates of the curve $a_1 b_1 I_1 c_1 d_1$, will also be constant, but that their distances from I, or the abscissæ of the curve will vary directly as the distance O I of the neutral axis from the upper surface of the beam.

In rectangular beams, or others, whose transverse sections are similar figures, the distance O I, and therefore the area of the curve

$O a_1 b_1 I_1$, will vary as the depth of the beam, and the solidity of the figure $O a_1 b_1 I_1 I_2 b_2 a_2 O$ will vary as the depth and the breadth, or $E \propto b d$; also, since the forms of the beams are similar, the distance δ of the center of gravity of the solid E from the neutral superficies, must vary as the depth of the beam; and therefore, in rectangular beams, the ultimate strength of the beam varies directly as the square of the depth multiplied by the breadth, and inversely as the length; or, putting $S (= P_1 + P_2 = P_3)$ for the breaking weight applied in the center, we have

$$S \propto \frac{d^2 b}{L} \quad \text{..... (II.)}$$

or, when the transverse sections are similar figures, d_1 being any similar dimensions in both,

$$S \propto \frac{d_1^3}{L} \quad \text{..... (III.)}$$

From the preceding investigation it is evident that we cannot deduce any general formulæ by which to determine the transverse strength of a beam of any given form of section, without knowing the position of the neutral axis, the limit to which the fibres may be extended before yielding, and the law which subsists between the extension and compression of the material and the force required to produce such extension and compression; upon none of which subjects have we any certain knowledge.

SECT. II. *Strength of Materials to resist tensile and crushing strains. Strength of Columns.*

The first principle alluded to in the foregoing section, viz., that the strength of a bar or rod to resist a tensile strain in the direction of its axis, is proportional to the area of its cross section, is not, for the reasons there mentioned, strictly correct. And although perhaps for ordinary practical purposes, or where the sections to be compared are large, no error of importance would result from the adoption of this principle, it is very necessary, in experiments whose object is the determination of the actual strength of any given material, that due regard should be had to all circumstances, (however trivial they may seem,) which may in any way affect the accuracy of the results.

The only late experiments upon the tensile strength of cast iron, now before the public, are those of Mr. Eaton Hodgkinson, first published in a paper communicated to the British Association, and subsequently in his "Experimental Researches." The mean tensile strength of a square inch of cast iron, as deduced from these experiments, is only 16,560 lbs., or 7 tons 7·85 cwt.; the lowest being 12,993 lbs., and the highest 21,907 lbs. In a series of experiments,

however, which have been tried by Mr. Thomas Cubitt*, during a period of several years, we find the strength of cast iron to be considerably higher than Mr. Hodgkinson's estimate: the average of twenty-five specimens, of which eight were unsound, gave 27,140 lbs. for the tensile strength per square inch section; and, excluding the unsound pieces, the average of the others gave 27,773 lbs., or 12 tons 8 cwt. The highest broke with 32,997 lbs. per square inch, and the *lowest*, even of the unsound pieces, broke with 21,471 lbs., or only a little under Mr. Hodgkinson's *best specimen*.

The anomaly between these results is very remarkable, since in both cases the experiments were conducted with the greatest care; we think, however, that Mr. Cubitt's results are entitled to the preference, for this reason, that while any cause might easily operate to *lessen* the strength of the iron, we cannot conceive it possible that any want of care in the experiments or bad arrangement of the apparatus could cause the iron to bear more than it otherwise would have done, providing that the means of measuring the force exerted in producing fracture were correct; and these, we do not hesitate to say, were not open to any exceptions. It is certainly difficult to account for so wide a difference in the results of these experiments, but we cannot but think that the form chosen by Mr. Hodgkinson, for the transverse section of the bars upon which he experimented, is open to objection. The form adopted by him is shown in fig. 234, being that of a cross; and the reason which induced him to choose this form, was "to obviate the objection made by Mr. Tredgold (Art. 79 and 80,) and others, against the conclusions of former experimenters." The objection of Mr. Tredgold here alluded to was, that if the strain did not pass through the center of gravity of the section in the place where the rod broke, the different parts of that section would be exposed to a different strain, and therefore, that the results obtained would be lower than the truth. We quite agree with Mr. Tredgold upon the importance of this objection, but we cannot but consider, that the form chosen by Mr. Hodgkinson is more open to this objection than the square or cylindrical form would have been. For, whatever might be the form of section of the rod, the same means of fixing the ends of it might be used, and since it depends entirely upon the accuracy with which that is done whether the axis of the rod coincides with the straight line between the points of attachment, (which is the direction of the straining force), we may suppose that the lateral distance between these two lines at the breaking section would be the same, whatever was the form of that section; such

* I am much indebted to Mr. Thomas Cubitt, for the liberal manner in which he has placed at my disposal the results of his very careful and elaborate experiments upon this subject. His experiments on the set, deflection, and ultimate strength of cast iron girders, extend over a period of many years, and possess the immense practical advantage of having been made upon large girders. These experiments, amounting in number to upwards of 2000, have been taken under the direction of Mr. Dines; and, having witnessed some of them myself, I can bear testimony to the care which has been taken to ensure accurate results, and to remove all imaginable causes of error.—H. L.

being the case then, we hold that the strength of the cruciform section would be more affected than the square, since in the former the metal is situated further from what would then be the neutral axis of the section, and exposed therefore to greater differences of strain. This form is also open to the practical objection that it is very difficult to obtain a perfectly sound casting from it; it usually happens that the iron is not perfect at the point of intersection, (*a*, fig. 234,) and also to our remarks at p. 371, that the proportion of the periphery to the area may influence the strength.

Mr. Cubitt has also tried a few experiments upon wrought iron, which gave for the average tensile strength per square inch, 58,952 lbs., or 26 tons 6·3 cwt.

The metals differ more widely from each other, in their elastic force and cohesive strength, than the several species of wood or vegetable fibres. Thus, the cohesion of fine steel is about 135,000 lbs. for the square inch, while that of cast lead amounts only to about the hundred and thirtieth part, or 1800 lbs.

According to the experiments of Mr. George Rennie, in 1817, the cohesive power of a rod an inch square of different metals, in pounds avoirdupois, with the corresponding length of the modulus of elasticity in feet, is as follows:—

	Pounds.		Feet.
Cast steel	134,256	...	39,455
Swedish malleable iron ...	72,064	...	19,740
English ditto	55,872	...	16,938
Cast iron.....	19,096	...	6,110
Cast copper	19,072	...	5,003
Yellow brass	17,958	...	5,180
Cast tin	4,736	...	1,496
Cast lead.....	1,824	...	348

It thus appears, as Professor *Leslie* remarks, that a vertical rod of lead 348 feet long would be rent asunder by its own weight. The best steel has nearly twice the strength of English soft iron, and this again is about three times stronger than cast iron. Copper and brass have almost the same cohesion as cast iron. This tenacity is sometimes considerably augmented by hammering or wire-drawing, that of copper being thus nearly doubled, and that of lead, according to Eytelwein, more than quadrupled. The consolidation is produced chiefly at the surface, and hence a slight notch with a file will materially weaken a hard metallic rod.

Professor Leslie has given the following tabular view of the absolute cohesion of the principal kinds of timber, or the load which would rend a prism of an inch square of each; and the altitude of the prism which would be severed by the action of its own weight.

	Pounds.		Feet.
Teak.....	12,915	...	36,049
Oak	11,880	...	32,900
Sycamore	9,630	...	35,800
Beech	12,225	...	38,940

	Pounds.		Feet.
Ash	14,130	...	42,080
Elm	9,720	...	39,050
Memel fir	9,540	...	40,500
Christiana deal	12,346	...	55,500
Larch	12,240	...	42,160

The following is the result of Professor Barlow's experiments upon the cohesive strength of various kinds of timber.

	Pounds per square inch.
Fir	12,203
Ash	17,077
Beech	11,467
Oak	10,389
Teak	15,090
Box	19,891
Pear	9,822
Mahogany	8,041

The cohesion of hempen fibres is, for every square inch of their transverse section, 6400 lbs. The best mode of estimating the strength of a rope of hemp is to multiply by 200 the square of its number of inches in girth, and the product will express in pounds the practical strain it may be safely loaded with; for cables, multiply by 120 instead of 200. The ultimate strain is probably double this, as will appear from the account following of Du Hamel's experiments. If yarns of 180 yards long be worked up into a rope of only 120 yards, it will lose one-fourth of its strength, the exterior fibres alone resisting the greatest part of the strain. The register cordage of the late Captain Huddart exerts nearly the whole force of the strands, since they suffer a contraction of only the eighth part in the process of combining.

For the *utmost* strength that a rope will bear before it breaks, a good estimate will be found by taking *one-fifth of the square of the girth of the rope,* to express the tons it will carry. This is about double the rule for practice which we have given above; and is, even for an ulterior measure, too great for tarred cordage, which is always weaker than white.

The following experiments were made by Mons. Du Hamel, at Rochfort, on cordage of three inches (French) circumference, made of the best Riga hemp, August 8th, 1741.

	White.		Tarred.
Broke with a strain of	4500 pounds	...	3400 pounds.
	4000 „	...	3300 „
	4800 „	...	3258 „
	August 25th, 1743.		
	4600 „	...	3500 „
	5000 „	...	3400 „
	5000 „	...	3400 „

September 23, 1746.

	White.		Tarred.	
Broke with a strain of	3880	pounds ...	3000	pounds.
	4000	„ ...	2700	„
	4200	„ ...	2800	„

A parcel of white and tarred cordage was taken out of a quantity which had been made February 12, 1746.

It was laid up in the Magazine, and comparisons were made from time to time, as follows:—

	White.		Tarred.		Difference.	
1746, April 14th,	2645	pounds	2312	pounds	333	pounds.
1747, May 18th,	2762	„	2155	„	607	„
1747, October 21st,	2710	„	2050	„	660	„
1748, June 19th,	2575	„	1752	„	823	„
1748, October 2nd,	2425	„	1837	„	588	„
1749, Sept. 25th,	2917	„	1865	„	1052	„

M. Du Hamel says, that it is decided by experience, that white cordage in continued service is one-third more durable than tarred; secondly, it retains its force much longer while kept in store; thirdly, it resists the ordinary injuries of the weather one-fourth longer. These observations deserve the attention of practical men.

Mr. *B. Bevan* has favoured the author with a tabular view of his results with regard to the *modulus of cohesion*, or the length in feet of any prismatic substance required to break its cohesion, or tear it asunder.

Bevan's Results.

	Feet.
Tanned cow's skin	10,250
Tanned calf skin	5,050
Tanned horse skin	7,000
Tanned cordovan	3,720
Tanned sheep skin	5,600
Untanned horse skin	8,900
Old harness of thirty years	5,000
Hempen twine	75,000
Catgut, some years old	23,000
Garden matting	27,000
Writing-paper, foolscap	8,000
Brown wrapping-paper, thin	6,700
Bent grass, (holcus)	79,000
Whalebone	14,000
Bricks, (Fenny Stratford)	970
Bricks, (Leighton)	144
Ice	300
Leicestershire slate	7,300

STRENGTH OF COLUMNS.

Upon the subject of the strength of columns, we are entirely indebted to Mr. Hodgkinson, who, after a very careful and accurate series of experiments, has deduced formulæ of immense practical value, and has clearly shown the principles upon which the strength of materials, submit to a compressing force, depends.

The results of his experiments are briefly as follows:—That when weights are applied to prisms, or cylinders, the shorter ones usually bore more than the longer ones, and that the strength may be considered to vary as the area. That, when the diameter or least lateral dimension is less than the height of the prism, fracture takes place, by the separation of either a pyramid, cone, or wedge, (depending upon the form of the prism,) the angle of whose sides is constant for the same material, and about 34° for cast iron. And further, that the mode of fracture and the ultimate strength of a prism varies but slightly with a variation in its height, so long as it is not less than its diameter or least lateral dimension, or greater than four or five times the same dimension; when higher, it begins to bend, and its strength decreases.

Mr. Hodgkinson finds that the strength of a column is very much influenced by the manner in which the ends are fixed; when both ends are rounded, so that the column may turn upon them as a center, its strength is only one-third of that of another column of precisely similar dimensions, but whose ends are flat and incapable of motion; and that, if one end is flat and immovable while the other is rounded, the strength is two-thirds of what it would have been had both ends been flat.

The following table exhibits the results of his experiments on solid and hollow cylindrical columns of cast iron, in which S is the breaking weight in tons; D the external diameter, d the internal diameter, both in inches; and l the length or height of the column in feet.

Kind of Column.	With both ends round, when the height of the column is not less than 15 times its diameter.	With both ends flat, when the height of the column is not less than 30 times its diameter.
Solid cylindrical cast iron columns	$S = 14{\cdot}9 \frac{D^{3{\cdot}6}}{l^{1{\cdot}7}}$.	$S = 44{\cdot}16 \frac{D^{3{\cdot}6}}{l^{1{\cdot}7}}$.
Hollow ditto ditto ...	$S = 13 \frac{D^{3{\cdot}6} - d^{3{\cdot}6}}{l^{1{\cdot}7}}$.	$S = 44{\cdot}3 \frac{D^{3{\cdot}6} - d^{3{\cdot}6}}{l^{1{\cdot}7}}$.

When the height of the column is less than that mentioned in this table, namely, fifteen times its diameter for columns with both ends rounded, and thirty times the same with both ends flat, the strength of the column becomes modified in consequence of its being then partially crushed as well as bent. If C equal the force which would be required to crush the column without flexure, S the strength as

calculated by the above formulæ for long columns, and S_1 the actual strength, we have

$$S_1 = \frac{Sc}{S + \frac{1}{4}c} \quad \text{..... (IV.)}$$

The value of c, or the weight required to crush a square inch of cast iron, as deduced from his experiments, is 107,750 lbs., or 48 tons 2 cwt.; therefore, the crushing force is to the tensile, according to Mr. Hodgkinson, as 6·507 is to 1; or, taking Mr. Hodgkinson's value for the crushing strength, and Mr. Cubitt's for the tensile, the ratio becomes as 3·88 is to 1.

The strength of a column of cast iron of given dimensions being 1000, the strength of a column of wrought iron of the same dimensions would be 1745, of cast steel 2518, of Dantzic oak 108·8, and of red deal 78·5.

The weights required to crush cubes of the quarter of an inch of certain metals, according to the experiments of Mr. Rennie, are as follows:—

	Pounds.
Iron cast vertically	11,140
Iron cast horizontally	10,110
Cast copper	7,318
Cast tin	966
Cast lead	483

Cubes of an inch are crushed by the weights annexed:—

	Pounds.
Elm	1,284
White deal	1,928
English oak	3,860
Craigleith freestone	8,688

Cubes of an inch and a half, and consequently presenting a section of two and a quarter times greater than the former, might be expected to resist compression in that ratio. They are crushed, however, with loads considerably less.

	Pounds.
Red brick	1,817
Yellow baked brick	2,254
Fire brick	3,864
Craigleith stone, direction of the strata	15,560
Ditto across the strata	12,346
White statuary marble	13,632
White-veined Italian marble	21,783
Purbeck limestone	20,610
Cornish granite	14,302
Peterhead granite	18,636
Aberdeen blue granite	24,536

These facts show the comparative firmness of different materials; but it is to be regretted that such results are not of much practical value, since they are confined to a very narrow scale, and applicable

only to cubical blocks. While the breadth remains the same, the resistance appears to depend on some unascertained ratio of the altitude of the column.

SECT. III. *Elasticity and Elongation of Bodies subjected to a crushing or tensile strain.*

The *modulus of the elasticity* of any substance is a column of the same substance, capable of producing a pressure on its base which is to the weight causing a certain degree of compression, as the length of the substance is to the diminution of its length.

The modulus of elasticity is the measure of the elastic force of any substance.

A practical notion of the *modulus of elasticity* may be readily obtained. Let ϵ be the quantity a bar of wood, iron, or other substance, an inch square and a foot in length would be extended or diminished by the force f; and let l be any other length of a bar of equal base and like substance; then

$$1 : l :: \epsilon : \Delta, \text{ or } l\epsilon = \Delta,$$

where Δ equals the extension or diminution in the length l.

The modulus of elasticity is found by this analogy: as the diminution of the length of any substance is to its length, so is the force that produced that diminution to the modulus of elasticity. Or, denoting the weight of the modulus in pounds for a base of an inch square by m, it is

$$\epsilon : f :: 1 : m = \frac{f}{\epsilon}.$$

And if w be the weight of a bar of the substance one inch square and one foot in length; then, if M be the height of the modulus of elasticity in feet, we have

$$\mathrm{M} = \frac{f}{w\epsilon} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(V.)}$$

The weight of the modulus of the elasticity of a column being m, a weight bending it in any manner f, the distance of the line of its application from any point of the axis D, and the depth of the column, d, the radius of curvature will be $\frac{d^2 m}{12\,\mathrm{D} f}$.

The distance of the point of greatest curvature of a prismatic beam, from the line of direction of the force, is twice the versed sine of that arc of the circle of greatest curvature, of which the extremity is parallel to that of the beam.

When the force is longitudinal, and the curvature inconsiderable, the form coincides with the harmonic curve, the curvature being proportional to the distance from the axis; and the distance of the point

of indifference from the axis becomes the secant of an arc proportional to the distance from the middle of the column.

If a beam is naturally of the form which a prismatic beam would acquire, if it were slightly bent by a longitudinal force, calling its depth d, its length l, the circumference of a circle of which the diameter is unity c, the weight of the modulus of elasticity m, the natural deviation from the rectilinear form Δ, and a force applied at the extremity of the axis f, the total deviation from the rectilinear form will be

$$\Delta' = \frac{d^2 c^2 \Delta' m}{d^2 c^2 m - 12 l^2 f} \quad \text{..... (VI.)}$$

It appears from this formula, that when the other quantities remain unaltered, Δ' varies in proportion to Δ, and if $\Delta = 0$, the beam cannot be retained in a state of inflection, while the denominator of the fraction remains a finite quantity; but when $d^2 c^2 m = 12 l^2 f$, Δ' becomes infinite, whatever may be the magnitude of Δ, and the force will overpower the beam, or will at least cause it to bend so much as to derange the operation of the forces concerned. In this case $f = \left(\frac{dc}{l}\right)^2 \cdot \frac{m}{12}$, $\cdot 8225 \frac{d^2}{l^2} m$, which is the force capable of holding the beam in equilibrium in any inconsiderable degree of curvature. Hence, the modulus being known for any substance, we may determine at once the weight which a given bar nearly straight is capable of supporting. For instance, in fir wood, supposing its height 10,000,000 feet, a bar an inch square and ten feet long may begin to bend with the weight of a bar of the same thickness, equal in length to $\cdot 8225 \times \frac{1}{120 \times 120} \times 10{,}000{,}000$ feet, or 571 feet; that is, with a weight of about 120 lbs.; neglecting the effect of the weight of the bar itself. In the same manner the strength of a bar of any other substance may be determined, either from direct experiments on its flexure, or from the sounds that it produces. If $f = \frac{m}{n}$, $\frac{l^2}{d^2} = \cdot 8225\, n$, and $\frac{l}{d} = \sqrt{(\cdot 8225\, n)} = \cdot 907 \sqrt{n}$; whence, if we know the force required to crush a bar or column, we may calculate what must be the proportion of its length to its depth, in order that it may begin to bend rather than be crushed.

When a longitudinal force is applied to the extremities of a straight prismatic beam, at the distance D from the axis, the deflection of the middle of the beam will be

$$D \cdot \left\{ \text{sec. arc} \left(\sqrt{\frac{3f}{m}} \cdot \frac{l}{d} \right) - 1 \right\} \quad \text{..... (VII.)}$$

If a column, subjected to a longitudinal force, be cut out of a plank or slab of equable depth, in order that the extension and compression of the surfaces may be initially every where equal, its outline must be a circular arc.

If a column be cut out of a plank of equable breadth, and the outline limiting its depth be composed of two triangles, joined at their bases, the tension of the surfaces produced by a longitudinal force will be every where equal, when the radius of curvature at the middle becomes equal to half the length of the column; and in this case the curve will be a cycloid.

When the curvature at the middle differs from that of the cycloid, the figure of the column becomes of more difficult investigation. It may, however, be delineated mechanically, making both the depth of the column and its radius of curvature proportional always to $\sqrt{a}$. If the breadth of the column vary in the same proportion as the depth, they must both be every where as the cube root of a, the ordinate. (*Young's Nat. Phil.* vol. ii.)

The modulus of elasticity has not yet been ascertained in reference to so many subjects as could be wished. Professor Leslie exhibits several, however, as below. That of white marble is 2,150,000 feet, or a weight of 2,520,000 pounds avoirdupois on the square inch; while that of Portland stone is only 1,570,000 feet, corresponding on the square inch to the weight of 1,530,000 pounds.

White marble and Portland stone are found to have, for every square inch of section, a cohesive power of 1811 lbs. and 857 lbs.; wherefore, suspended columns of these stones, of the altitude of 1542 and 945 feet, or only the 1394th and 1789th part of their respective measure of elasticity, would be torn asunder by their own weight.

Of the principal kinds of timber employed in building and carpentry, the annexed table will exhibit their respective Modulus of Elasticity, and the portion of it which limits their cohesion, or which lengthwise would tear them asunder.

	Feet.	
Teak	6,040,000	168th
Oak	4,150,000	144th
Sycamore	3,860,000	108th
Beech	4,180,000	107th
Ash	4,617,000	109th
Elm	5,680,000	146th
Memel fir	8,292,000	205th
Christiana deal	8,118,000	146th
Larch	5,096,000	121th

The following, also, exhibits Mr. *Bevan's* results as to the *modulus of elasticity*.

	Feet.
Platinum	2,390,000
Gold (pure)	1,390,000
Steel	9,300,000
Bar iron	9,000,000
Ditto	8,450,000
Yellow pine	9,150,000
Ditto	11,840,000

	Feet.
Finland deal	6,000,000
Mahogany	7,500,000
Rose wood	3,600,000
Oak, dry	5,100,000
Fir bottom, 25 years old	7,400,000
Petersburg deal	6,000,000
Lance wood	5,100,000
Willow	6,200,000
Oak	4,350,000
Satin wood	2,290,000
Lincolnshire bog oak	1,710,000
Lignum Vitæ	1,850,000
Teak wood	4,780,000
Yew	2,220,000
Whalebone	1,000,000
Cane	1,400,000
Glass tube	4,440,000
Ice	6,000,000
Limestone.	
„ Dinton, Buck	2,400,000
„ Ketton	1,600,000
„ Jetternoe	635,000
Ryegate	621,000
Yorkshire paving	1,320,000
Cork	3,300
Slate, Leicestershire	7,800,000

The following is the weight of the *modulus of elasticity* of various substances employed very generally in construction, according to Mr. Tredgold.

	Pounds.
Ash	1,640,000
Beech	1,345,000
Brass, cast	8,930,000
Elm	1,340,000
Fir, red or yellow	2,016,000
Fir, white	1,830,000
Iron, cast	18,400,000
Iron, malleable	24,920,000
Larch	10,740,000
Lead, cast	720,000
Mahogany	1,596,000
Oak, good English	1,700,000
Pine, yellow American	1,600,000
Steel	29,000,000
Tin, cast	4,608,000
Steel, cast	13,680,000

SECT. IV. *On the Strength of Materials subjected to a transverse strain.*

From the observations which we have made in the first section, it appears that we are not possessed of all the data which are required for determining generally the strength of a cast-iron beam of any given form of section; and consequently, that although rules have been given for that purpose, their results are in many cases, (as might be expected,) at utter variance with the actual strengths as obtained by experiment. Such, then, being the state of our knowledge upon the transverse strength of materials, we have abstained from giving any rules deduced *alone from theory*, being well assured that to have an *erroneous rule* is much worse than to be without *any rule at all.* Nevertheless, the strengths of a few forms having been determined by actual experiment, we are enabled to calculate with certainty, by means of formula (III.), page 374, the strength of a beam of any dimensions, whose form of section is *similar* to any of those forms, using the word similar here in its strict geometrical sense, as explained in definition 12, page 103.

1. For the rectangular form of section:—From the mean of 265 experiments of Messrs. Hodgkinson and Fairbairn, it appears that a weight of 454·4 lbs. applied at the center of a bar of cast iron 1 inch square and with a bearing of 4·5 feet produced fracture; therefore, for a bar of any other dimensions, putting W for the breaking weight applied in the center, L the length of the beam in feet, b and d its breadth and depth in inches, we have

$$\left.\begin{aligned} \frac{2045\, b d^2}{L} &= W, \text{ in lbs.} \\ \frac{18{\cdot}25\, b d^2}{L} &= W, \text{ in cwts.} \\ \frac{{\cdot}912\, b\, d^2}{L} &= W, \text{ in tons.} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \text{(VIII.)}$$

2. For Mr. Hodgkinson's form of section (fig. 235) in which the area (a) of the lower flange is made six times that of the upper, we have

$$\left.\begin{aligned} \frac{4852\, a\, d}{L} &= W, \text{ in lbs.} \\ \frac{43{\cdot}3\dot{3}\, a\, d}{L} &= W, \text{ in cwts.} \\ \frac{2{\cdot}166\, a\, d}{L} &= W, \text{ in tons.} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \text{(IX.)}$$

3. When the form of section does not materially differ from that shown in figure 235, the following formula is found by Mr. Hodgkinson to agree moderately well with the results of experiments. If b_1 equal the entire breadth of the bottom flange, b_2 the thickness of the vertical part, d_1 equal the whole depth of the girder, d_2 the depth without the lower flange, all in inches, and L equal the length in feet, we have

$$\left.\begin{aligned} \frac{1493}{d_1 L}\{b_1 d_1^3 - (b_1 - b_2) d_2^3\} &= W, \text{ in lbs.} \\ \frac{40}{3 d_1 L}\{b_1 d_1^3 - (b_1 - b_2) d_2^3\} &= W, \text{ in cwts.} \\ \frac{2}{3 d_1 L}\{b_1 d_1^3 - (b_1 - b_2) d_2^3\} &= W, \text{ in tons.} \end{aligned}\right\} \quad \ldots \text{ (X.)}$$

4. We have also been favoured with the following arbitrary formula by Mr. Dines, which he has found to be tolerably correct in all cases where the length of the girder did not exceed 25 feet, its depth in the center was not greater than 20 inches, nor the breadth of its bottom flange less than one-third, nor more than half the depth, and the thickness of the metal not less than $\frac{1}{16}$th of the depth. Then the letters expressing the same quantities as before, we have

$$\left.\begin{aligned} \frac{1792}{L}\{b_1 d_1^2 - (b_1 - b_2) d_2^2\} &= W, \text{ in lbs.} \\ \frac{80}{5 L}\{b_1 d_1^2 - (b_1 - b_2) d_2^2\} &= W, \text{ in cwts.} \\ \frac{4}{5 L}\{b_1 d_1^2 - (b_1 - b_2) d_2^2\} &= W, \text{ in tons.} \end{aligned}\right\} \quad \ldots \text{ (XI.)}$$

LONGITUDINAL FORM OF BEAM OF UNIFORM STRENGTH.

1. *If the* depth *of the beam is uniform:*—

When the whole load is collected in one point, the sides of the beam should be straight lines, the breadth at the ends being half the breadth where the load is applied, as in fig. 236.

When the load is uniformly distributed, the sides should be portions of a circle, the radius of which should equal the square of the length of the beam, divided by twice its breadth, as in fig. 237.

2. *When the* breadth *of the beam is uniform:*—

When the load is collected in one point, the extended side of the beam should be straight, the depth at the point where the load is applied twice that at the ends, and the lines connecting them straight, as in fig. 238.

When the load is uniformly distributed, the extended side should be straight, and the compressed side a portion of a circle whose radius equals the square of half the length of the beam divided by its depth, as in fig. 239.

3. *When the* transverse section *of the beam is a* similar figure *throughout its whole length:*—

When the load is collected in one point, the depth at this point should be to the depth at its extremities as 3 : 2; the sides of the beam being all straight lines, as in fig. 240.

When the load is uniformly distributed, the depth in the center should be to the depth at the end as 3 : 1; the sides of the beam being all straight lines, as in fig. 241.

TRANSVERSE STRENGTH OF OTHER MATERIALS THAN CAST IRON.

The only form of beam which is employed of any other material than cast-iron is the rectangular form, the strength of which may be immediately obtained from the following formula, in which b is the breadth, d the depth, both in inches, L the length in feet, and W the breaking weight applied in the center; then

$$\left.\begin{aligned} \frac{a_1 b d^2}{L} &= W, \text{ in lbs.} \\ \frac{a_2 b d^2}{L} &= W, \text{ in cwts.} \\ \frac{a_3 b d^2}{L} &= W, \text{ in tons.} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(XII.)}$$

The following table exhibits the values of the constant coefficients a_1, a_2, and a_3, according to Professor Barlow, for several different kinds of timber.

Name of Material.	Value of a_1.	Value of a_2.	Value of a_3.	Authority.
Wrought iron	2290	20·45	1·022	Tredgold.
Brass, cast	890	7·95	·397	,,
Lead, cast	196	1·75	·087	,,
Tin, cast	372	3·32	·166	,,
Zinc, cast	746	6·66	·333	,,
Teak	821	7·33	·366	Barlow.
Poon	740	6·61	·330	,,
Ash	675	6·03	·301	,,
Canadian oak	589	5·26	·263	,,
English oak	557	4·98	·249	,,
Pitch pine	544	4·86	·243	,,

Name of Material.	Value of a_1.	Value of a_2.	Value of a_3.	Authority.
Beech	519	4·63	·232	Barlow.
Dantzic oak	486	4·33	·216	,,
Adriatic oak	461	4·12	·206	,,
Red pine	447	4·00	·200	,,
Mar Forest fir	408	3·64	·182	,,
New England fir	367	3·28	·164	,,
Riga fir	359	3·21	·160	,,
Elm	338	3·02	·151	,,
Larch	330	2·94	·147	,,

THE STRENGTH OF BEAMS ACCORDING TO THE MANNER IN WHICH THE LOAD IS DISTRIBUTED.

In the same beam, the weight which will be required to break it depends very much upon the point at which it is applied; and upon the way in which it is distributed; as also upon the manner in which the beam is supported.

1. When the beam is supported at both ends:—

If the weight which must be applied at the center to produce fracture, equals ..... } W;

Then that which must be applied at any other point, (where l_1 and l_2 equal the distances of that point from each support, and L the distance between the supports,) will be equal to ..... } $\frac{L^2}{4\,l_1\,l_2}$. W ;

And that which would be required if distributed uniformly along the beam, would be equal to ..... } 2 W.

2. When the beam is supported only at one end:—

If the weight is applied at the other, it must be equal to ..... } $\frac{1}{4}$ W;

And if distributed uniformly along the beam, it must be equal to ..... } $\frac{1}{2}$ W.

SECT. V. *Elasticity of Bodies subjected to a transverse strain.*

The deflection of rectangular beams when supported at each extremity and loaded in the middle, is found by the following formula, in which δ equals the deflection in inches, W_1 the weight in lbs. producing it, and m the weight of the modulus of elasticity in lbs. for the given material, which is given at pages 383 and 384; the other letters represent the same quantities as before, then we have

$$\delta = \frac{432\ L^3\ W_1}{m\ b\ d^3} \quad \text{..... (XIII.)}$$

If the weight be uniformly distributed, instead of being collected in the center, it will only produce $\frac{5}{8}$ths of the deflection given by the above formula.

For cast-iron girders whose dimensions are limited, as described in § 4, page 386, Mr. Dines finds that the following formulæ may be made use of to determine the deflection of the beam when loaded in the center with $\frac{3}{5}$ths of the breaking weight; in which d is the depth of the beam in the center in inches, and L the length in feet.

When the top and bottom flanges are equal, and the girder parallel, or of equal depth throughout—

$$\delta = \frac{L^2}{40\,d} \quad \text{(XIV.)}$$

When the flanges are not equal, and the girder is not parallel—

$$\delta = \frac{L^2}{35\,d} \quad \text{(XV.)}$$

When the beam has no top flange, and the depth varies, then

$$\delta = \frac{L^2}{30\,d} \quad \text{(XVI.)}$$

These formulæ are the result of upwards of 2000 experiments upon beams of a moderately large size.

In the first section we stated that the elasticity of a cast-iron beam becomes injured with a strain very much smaller than the breaking weight, so that, when the strain is removed, the beam does not recover its former shape, but remains permanently bent. Mr. Hodgkinson was, we believe, the first writer who noticed that such was the case, which he has proved by his experiments, and which result has been fully confirmed by those of Mr. Cubitt.

The greater part of Mr. Hodgkinson's experiments were made upon beams having only a bearing of 4 feet 6 inches; and he found that the amount of the permanent set varied as the square of the weight applied. This rule, however, although correct for beams about 5 feet in length, does not apply when the length becomes much greater, for Mr. Cubitt found by his experiments, that when the length became equal to about 20 feet, the set was only as the weight, and that with larger beams the set was still less. As these experiments are important from the size of the girders, we have tabulated a few of them, as on the following page.

No. of experiments.	Weight in tons applied in the center.	Deflection in inches in the center.	Differences.	Set in inches in the center.	Differences.	REMARKS.
1.	2	·086	·086	·0160	·0035	The form and dimensions of these girders are shown in fig. 242. The results here given are the mean of two girders, one of which weighed 41 cwt. 0 qrs. 20 lbs., and the other 41 cwt. 2 qrs. 6 lbs. The total length of the girders was 30 ft. 4 in., the length of bearing 28 ft.
	4	·172	·123	·0195	·0070	
	6	·295	·100	·0265	·0070	
	8	·395	·094	·0335	·0070	
	10	·489	·100	·0405	·0075	
	12	·589	·095	·0480	·0095	
	14	·684	·098	·0575	·0095	
	16	·782	·102	·0670	·0100	
	18	·884		·0770		
2.	3	·081	·041	·0210	·0050	Cast from the same pattern as No. 1. The results are the mean of two girders, one of which weighed 42 cwt. 0 qrs. 14 lbs., the other 41 cwt. 3 qrs. 16 lbs. The length of bearing was 27 ft. 10 in.
	4	·122	·099	·0260	·0050	
	6¼	·221	·084	·0310	·0045	
	8	·305	·091	·0355	·0070	
	10	·396	·107	·0425	·0105	
	12	·503	·102	·0530	·0105	
	14	·605	·104	·0635	·0165	
	16	·709	·091	...		
	18	·800		·0800		
3.	2	·117	·096	·020	·005	The form and dimensions of these girders are shown in fig. 243. The results are the mean of two girders, one of which weighed 26 cwt. 1 qr. 26 lbs., and the other 25 cwt. 2 qrs. 22 lbs. The length of the girders was 24 ft. 6 in., the length of bearing 23 ft. 10 in.
	4	·213	·133	·025	·007	
	6	·346	·120	·032	·013	
	8	·466	·124	·045	·012	
	10	·590	·118	·057	·011	
	12	·708		·068		
4.	2	·125	·140	·0150	·0095	Cast from the same pattern as No. 3. The results are the mean of two girders, one of which weighed 25 cwt. 2 qrs. 6 lbs., and the other 24 cwt. 3 qrs. 17 lbs. The length of the bearing was 23 ft.
	4	·265	·109	·0245	·0105	
	6	·374	·119	·0350	·0130	
	8	·493	·119	·0480	·0140	
	10	·612	·059	·0620	·0070	
	11	·671		·0690		
5.	4	·24	·12	·035	·015	The form and dimensions of the girder are shown in fig. 244. The weight of the girder was 19 cwt. 0 qrs. 18 lbs., its total length 2 feet, and the distance between the bearings 20 ft. 7 in.
	6	·36	·14	·050	·015	
	8	·50	·14	·065	·015	
	10	·64	·10	·080		
	12	·74				
6.	1	·085	·089	·018	·017	Rectangular bar of cast iron, 5·1 inches in depth, 1·04 inches in breadth, and 5 ft. 6 in. length of bearing. Broke at 5 tons.
	2	·174	·106	·035	·045	
	3	·280	·122	·080	·023	
	4	·402	·073	·103		
	4½	·475				
7.	1	·083	·078	·013	·015	Rectangular bar of cast iron, 5·1 inches in depth, 1·05 inches in breadth, and 5 feet 6 inches length of bearing. Broke with 4½ tons.
	2	·161	·099	·028	·032	
	3	·260	·120	·060	·030	
	4	·380	·060	·090		
	4½	·440				

No. of experiments.	Weight in tons applied in the center.	Deflection in inches in the center.	Differences.	Set in inches in the center.	Differences.	REMARKS
8.	1	·047				Rectangular bar of wrought iron, 4·97 inches in depth, 1 inch in breadth, and 5 ft. 6 in. length of bearing.
	2	·100	·053	No set.		
	3	·145	·045	·005	·004	
	4	·195	·050	·009	·011	
	5	·250	·055	·020	·160	
	6	·350	·100	·180		
9.	1	·056				Rectangular bar of wrought iron, 5·03 inches in depth, 1 inch in breadth, and 5 ft. 6 in. length of bearing.
	2	·110	·054	No set.		
	3	·155	·045	·006	·006	
	4	·210	·055	·012	·038	
	5	·300	·090	·050	·200	
	6	·530	·230	·250		
10.	2	·100		Trifling set.		Rectangular bar of wrought iron, 5 inches in depth, ·97 inches in breadth, and 5 ft. 6 in. length of bearing.
	3	·145	·045	·005	·013	
	4	·205	·060	·018	·136	
	5	·402	·197	·154	·126	
	5½	·540	·138	·280		
11.	2	·106		Trifling set.		Rectangular bar of wrought iron, 4·97 inches in depth, 1·02 inches in breadth, and 5 ft. 6 in. length of bearing.
	3	·158	·052	·008	·012	
	4	·218	·060	·020	·024	
	5	·310	·092	·044	·061	
	5½	·380	·070	·105		
12.	1	·050		Set.		Rectangular bar of wrought iron, 4·97 inches in depth, 1·02 inches in breadth, and 5 ft. 6 in. length of bearing.
	2	·105	·055	·004	·004	
	3	·150	·045	·008	·007	
	4	·206	·056	·015	·005	
	4½	·238	·032	·020	·008	
	5	·268	·030	·028		
13.	1	·050		Set.		Rectangular bar of wrought iron, 4·97 inches in depth, 1·02 inches in breadth, and 5 ft. 6 in. length of bearing.
	2	·103	·053			
	3	·148	·045	·004		
	4	·200	·052			
	4½	·220	·020			
14.	1	·102				Rectangular bar of wrought iron, 4 inches in depth, 1·01 inches in breadth, and 5 ft. 6 in. length of bearing.
	1½	·152	·050	·002	·001	
	2	·202	·050	·003	·005	
	2½	·248	·046	·008	·002	
	3	·300	·052	·010	·045	
	3½	·374	·074	·055	·151	
	4	·560	·186	·206		
15.	1	·100				Rectangular bar of wrought iron, 4 inches in depth, 1·01 inches in breadth, and 5 ft. 6 in. length of bearing.
	1½	·150	·050	·003	·007	
	2	·190	·040	·010	·006	
	2½	·235	·045	·016	·007	
	3	·295	·060	·023	·023	
	3½	·355	·060	·046	·056	
	4	·450	·095	·102	·163	
	4½	·658	·208	·265		

SECT. VI. *Strength of Materials to resist Torsion.*

The power of a bar or rod of any material to resist torsion, may be measured by the angle through which the end of a lever attached to the same will be moved by a given weight. If D equal the distance from the fixed end of the bar or prism to the point of application of the lever used to twist it, l the length of the lever in inches, w the weight in lbs. applied to the end of the lever, r the radius of the prism if cylindrical, and b, d, its breadth and depth if rectangular, all in inches, θ the angle of torsion at the point of application, F a constant for each material, representing the specific resistance to flexure by torsion, and W a constant weight expressing the resistance to torsion, with regard to a unit of surface at the time of fracture; then the following table will exhibit the relations subsisting between these quantities.

Form of section of the prism.	Resistance to angular flexure by a force of torsion.	Resistance to fracture by a force of torsion.
Cylindrical	$F = wl\frac{2L}{\pi r^4\theta}$.	$W = wl\frac{2}{\pi r^3}$.
Square	$F = wl\frac{6L}{d^4\theta}$.	$W = wl\frac{6}{\sqrt{2}d^3}$.
Rectangular	$F = wl\frac{3(b^2+d^2)L}{b^3d^3\theta}$.	$W = wl\frac{3\sqrt{b^2+d^2}}{b^2d^2}$.

The mean value of W for cast iron, as obtained from the experiments of Messrs. George Rennie, Bramah, and Dunlop, is 32,503 lbs., and from Mr. Bevan's experiments, the mean value of F for cast iron is 5,709,600 lbs., and for wrought iron and steel 10,674,540 lbs., and, these values being substituted in the above table, it becomes

Form of section of the prism.	Resistance to angular flexure by a force of torsion.		Resistance to fracture by a force of torsion.
	Cast iron.	Wrought iron or steel.	Cast iron.
Cylindrical	$wl = 8968020\frac{r^4\theta}{L}$.	$wl = 16767770\frac{r^4\theta}{L}$.	$wl = 51055\,r^3$.
Square	$wl = 951600\frac{d^4\theta}{L}$.	$wl = 1779090\frac{d^4\theta}{L}$.	$wl = 7661\,d^3$.
Rectangular	$wl = 1903200\frac{b^3d^3\theta}{(b^2+d^2)L}$.	$wl = 3558180\frac{b^3d^3\theta}{(b^2+d^2)L}$.	$wl = 10834\frac{b^2d^2}{\sqrt{b^2+}}$

APPENDIX.

TABLE III. [No. 1499 L. 175802

2	3	4	5	6	7	8	9	N.
68	1301	1734	2166	2598	3029	3461	3891	100
81	5609	6038	6466	6894	7321	7748	8174	1
51	9876	0300	0724	1147	1570	1993	2415	2
80	4100	4521	4940	5360	5779	6197	6616	3
68	8284	8700	9116	9532	9947	0361	0775	4
16	2428	2841	3252	3664	4075	4486	4896	5
25	6533	6942	7350	7757	8164	8571	8978	6
95	0600	1004	1408	1812	2216	2619	3021	7
27	4628	5029	5430	5830	6230	6629	7028	8
23	8620	9017	9414	9811	0207	0602	0998	9
2	3	4	5	6	7	8	9	
82	2576	2969	3362	3755	4148	4540	4932	110
05	6495	6885	7275	7664	8053	8442	8830	1
93	0380	0766	1153	1538	1924	2309	2694	2
46	4230	4613	4996	5378	5760	6142	6524	3
66	8046	8426	8805	9185	9563	9942	0320	4
52	1829	2206	2582	2958	3333	3709	4083	5
06	5580	5953	6326	6699	7071	7443	7815	6
28	9298	9668	0038	0407	0776	1145	1514	7
17	2985	3352	3718	4085	4451	4816	5182	8
76	6640	7004	7368	7731	8094	8457	8819	9
2	3	4	5	6	7	8	9	
04	0266	0626	0987	1347	1707	2067	2426	120
03	3861	4219	4576	4934	5291	5647	6004	1
71	7426	7781	8136	8490	8845	9198	9552	2
11	0963	1315	1667	2018	2370	2721	3071	3
22	4471	4820	5169	5518	5866	6215	6562	4
04	7951	8298	8644	8990	9335	9681	0026	5
59	1403	1747	2091	2434	2777	3119	3462	6
87	4828	5169	5510	5851	6191	6531	6871	7
88	8227	8565	8903	9241	9579	9916	0253	8
63	1599	1934	2270	2605	2940	3275	3609	9
2	3	4	5	6	7	8	9	
11	4644	5278	5611	5943	6276	6608	6940	130
34	8265	8595	8926	9256	9586	9915	0245	1
31	1560	1888	2216	2544	2871	3198	3525	2
04	4830	5156	5481	5806	6131	6456	6781	3
53	8076	8399	8722	9045	9368	9690	0012	4
77	1298	1619	1939	2260	2580	2900	3219	5
77	4496	4814	5133	5451	5769	6086	6403	6
54	7671	7987	8303	8618	8934	9249	9564	7
08	0822	1136	1450	1763	2076	2389	2702	8
39	3951	4263	4574	4885	5196	5507	5818	9
2	3	4	5	6	7	8	9	
48	7058	7367	7676	7985	8294	8603	8911	140
35	0142	0449	0756	1063	1370	1676	1982	1
00	3205	3510	3815	4120	4424	4728	5032	2
43	6246	6549	6852	7154	7457	7759	8061	3
65	9266	9567	9868	0168	0469	0769	1068	4
67	2266	2564	2863	3161	3460	3758	4055	5
47	5244	5541	5838	6134	6430	6726	7022	6
08	8203	8497	8792	9086	9380	9674	9968	7
48	1141	1434	1726	2019	2311	2603	2895	8
69	4060	4351	4641	4932	5222	5512	5802	9

No. 1500 L. 176091] TABLE III. [No. 1999 L. 300813

N.	0	1	2	3	4	5	6	7	8	9	N.
150	176091	6381	6670	6959	7248	7536	7825	8113	8401	8689	150
1	8977	9264	9552	9839	0126	0413	0699	0986	1272	1558	1
2	181844	2129	2415	2700	2985	3270	3555	3839	4123	4407	2
3	4691	4975	5259	5542	5825	6108	6391	6674	6956	7239	3
4	7521	7803	8084	8366	8647	8928	9209	9490	9771	0051	4
5	190332	0612	0892	1171	1451	1730	2010	2289	2567	2846	5
6	3125	3403	3681	3959	4237	4514	4792	5069	5346	5623	6
7	5900	6176	6453	6729	7005	7281	7556	7832	8107	8382	7
8	8657	8932	9206	9481	9755	0029	0303	0577	0850	1124	8
9	201397	1670	1943	2216	2488	2761	3033	3305	3577	3848	9

N.	0	1	2	3	4	5	6	7	8	9	N.
160	4120	4391	4663	4934	5204	5475	5746	6016	6286	6556	160
1	6826	7096	7365	7634	7904	8173	8441	8710	8979	9247	1
2	9515	9783	0051	0319	0586	0853	1121	1388	1654	1921	2
3	212188	2454	2720	2986	3252	3518	3783	4049	4314	4579	3
4	4844	5109	5373	5638	5902	6166	6430	6694	6957	7221	4
5	7484	7747	8010	8273	8536	8798	9060	9323	9585	9846	5
6	220108	0370	0631	0892	1153	1414	1675	1936	2196	2456	6
7	2716	2976	3236	3496	3755	4015	4274	4533	4792	5051	7
8	5309	5568	5826	6084	6342	6600	6858	7115	7372	7630	8
9	7887	8144	8400	8657	8913	9170	9426	9682	9938	0193	9

N.	0	1	2	3	4	5	6	7	8	9	N.
170	230449	0704	0960	1215	1470	1724	1979	2234	2488	2742	170
1	2996	3250	3504	3757	4011	4264	4517	4770	5023	5276	1
2	5528	5781	6033	6285	6537	6789	7041	7292	7544	7795	2
3	8046	8297	8548	8799	9049	9299	9550	9800	0050	0300	3
4	240549	0799	1048	1297	1546	1795	2044	2293	2541	2790	4
5	3038	3286	3534	3782	4030	4277	4525	4772	5019	5266	5
6	5513	5759	6006	6252	6499	6745	6991	7237	7482	7728	6
7	7973	8219	8464	8709	8954	9198	9443	9687	9932	0176	7
8	250420	0664	0908	1151	1395	1638	1881	2125	2368	2610	8
9	2853	3096	3338	3580	3822	4064	4306	4548	4790	5031	9

N.	0	1	2	3	4	5	6	7	8	9	N.
180	5273	5514	5755	5996	6237	6477	6718	6958	7198	7439	180
1	7679	7918	8158	8398	8637	8877	9116	9355	9594	9833	1
2	260071	0310	0548	0787	1025	1263	1501	1739	1976	2214	2
3	2451	2688	2925	3162	3399	3636	3873	4109	4346	4582	3
4	4818	5054	5290	5525	5761	5996	6232	6467	6702	6937	4
5	7172	7406	7641	7875	8110	8344	8578	8812	9046	9279	5
6	9513	9746	9980	0213	0446	0679	0912	1144	1377	1609	6
7	271842	2074	2306	2538	2770	3001	3233	3464	3696	3927	7
8	4158	4389	4620	4850	5081	5311	5542	5772	6002	6232	8
9	6462	6692	6921	7151	7380	7609	7838	8067	8296	8525	9

N.	0	1	2	3	4	5	6	7	8	9	N.
190	8754	8982	9211	9439	9667	9895	0123	0351	0578	0806	190
1	281033	1261	1488	1715	1942	2169	2396	2622	2849	3075	1
2	3301	3527	3753	3979	4205	4431	4656	4882	5107	5332	2
3	5557	5782	6007	6232	6456	6681	6905	7130	7354	7578	3
4	7802	8026	8249	8473	8696	8920	9143	9366	9589	9812	4
5	290035	0257	0480	0702	0925	1147	1369	1591	1813	2034	5
6	2256	2478	2699	2920	3141	3363	3584	3804	4025	4246	6
7	4466	4687	4907	5127	5347	5567	5787	6007	6226	6446	7
8	6665	6884	7104	7323	7542	7761	7979	8198	8416	8635	8
9	8853	9071	9289	9507	9725	9943	0161	0378	0595	0813	9

No. 2000 L. 801030] TABLE III. [No. 2499 L. 397766

N.	0	1	2	3	4	5	6	7	8	9	N.
200	301030	1247	1464	1681	1898	2114	2331	2547	2764	2980	200
1	3196	3412	3628	3844	4059	4275	4491	4706	4921	5136	1
2	5351	5566	5781	5996	6211	6425	6639	6854	7068	7282	2
3	7496	7710	7924	8137	8351	8564	8778	8991	9204	9417	3
4	9630	9843	0056	0268	0481	0693	0906	1118	1330	1542	4
5	311754	1966	2177	2389	2600	2812	3023	3234	3445	3656	5
6	3867	4078	4289	4499	4710	4920	5130	5340	5551	5760	6
7	5970	6180	6390	6599	6809	7018	7227	7436	7646	7854	7
8	8063	8272	8481	8689	8898	9106	9314	9522	9730	9938	8
9	320146	0354	0562	0769	0977	1184	1391	1598	1805	2012	9
	0	1	2	3	4	5	6	7	8	9	
210	2219	2426	2633	2839	3046	3252	3458	3665	3871	4077	210
1	4282	4488	4694	4899	5105	5310	5516	5721	5926	6131	1
2	6336	6541	6745	6950	7155	7359	7563	7767	7972	8176	2
3	8380	8583	8787	8991	9194	9398	9601	9805	0008	0211	3
4	330414	0617	0819	1022	1225	1427	1630	1832	2034	2236	4
5	2438	2640	2842	3044	3246	3447	3649	3850	4051	4253	5
6	4454	4655	4856	5057	5257	5458	5658	5859	6059	6260	6
7	6460	6660	6860	7060	7260	7459	7659	7858	8058	8257	7
8	8456	8656	8855	9054	9253	9451	9650	9849	0047	0246	8
9	340444	0642	0841	1039	1237	1435	1632	1830	2028	2225	9
	0	1	2	3	4	5	6	7	8	9	
220	2423	2620	2817	3014	3212	3409	3606	3802	3999	4196	220
1	4392	4589	4785	4981	5178	5374	5570	5766	5962	6157	1
2	6353	6549	6744	6939	7135	7330	7525	7720	7915	8110	2
3	8305	8500	8694	8889	9083	9278	9472	9666	9860	0054	3
4	350248	0442	0636	0829	1023	1216	1410	1603	1796	1989	4
5	2183	2375	2568	2761	2954	3147	3339	3532	3724	3916	5
6	4108	4301	4493	4685	4876	5068	5260	5452	5643	5834	6
7	6026	6217	6408	6599	6790	6981	7172	7363	7554	7744	7
8	7935	8125	8316	8506	8696	8886	9076	9266	9456	9646	8
9	9835	0025	0215	0404	0593	0783	0972	1161	1350	1539	9
	0	1	2	3	4	5	6	7	8	9	
230	361728	1917	2105	2294	2482	2671	2859	3048	3236	3424	230
1	3612	3800	3988	4176	4363	4551	4739	4926	5113	5301	1
2	5488	5675	5862	6049	6236	6423	6610	6796	6983	7169	2
3	7356	7542	7729	7915	8101	8287	8473	8659	8845	9030	3
4	9216	9401	9587	9772	9958	0143	0328	0513	0698	0883	4
5	371068	1253	1437	1622	1806	1991	2175	2360	2544	2728	5
6	2912	3096	3280	3464	3647	3831	4015	4198	4382	4565	6
7	4748	4932	5115	5298	5481	5664	5846	6029	6212	6394	7
8	6577	6759	6942	7124	7306	7488	7670	7852	8034	8216	8
9	8398	8580	8761	8943	9124	9306	9487	9668	9849	0030	9
	0	1	2	3	4	5	6	7	8	9	
240	380211	0392	0573	0754	0934	1115	1296	1476	1656	1837	240
1	2017	2197	2377	2557	2737	2917	3097	3277	3456	3636	1
2	3815	3995	4174	4353	4533	4712	4891	5070	5249	5428	2
3	5606	5785	5964	6142	6321	6499	6677	6856	7034	7212	3
4	7390	7568	7746	7924	8101	8279	8456	8634	8811	8989	4
5	9166	9343	9520	9698	9875	0051	0228	0405	0582	0759	5
6	390935	1112	1288	1464	1641	1817	1993	2169	2345	2521	6
7	2697	2873	3048	3224	3400	3575	3751	3926	4101	4277	7
8	4452	4627	4802	4977	5152	5326	5501	5676	5850	6025	8
9	6199	6374	6548	6722	6896	7071	7245	7419	7592	7766	9

N.	0	1	2	3	4	5	6	7	8	9	N.
250	397940	8114	8287	8461	8634	8808	8981	9154	9328	9501	250
1	9674	9847	0020	0192	0365	0538	0711	0883	1056	1228	1
2	401401	1573	1745	1917	2089	2261	2433	2605	2777	2949	2
3	3121	3292	3464	3635	3807	3978	4149	4320	4492	4663	3
4	4834	5005	5176	5346	5517	5688	5858	6029	6199	6370	4
5	6540	6710	6881	7051	7221	7391	7561	7731	7901	8070	5
6	8240	8410	8579	8749	8918	9087	9257	9426	9595	9764	6
7	9933	0102	0271	0440	0609	0777	0946	1114	1283	1451	7
8	411620	1788	1956	2124	2293	2461	2629	2796	2964	3132	8
9	3300	3467	3635	3803	3970	4137	4305	4472	4639	4806	9
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260	4973	5140	5307	5474	5641	5808	5974	6141	6308	6474	260
1	6641	6807	6973	7139	7306	7472	7638	7804	7970	8135	1
2	8301	8467	8633	8798	8964	9129	9295	9460	9625	9791	2
3	9956	0121	0286	0451	0616	0781	0945	1110	1275	1439	3
4	421604	1768	1933	2097	2261	2426	2590	2754	2918	3082	4
5	3246	3410	3574	3737	3901	4065	4228	4392	4555	4718	5
6	4882	5045	5208	5371	5534	5697	5860	6023	6186	6349	6
7	6511	6674	6836	6999	7161	7324	7486	7648	7811	7973	7
8	8135	8297	8459	8621	8783	8944	9106	9268	9429	9591	8
9	9752	9914	0075	0236	0398	0559	0720	0881	1042	1203	9
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1	2969	3130	3290	3450	3610	3770	3930	4090	4249	4409	1
2	4569	4729	4888	5048	5207	5367	5526	5685	5844	6004	2
3	6163	6322	6481	6640	6799	6957	7116	7275	7433	7592	3
4	7751	7909	8067	8226	8384	8542	8701	8859	9017	9175	4
5	9333	9491	9648	9806	9964	0122	0279	0437	0594	0752	5
6	440909	1066	1224	1381	1538	1695	1852	2009	2166	2323	6
7	2480	2637	2793	2950	3106	3263	3419	3576	3732	3889	7
8	4045	4201	4357	4513	4669	4825	4981	5137	5293	5449	8
9	5604	5760	5915	6071	6226	6382	6537	6692	6848	7003	9
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1	8706	8861	9015	9170	9324	9478	9633	9787	9941	0095	1
2	450249	0403	0557	0711	0865	1018	1172	1326	1479	1633	2
3	1786	1940	2093	2247	2400	2553	2706	2859	3012	3165	3
4	3318	3471	3624	3777	3930	4082	4235	4387	4540	4692	4
5	4845	4997	5150	5302	5454	5606	5758	5910	6062	6214	5
6	6366	6518	6670	6821	6973	7125	7276	7428	7579	7731	6
7	7882	8033	8184	8336	8487	8638	8789	8940	9091	9242	7
8	9392	9543	9694	9845	9995	0146	0296	0447	0597	0748	8
9	460898	1048	1198	1348	1499	1649	1799	1948	2098	2248	9
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1	3893	4042	4191	4340	4490	4639	4788	4936	5085	5234	1
2	5383	5532	5680	5829	5977	6126	6274	6423	6571	6719	2
3	6868	7016	7164	7312	7460	7608	7756	7904	8052	8200	3
4	8347	8495	8643	8790	8938	9085	9233	9380	9527	9675	4
5	9822	9969	0116	0263	0410	0557	0704	0851	0998	1145	5
6	471292	1438	1585	1732	1878	2025	2171	2318	2464	2610	6
7	2756	2903	3049	3195	3341	3487	3633	3779	3925	4071	7
8	4216	4362	4508	4653	4799	4944	5090	5235	5381	5526	8
9	5671	5816	5962	6107	6252	6397	6542	6687	6832	6976	9

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477121	7266	7411	7555	7700	7844	7989	8133	8278	8422	300
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480007	0151	0294	0438	0582	0725	0869	1012	1156	1299	2
1443	1586	1729	1872	2016	2159	2302	2445	2588	2731	3
2874	3016	3159	3302	3445	3587	3730	3872	4015	4157	4
4300	4442	4585	4727	4869	5011	5153	5295	5437	5579	5
5721	5863	6005	6147	6289	6430	6572	6714	6855	6997	6
7138	7280	7421	7563	7704	7845	7986	8127	8269	8410	7
8551	8692	8833	8974	9114	9255	9396	9537	9677	9818	8
9958	0099	0239	0380	0520	0661	0801	0941	1081	1222	9
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491362	1502	1642	1782	1922	2062	2201	2341	2481	2621	310
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5544	5683	5822	5960	6099	6238	6376	6515	6653	6791	3
6930	7068	7206	7344	7483	7621	7759	7897	8035	8173	4
8311	8448	8586	8724	8862	8999	9137	9275	9412	9550	5
9687	9824	9962	0099	0236	0374	0511	0648	0785	0922	6
501059	1196	1333	1470	1607	1744	1880	2017	2154	2291	7
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5150	5286	5421	5557	5693	5828	5964	6099	6234	6370	320
6505	6640	6776	6911	7046	7181	7316	7451	7586	7721	1
7856	7991	8126	8260	8395	8530	8664	8799	8934	9068	2
9203	9337	9471	9606	9740	9874	0009	0143	0277	0411	3
510545	0679	0813	0947	1081	1215	1349	1482	1616	1750	4
1883	2017	2151	2284	2418	2551	2684	2818	2951	3084	5
3218	3351	3484	3617	3750	3883	4016	4149	4282	4415	6
4548	4681	4813	4946	5079	5211	5344	5476	5609	5741	7
5874	6006	6139	6271	6403	6535	6668	6800	6932	7064	8
7196	7328	7460	7592	7724	7855	7987	8119	8251	8382	9
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8514	8646	8777	8909	9040	9171	9303	9434	9566	9697	330
9828	9959	0090	0221	0353	0485	0615	0745	0876	1007	1
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2444	2575	2705	2835	2966	3096	3226	3356	3486	3616	3
3746	3876	4006	4136	4266	4396	4526	4656	4785	4915	4
5045	5174	5304	5434	5563	5693	5822	5951	6081	6210	5
6339	6469	6598	6727	6856	6985	7114	7243	7372	7501	6
7630	7759	7888	8016	8145	8274	8402	8531	8660	8788	7
8917	9045	9174	9302	9430	9559	9687	9815	9943	0072	8
530200	0328	0456	0584	0712	0840	0968	1096	1223	1351	9
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1479	1607	1734	1862	1990	2117	2245	2372	2500	2627	340
2754	2882	3009	3136	3264	3391	3518	3645	3772	3899	1
4026	4153	4280	4407	4534	4661	4787	4914	5041	5167	2
5294	5421	5547	5674	5800	5927	6053	6179	6306	6432	3
6558	6685	6811	6937	7063	7189	7315	7441	7567	7693	4
7819	7945	8071	8197	8322	8448	8574	8699	8825	8951	5
9076	9202	9327	9452	9578	9703	9829	9954	0079	0204	6
540329	0455	0580	0705	0830	0955	1080	1205	1330	1454	7
1579	1704	1829	1953	2078	2203	2327	2452	2576	2701	8
2825	2950	3074	3199	3323	3447	3571	3696	3820	3944	9

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350	544068	4192	4316	4440	4564	4688	4812	4936	5060	5183	350
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2	6543	6666	6789	6913	7036	7159	7282	7405	7529	7652	2
3	7775	7898	8021	8144	8267	8389	8512	8635	8758	8881	3
4	9003	9126	9249	9371	9494	9616	9739	9861	9984	0106	4
5	550228	0351	0473	0595	0717	0840	0962	1084	1206	1328	5
6	1450	1572	1694	1816	1938	2060	2181	2303	2425	2547	6
7	2668	2790	2911	3033	3155	3276	3398	3519	3640	3762	7
8	3883	4004	4126	4247	4368	4489	4610	4731	4852	4973	8
9	5094	5215	5336	5457	5578	5699	5820	5940	6061	6182	9

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360	6303	6423	6544	6664	6785	6905	7026	7146	7267	7387	360
1	7507	7627	7748	7868	7988	8108	8228	8349	8469	8589	1
2	8709	8829	8948	9068	9188	9308	9428	9548	9667	9787	2
3	9907	0026	0146	0265	0385	0504	0624	0743	0863	0982	3
4	561101	1221	1340	1459	1578	1698	1817	1936	2055	2174	4
5	2293	2412	2531	2650	2769	2887	3006	3125	3244	3362	5
6	3481	3600	3718	3837	3955	4074	4192	4311	4429	4548	6
7	4666	4784	4903	5021	5139	5257	5376	5494	5612	5730	7
8	5848	5966	6084	6202	6320	6437	6555	6673	6791	6909	8
9	7026	7144	7262	7379	7497	7614	7732	7849	7967	8084	9

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370	8202	8319	8436	8554	8671	8788	8905	9023	9140	9257	370
1	9374	9491	9608	9725	9842	9959	0076	0193	0309	0426	1
2	570543	0660	0776	0893	1010	1126	1243	1359	1476	1592	2
3	1709	1825	1942	2058	2174	2291	2407	2523	2639	2755	3
4	2872	2988	3104	3220	3336	3452	3568	3684	3800	3915	4
5	4031	4147	4263	4379	4494	4610	4726	4841	4957	5072	5
6	5188	5303	5419	5534	5650	5765	5880	5996	6111	6226	6
7	6341	6457	6572	6687	6802	6917	7032	7147	7262	7377	7
8	7492	7607	7722	7836	7951	8066	8181	8295	8410	8525	8
9	8639	8754	8868	8983	9097	9212	9326	9441	9555	9669	9

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380	9784	9898	0012	0126	0241	0355	0469	0583	0697	0811	380
1	580925	1039	1153	1267	1381	1495	1608	1722	1836	1950	1
2	2063	2177	2291	2404	2518	2631	2745	2858	2972	3085	2
3	3199	3312	3426	3539	3652	3765	3879	3992	4105	4218	3
4	4331	4444	4557	4670	4783	4896	5009	5122	5235	5348	4
5	5461	5574	5686	5799	5912	6024	6137	6250	6362	6475	5
6	6587	6700	6812	6925	7037	7149	7262	7374	7486	7599	6
7	7711	7823	7935	8047	8160	8272	8384	8496	8608	8720	7
8	8832	8944	9056	9167	9279	9391	9503	9615	9726	9838	8
9	9950	0061	0173	0284	0396	0507	0619	0730	0842	0953	9

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390	591065	1176	1287	1399	1510	1621	1732	1843	1955	2066	390
1	2177	2288	2399	2510	2621	2732	2843	2954	3064	3175	1
2	3286	3397	3508	3618	3729	3840	3950	4061	4171	4282	2
3	4393	4503	4614	4724	4834	4945	5055	5165	5276	5386	3
4	5496	5606	5717	5827	5937	6047	6157	6267	6377	6487	4
5	6597	6707	6817	6927	7037	7146	7256	7366	7476	7586	5
6	7695	7805	7914	8024	8134	8243	8353	8462	8572	8681	6
7	8791	8900	9009	9119	9228	9337	9446	9556	9665	9774	7
8	9883	9992	0101	0210	0319	0428	0537	0646	0755	0864	8
9	600973	1082	1191	1299	1408	1517	1625	1734	1843	1951	9

No. 4000 L. 602060]				TABLE III.						[No. 4499 L. 653116	
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2	4226	4334	4442	4550	4658	4766	4874	4982	5089	5197	2
3	5305	5413	5521	5628	5736	5844	5951	6059	6166	6274	3
4	6381	6489	6596	6704	6811	6919	7026	7133	7241	7348	4
5	7455	7562	7669	7777	7884	7991	8098	8205	8312	8419	5
6	8526	8633	8740	8847	8954	9061	9167	9274	9381	9488	6
7	9594	9701	9808	9914	0021	0128	0234	0341	0447	0554	7
8	610660	0767	0873	0979	1086	1192	1298	1405	1511	1617	8
9	1723	1829	1936	2042	2148	2254	2360	2466	2572	2678	9
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410	2784	2890	2996	3102	3207	3313	3419	3525	3630	3736	410
1	3842	3947	4053	4159	4264	4370	4475	4581	4686	4792	1
2	4897	5003	5108	5213	5319	5424	5529	5634	5740	5845	2
3	5950	6055	6160	6265	6370	6476	6581	6686	6790	6895	3
4	7000	7105	7210	7315	7420	7525	7629	7734	7839	7943	4
5	8048	8153	8257	8362	8466	8571	8676	8780	8884	8989	5
6	9093	9198	9302	9406	9511	9615	9719	9824	9928	0032	6
7	620136	0240	0344	0448	0552	0656	0760	0864	0968	1072	7
8	1176	1280	1384	1488	1592	1695	1799	1903	2007	2110	8
9	2214	2318	2421	2525	2628	2732	2835	2939	3042	3146	9
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420	3249	3353	3456	3559	3663	3766	3869	3973	4076	4179	420
1	4282	4385	4488	4591	4695	4798	4901	5004	5107	5210	1
2	5312	5415	5518	5621	5724	5827	5929	6032	6135	6238	2
3	6340	6443	6546	6648	6751	6853	6956	7058	7161	7263	3
4	7366	7468	7571	7673	7775	7878	7980	8082	8185	8287	4
5	8389	8491	8593	8695	8797	8900	9002	9104	9206	9308	5
6	9410	9512	9613	9715	9817	9919	0021	0123	0224	0326	6
7	630428	0530	0631	0733	0835	0936	1038	1139	1241	1342	7
8	1444	1545	1647	1748	1849	1951	2052	2153	2255	2356	8
9	2457	2559	2660	2761	2862	2963	3064	3165	3266	3367	9
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430	3468	3569	3670	3771	3872	3973	4074	4175	4276	4376	430
1	4477	4578	4679	4779	4880	4981	5081	5182	5283	5383	1
2	5484	5584	5685	5785	5886	5986	6087	6187	6287	6388	2
3	6488	6588	6688	6789	6889	6989	7089	7189	7290	7390	3
4	7490	7590	7690	7790	7890	7990	8090	8190	8290	8389	4
5	8489	8589	8689	8789	8888	8988	9088	9188	9287	9387	5
6	9486	9586	9686	9785	9885	9984	0084	0183	0283	0382	6
7	640481	0581	0680	0779	0879	0978	1077	1177	1276	1375	7
8	1474	1573	1672	1771	1871	1970	2069	2168	2267	2306	8
9	2465	2563	2662	2761	2860	2959	3058	3156	3255	3354	9
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1	4439	4537	4636	4734	4832	4931	5029	5127	5226	5324	1
2	5422	5521	5619	5717	5815	5913	6011	6110	6208	6306	2
3	6404	6502	6600	6698	6796	6894	6992	7089	7187	7285	3
4	7383	7481	7579	7676	7774	7872	7969	8067	8165	8262	4
5	8360	8458	8555	8653	8750	8848	8945	9043	9140	9237	5
6	9335	9432	9530	9627	9724	9821	9919	0016	0113	0210	6
7	650308	0405	0502	0599	0696	0793	0890	0987	1084	1181	7
8	1278	1375	1472	1569	1666	1762	1859	1956	2053	2150	8
9	2246	2343	2440	2536	2633	2730	2826	2923	3019	3116	9

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2	5138	5235	5331	5427	5523	5619	5715	5810	5906	6002	
3	6098	6194	6290	6386	6482	6577	6673	6769	6864	6960	
4	7056	7152	7247	7343	7438	7534	7629	7725	7820	7916	
5	8011	8107	8202	8298	8393	8488	8584	8679	8774	8870	
6	8965	9060	9155	9250	9346	9441	9536	9631	9726	9821	
7	9916										
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8	660865	0960	1055	1150	1245	1339	1434	1529	1623	1718	
9	1813	1907	2002	2096	2191	2286	2380	2475	2569	2663	
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460	2758	2852	2947	3041	3135	3230	3324	3418	3512	3607	46
1	3701	3795	3889	3983	4078	4172	4266	4360	4454	4548	
2	4642	4736	4830	4924	5018	5112	5206	5299	5393	5487	
3	5581	5675	5769	5862	5956	6050	6143	6237	6331	6424	
4	6518	6612	6705	6799	6892	6986	7079	7173	7266	7360	
5	7453	7546	7640	7733	7826	7920	8013	8106	8199	8293	
6	8386	8479	8572	8665	8759	8852	8945	9038	9131	9224	
7	9317	9410	9503	9596	9689	9782	9875	9967			
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8	670246	0339	0431	0524	0617	0710	0802	0895	0988	1080	
9	1173	1265	1358	1451	1543	1636	1728	1821	1913	2005	
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470	2098	2190	2283	2375	2467	2560	2652	2744	2836	2929	47
1	3021	3113	3205	3297	3390	3482	3574	3666	3758	3850	
2	3942	4034	4126	4218	4310	4402	4494	4586	4677	4769	
3	4861	4953	5045	5137	5228	5320	5412	5503	5595	5687	
4	5778	5870	5962	6053	6145	6236	6328	6419	6511	6602	
5	6694	6785	6876	6968	7059	7151	7242	7333	7424	7516	
6	7607	7698	7789	7881	7972	8063	8154	8245	8336	8427	
7	8518	8609	8700	8791	8882	8973	9064	9155	9246	9337	
8	9428	9519	9610	9700	9791	9882	9973				
									0063	0154	0245	
9	680336	0426	0517	0607	0698	0789	0879	0970	1060	1151	
	0	1	2	3	4	5	6	7	8	9	
480	1241	1332	1422	1513	1603	1693	1784	1874	1964	2055	48
1	2145	2235	2326	2416	2506	2596	2686	2777	2867	2957	
2	3047	3137	3227	3317	3407	3497	3587	3677	3767	3857	
3	3947	4037	4127	4217	4307	4396	4486	4576	4666	4756	
4	4845	4935	5025	5114	5204	5294	5383	5473	5563	5652	
5	5742	5831	5921	6010	6100	6189	6279	6368	6458	6547	
6	6636	6726	6815	6904	6994	7083	7172	7261	7351	7440	
7	7529	7618	7707	7796	7886	7975	8064	8153	8242	8331	
8	8420	8509	8598	8687	8776	8865	8953	9042	9131	9220	
9	9309	9398	9486	9575	9664	9753	9841	9930			
									0019	0107	
	0	1	2	3	4	5	6	7	8	9	
490	690196	0285	0373	0462	0550	0639	0728	0816	0905	0993	49
1	1081	1170	1258	1347	1435	1524	1612	1700	1789	1877	
2	1965	2053	2142	2230	2318	2406	2494	2583	2671	2759	
3	2847	2935	3023	3111	3199	3287	3375	3463	3551	3639	
4	3727	3815	3903	3991	4078	4166	4254	4342	4430	4517	
5	4605	4693	4781	4868	4956	5044	5131	5219	5307	5394	
6	5482	5569	5657	5744	5832	5919	6007	6094	6182	6269	
7	6356	6444	6531	6618	6706	6793	6880	6968	7055	7142	
8	7229	7317	7404	7491	7578	7665	7752	7839	7926	8014	
9	8100	8188	8275	8362	8449	8535	8622	8709	8796	8883	

N. 5000 L. 698970] TABLE III. [No. 5499 L. 740284

N.	0	1	2	3	4	5	6	7	8	9	N.
500	698970	9057	9144	9231	9317	9404	9491	9578	9664	9751	500
1	9838	9924	0011	0098	0184	0271	0358	0444	0531	0617	1
2	700704	0790	0877	0963	1050	1136	1222	1309	1395	1482	2
3	1568	1654	1741	1827	1913	1999	2086	2172	2258	2344	3
4	2431	2517	2603	2689	2775	2861	2947	3033	3119	3205	4
5	3291	3377	3463	3549	3635	3721	3807	3893	3979	4065	5
6	4151	4236	4322	4408	4494	4579	4665	4751	4837	4922	6
7	5008	5094	5179	5265	5350	5436	5522	5607	5693	5778	7
8	5864	5949	6035	6120	6206	6291	6376	6462	6547	6632	8
9	6718	6803	6888	6974	7059	7144	7229	7315	7400	7485	9
	0	1	2	3	4	5	6	7	8	9	
510	7570	7655	7740	7826	7911	7996	8081	8166	8251	8336	510
1	8421	8506	8591	8676	8761	8846	8931	9015	9100	9185	1
2	9270	9355	9440	9524	9609	9694	9779	9863	9948	0033	2
3	710117	0202	0287	0371	0456	0540	0625	0710	0794	0879	3
4	0963	1048	1132	1217	1301	1385	1470	1554	1639	1723	4
5	1807	1892	1976	2060	2144	2229	2313	2397	2481	2566	5
6	2650	2734	2818	2902	2986	3070	3154	3238	3323	3407	6
7	3491	3575	3659	3742	3826	3910	3994	4078	4162	4246	7
8	4330	4414	4497	4581	4665	4749	4833	4916	5000	5084	8
9	5167	5251	5335	5418	5502	5586	5669	5753	5836	5920	9
	0	1	2	3	4	5	6	7	8	9	
520	6003	6087	6170	6254	6337	6421	6504	6588	6671	6754	520
1	6838	6921	7004	7088	7171	7254	7338	7421	7504	7587	1
2	7671	7754	7837	7920	8003	8086	8169	8253	8336	8419	2
3	8502	8585	8668	8751	8834	8917	9000	9083	9165	9248	3
4	9331	9414	9497	9580	9663	9745	9828	9911	9994	0077	4
5	720159	0242	0325	0407	0490	0573	0655	0738	0821	0903	5
6	0986	1068	1151	1233	1316	1398	1481	1563	1646	1728	6
7	1811	1893	1975	2058	2140	2222	2305	2387	2469	2552	7
8	2634	2716	2798	2881	2963	3045	3127	3209	3291	3374	8
9	3456	3538	3620	3702	3784	3866	3948	4030	4112	4194	9
	0	1	2	3	4	5	6	7	8	9	
530	4276	4358	4440	4522	4604	4685	4767	4849	4931	5013	530
1	5095	5176	5258	5340	5422	5503	5585	5667	5748	5830	1
2	5912	5993	6075	6156	6238	6320	6401	6483	6564	6646	2
3	6727	6809	6890	6972	7053	7134	7216	7297	7379	7460	3
4	7541	7623	7704	7785	7866	7948	8029	8110	8191	8273	4
5	8354	8435	8516	8597	8678	8759	8841	8922	9003	9084	5
6	9165	9246	9327	9408	9489	9570	9651	9732	9813	9893	6
7	9974	0055	0136	0217	0298	0378	0459	0540	0621	0702	7
8	730782	0863	0944	1024	1105	1186	1266	1347	1428	1508	8
9	1589	1669	1750	1830	1911	1991	2072	2152	2233	2313	9
	0	1	2	3	4	5	6	7	8	9	
540	2394	2474	2555	2635	2715	2796	2876	2956	3037	3117	540
1	3197	3278	3358	3438	3518	3598	3679	3759	3839	3919	1
2	3999	4079	4160	4240	4320	4400	4480	4560	4640	4720	2
3	4800	4880	4960	5040	5120	5199	5279	5359	5439	5519	3
4	5599	5679	5759	5838	5918	5998	6078	6157	6237	6317	4
5	6397	6476	6556	6635	6715	6795	6874	6954	7034	7113	5
6	7193	7272	7352	7431	7511	7590	7670	7749	7829	7908	6
7	7987	8067	8146	8225	8305	8384	8463	8543	8622	8701	7
8	8781	8860	8939	9018	9097	9177	9256	9335	9414	9493	8
9	9572	9651	9731	9810	9889	9968	0047	0126	0205	0284	9

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[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	4684	4762	4840	4919	4997	5
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	5465	5543	5621	5699	5777	6
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[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	7801	7878	7955	8033	8110	9
	0	1	2	3	4	5	6	7	8	9	
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[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	9350	9427	9504	9582	9659	1
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]							
					0045	0123	0200	0277	0354	0431	2
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0894	0971	1048	1125	1202	3
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1587	1664	1741	1818	1895	1972	4
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[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	3889	3966	4042	4119	4195	4272	7
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	4654	4730	4807	4883	4960	5036	8
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	5417	5494	5570	5646	5722	5799	9
	0	1	2	3	4	5	6	7	8	9	
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	6256	6332	6408	6484	6560	570
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	6940	7016	7092	7168	7244	7320	1
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	7700	7775	7851	7927	8003	8079	2
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	8458	8533	8609	8685	8761	8836	3
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	9214	9290	9366	9441	9517	9592	4
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	9970						
						0045	0121	0196	0272	0347	5
6	7 60422	0498	0573	0649	0724	0799	0875	0950	1025	1101	6
7	1176	1251	1326	1402	1477	1552	1627	1702	1778	1853	7
8	1928	2003	2078	2153	2228	2303	2378	2453	2529	2604	8
9	2679	2754	2829	2904	2978	3053	3128	3203	3278	3353	9
	0	1	2	3	4	5	6	7	8	9	
580	3428	3503	3578	3653	3727	3802	3877	3952	4027	4101	580
1	4176	4251	4326	4400	4475	4550	4624	4699	4774	4848	1
2	4923	4998	5072	5147	5221	5296	5370	5445	5520	5594	2
3	5669	5743	5818	5892	5966	6041	6115	6190	6264	6338	3
4	6413	6487	6562	6636	6710	6785	6859	6933	7007	7082	4
5	7156	7230	7304	7379	7453	7527	7601	7675	7749	7823	5
6	7898	7972	8046	8120	8194	8268	8342	8416	8490	8564	6
7	8638	8712	8786	8860	8934	9008	9082	9156	9230	9303	7
8	9377	9451	9525	9599	9673	9746	9820	9894	9968		
										0042	8
9	770115	0189	0263	0336	0410	0484	0557	0631	0705	0778	9
	0	1	2	3	4	5	6	7	8	9	
590	0852	0926	0999	1073	1146	1220	1293	1367	1440	1514	590
1	1587	1661	1734	1808	1881	1955	2028	2102	2175	2248	1
2	2322	2395	2468	2542	2615	2688	2762	2835	2908	2981	2
3	3055	3128	3201	3274	3348	3421	3494	3567	3640	3713	3
4	3786	3860	3933	4006	4079	4152	4225	4298	4371	4444	4
5	4517	4590	4663	4736	4809	4882	4955	5028	5100	5173	5
6	5246	5319	5392	5465	5538	5610	5683	5756	5829	5902	6
7	5974	6047	6120	6193	6265	6338	6411	6483	6556	6629	7
8	6701	6774	6846	6919	6992	7064	7137	7209	7282	7354	8
9	7427	7499	7572	7644	7717	7789	7862	7934	8006	8079	9

N.	0	1	2	3	4	5	6	7	8	9	N.
600	778151	8224	8296	8368	8441	8513	8585	8658	8730	8802	600
1	8874	8947	9019	9091	9163	9236	9308	9380	9452	9524	1
2	9596	9669	9741	9813	9885	9957					
							0029	0101	0173	9245	2
3	780317	0389	0461	0533	0605	0677	0749	0821	0893	0965	3
4	1037	1109	1181	1253	1324	1396	1468	1540	1612	1684	4
5	1755	1827	1899	1971	2042	2114	2186	2258	2329	2401	5
6	2473	2544	2616	2688	2759	2831	2902	2974	3046	3117	6
7	3189	3260	3332	3403	3475	3546	3618	3689	3761	3832	7
8	3904	3975	4046	4118	4189	4261	4332	4403	4475	4546	8
9	4617	4689	4760	4831	4902	4974	5045	5116	5187	5259	9
	0	1	2	3	4	5	6	7	8	9	
610	5330	5401	5472	5543	5615	5686	5757	5828	5899	5970	610
1	6041	6112	6183	6254	6325	6396	6467	6538	6609	6680	1
2	6751	6822	6893	6964	7035	7106	7177	7248	7319	7390	2
3	7460	7531	7602	7673	7744	7815	7885	7956	8027	8098	3
4	8168	8239	8310	8381	8451	8522	8593	8663	8734	8804	4
5	8875	8946	9016	9087	9157	9228	9299	9369	9440	9510	5
6	9581	9651	9722	9792	9863	9933					
							0004	0074	0144	0215	6
7	790285	0356	0426	0496	0567	0637	0707	0778	0848	0918	7
8	0988	1059	1129	1199	1269	1340	1410	1480	1550	1620	8
9	1691	1761	1831	1901	1971	2041	2111	2181	2252	2322	9
	0	1	2	3	4	5	6	7	8	9	
620	2392	2462	2532	2602	2672	2742	2812	2882	2952	3022	620
1	3092	3162	3231	3301	3371	3441	3511	3581	3651	3721	1
2	3790	3860	3930	4000	4070	4139	4209	4279	4349	4418	2
3	4488	4558	4627	4697	4767	4836	4906	4976	5045	5115	3
4	5185	5254	5324	5393	5463	5532	5602	5672	5741	5811	4
5	5880	5949	6019	6088	6158	6227	6297	6366	6436	6505	5
6	6574	6644	6713	6782	6852	6921	6990	7060	7129	7198	6
7	7268	7337	7406	7475	7545	7614	7683	7752	7821	7890	7
8	7960	8029	8098	8167	8236	8305	8374	8443	8513	8582	8
9	8651	8720	8789	8858	8927	8996	9065	9134	9203	9272	9
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630	9341	9409	9478	9547	9616	9685	9754	9823	9892	9961	630
1	800029	0098	0167	0236	0305	0373	0442	0511	0580	0648	1
2	0717	0786	0854	0923	0992	1061	1129	1198	1266	1335	2
3	1404	1472	1541	1609	1678	1747	1815	1884	1952	2021	3
	2089	2158	2226	2295	2363	2432	2500	2568	2637	2705	4
	2774	2842	2910	2979	3047	3116	3184	3252	3321	3389	5
	3457	3525	3594	3662	3730	3798	3867	3935	4003	4071	6
	4139	4208	4276	4344	4412	4480	4548	4616	4685	4753	7
	4821	4889	4957	5025	5093	5161	5229	5297	5365	5433	8
	5501	5569	5637	5705	5773	5841	5908	5976	6044	6112	9
	0	1	2	3	4	5	6	7	8	9	
	6180	6248	6316	6384	6451	6519	6587	6655	6723	6790	640
	6858	6926	6994	7061	7129	7197	7264	7332	7400	7467	1
	7535	7603	7670	7738	7806	7873	7941	8008	8076	8143	2
	8211	8279	8346	8414	8481	8549	8616	8684	8751	8818	3
	8886	8953	9021	9088	9156	9223	9290	9358	9425	9492	4
	9560	9627	9694	9762	9829	9896	9964				
								0031	0098	0165	5
	233	0300	0367	0434	0501	0569	0636	0703	0770	0837	6
	904	0971	1039	1106	1173	1240	1307	1374	1441	1508	7
	575	1642	1709	1776	1843	1910	1977	2044	2111	2178	8
	245	2312	2379	2445	2512	2579	2646	2713	2780	2847	9

No. 6500 L. 812913] TABLE III. [No. 6999 L. 845036

N.	0	1	2	3	4	5	6	7	8	9	N.
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2	4248	4314	4381	4447	4514	4581	4647	4714	4780	4847	2
3	4913	4980	5046	5113	5179	5246	5312	5378	5445	5511	3
4	5578	5644	5711	5777	5843	5910	5976	6042	6109	6175	4
5	6241	6308	6374	6440	6506	6573	6639	6705	6771	6838	5
6	6904	6970	7036	7102	7169	7235	7301	7367	7433	7499	6
7	7565	7631	7698	7764	7830	7896	7962	8028	8094	8160	7
8	8226	8292	8358	8424	8490	8556	8622	8688	8754	8820	8
9	8885	8951	9017	9083	9149	9215	9281	9346	9412	9478	9
	0	1	2	3	4	5	6	7	8	9	
660	9544	9610	9676	9741	9807	9873	9939	0004	0070	0136	660
1	820201	0267	0333	0399	0464	0530	0595	0661	0727	0792	1
2	0858	0924	0989	1055	1120	1186	1251	1317	1382	1448	2
3	1514	1579	1645	1710	1775	1841	1906	1972	2037	2103	3
4	2168	2233	2299	2364	2430	2495	2560	2626	2691	2756	4
5	2822	2887	2952	3018	3083	3148	3213	3279	3344	3409	5
6	3474	3539	3605	3670	3735	3800	3865	3930	3996	4061	6
7	4126	4191	4256	4321	4386	4451	4516	4581	4646	4711	7
8	4776	4841	4906	4971	5036	5101	5166	5231	5296	5361	8
9	5426	5491	5556	5621	5686	5751	5815	5880	5945	6010	9
	0	1	2	3	4	5	6	7	8	9	
670	6075	6140	6204	6269	6334	6399	6464	6528	6593	6658	670
1	6723	6787	6852	6917	6981	7046	7111	7175	7240	7305	1
2	7369	7434	7499	7563	7628	7692	7757	7821	7886	7951	2
3	8015	8080	8144	8209	8273	8338	8402	8467	8531	8595	3
4	8660	8724	8789	8853	8918	8982	9046	9111	9175	9239	4
5	9304	9368	9432	9497	9561	9625	9690	9754	9818	9882	5
6	9947										
		0011	0075	0139	0204	0268	0332	0396	0460	0525	6
7	830589	0653	0717	0781	0845	0909	0973	1037	1102	1166	7
8	1230	1294	1358	1422	1486	1550	1614	1678	1742	1806	8
9	1870	1934	1998	2062	2126	2189	2253	2317	2381	2445	9
	0	1	2	3	4	5	6	7	8	9	
680	2509	2573	2637	2700	2764	2828	2892	2956	3020	3083	680
1	3147	3211	3275	3338	3402	3466	3530	3593	3657	3721	1
2	3784	3848	3912	3975	4039	4103	4166	4230	4294	4357	2
3	4421	4484	4548	4611	4675	4739	4802	4866	4929	4993	3
4	5056	5120	5183	5247	5310	5373	5437	5500	5564	5627	4
5	5691	5754	5817	5881	5944	6007	6071	6134	6197	6261	5
6	6324	6387	6451	6514	6577	6641	6704	6767	6830	6894	6
7	6957	7020	7083	7146	7210	7273	7336	7399	7462	7525	7
8	7588	7652	7715	7778	7841	7904	7967	8030	8093	8156	8
9	8219	8282	8345	8408	8471	8534	8597	8660	8723	8786	9
	0	1	2	3	4	5	6	7	8	9	
690	8849	8912	8975	9038	9101	9164	9227	9289	9352	9415	690
1	9478	9541	9604	9667	9729	9792	9855	9918	9981		
										0043	1
2	840106	0169	0232	0294	0357	0420	0482	0545	0608	0671	2
3	0733	0796	0859	0921	0984	1046	1109	1172	1234	1297	3
4	1359	1422	1485	1547	1610	1672	1735	1797	1860	1922	4
5	1985	2047	2110	2172	2235	2297	2360	2422	2484	2547	5
6	2609	2672	2734	2796	2859	2921	2983	3046	3108	3170	6
7	3233	3295	3357	3420	3482	3544	3606	3669	3731	3793	7
8	3855	3918	3980	4042	4104	4166	4229	4291	4353	4415	8
9	4477	4539	4601	4664	4726	4788	4850	4912	4974	5036	9

No. 7000 L. 845098] TABLE III. [No. 7499 L. 875003

N.	0	1	2	3	4	5	6	7	8	9	N.
700	845098	5160	5222	5284	5346	5408	5470	5532	5594	5656	700
1	5718	5780	5842	5904	5966	6028	6090	6151	6213	6275	1
2	6337	6399	6461	6523	6585	6646	6708	6770	6832	6894	2
3	6955	7017	7079	7141	7202	7264	7326	7388	7449	7511	3
4	7573	7634	7696	7758	7819	7881	7943	8004	8066	8128	4
5	8189	8251	8312	8374	8435	8497	8559	8620	8682	8743	5
6	8805	8866	8928	8989	9051	9112	9174	9235	9297	9358	6
7	9419	9481	9542	9604	9665	9726	9788	9849	9911	9972	7
8	850033	0095	0156	0217	0279	0340	0401	0462	0524	0585	8
9	0646	0707	0769	0830	0891	0952	1014	1075	1136	1197	9

	0	1	2	3	4	5	6	7	8	9	
710	1258	1320	1381	1442	1503	1564	1625	1686	1747	1809	710
1	1870	1931	1992	2053	2114	2175	2236	2297	2358	2419	1
2	2480	2541	2602	2663	2724	2785	2846	2907	2968	3029	2
3	3090	3150	3211	3272	3333	3394	3455	3516	3577	3637	3
4	3698	3759	3820	3881	3941	4002	4063	4124	4185	4245	4
5	4306	4367	4428	4488	4549	4610	4670	4731	4792	4852	5
6	4913	4974	5034	5095	5156	5216	5277	5337	5398	5459	5
7	5519	5580	5640	5701	5761	5822	5882	5943	6003	6064	7
8	6124	6185	6245	6306	6366	6427	6487	6548	6608	6668	8
9	6729	6789	6850	6910	6970	7031	7091	7152	7212	7272	9

	0	1	2	3	4	5	6	7	8	9	
720	7332	7393	7453	7513	7574	7634	7694	7755	7815	7875	720
1	7935	7995	8056	8116	8176	8236	8297	8357	8417	8477	1
2	8537	8597	8657	8718	8778	8838	8898	8958	9018	9078	2
3	9138	9198	9258	9318	9379	9439	9499	9559	9619	9679	3
4	9739	9799	9859	9918	9978						
5	860338	0398	0458	0518	0578	0038	0098	0158	0218	0278	4
6	0937	0996	1056	1116	1176	0637	0697	0757	0817	0877	5
7	1534	1594	1654	1714	1773	1236	1295	1355	1415	1475	6
8	2131	2191	2251	2310	2370	1833	1893	1952	2012	2072	7
9	2728	2787	2847	2906	2966	2430	2489	2549	2608	2668	8
						3025	3085	3144	3204	3263	9

	0	1	2	3	4	5	6	7	8	9	
730	3323	3382	3442	3501	3561	3620	3680	3739	3799	3858	730
1	3917	3977	4036	4096	4155	4214	4274	4333	4392	4452	1
2	4511	4570	4630	4689	4748	4808	4867	4926	4985	5045	2
3	5104	5163	5222	5282	5341	5400	5459	5519	5578	5637	3
4	5696	5755	5814	5874	5933	5992	6051	6110	6169	6228	4
5	6287	6346	6405	6465	6524	6583	6642	6701	6760	6819	5
6	6878	6937	6996	7055	7114	7173	7232	7291	7350	7409	6
7	7467	7526	7585	7644	7703	7762	7821	7880	7939	7998	7
8	8056	8115	8174	8233	8292	8350	8409	8468	8527	8586	8
9	8644	8703	8762	8821	8879	8938	8997	9056	9114	9173	9

	0	1	2	3	4	5	6	7	8	9	
740	9232	9290	9349	9408	9466	9525	9584	9642	9701	9760	740
1	9818	9877	9935	9994							
2	870404	0462	0521	0579	0053	0111	0170	0228	0287	0345	1
3	0989	1047	1106	1164	0638	0696	0755	0813	0872	0930	2
4	1573	1631	1690	1748	1223	1281	1339	1398	1456	1515	3
5	2156	2215	2273	2331	1806	1865	1923	1981	2040	2098	4
6	2739	2797	2855	2913	2389	2448	2506	2564	2622	2681	5
7	3321	3379	3437	3495	2972	3030	3088	3146	3204	3262	6
8	3902	3960	4018	4076	3553	3611	3669	3727	3785	3844	7
9	4482	4540	4598	4656	4134	4192	4250	4308	4366	4424	8
					4714	4772	4830	4888	4945	5003	9

No. 7560 L. 875061] TABLE III. [No. 7999 L. 903036

N.	0	1	2	3	4	5	6	7	8	9	N.
750	875061	5119	5177	5235	5293	5351	5409	5466	5524	5582	750
1	5640	5698	5756	5813	5871	5929	5987	6045	6102	6160	1
2	6218	6276	6333	6391	6449	6507	6564	6622	6680	6737	2
3	6795	6853	6910	6968	7026	7083	7141	7199	7256	7314	3
4	7371	7429	7487	7544	7602	7659	7717	7774	7832	7889	4
5	7947	8004	8062	8119	8177	8234	8292	8349	8407	8464	5
6	8522	8579	8637	8694	8752	8809	8866	8924	8981	9039	6
7	9096	9153	9211	9268	9325	9383	9440	9497	9555	9612	7
8	9669	9726	9784	9841	9898	9956					
							0013	0070	0127	0185	8
9	880242	0299	0356	0413	0471	0528	0585	0642	0699	0756	9
	0	1	2	3	4	5	6	7	8	9	
760	0814	0871	0928	0985	1042	1099	1156	1213	1271	1328	760
1	1385	1442	1499	1556	1613	1670	1727	1784	1841	1898	1
2	1955	2012	2069	2126	2183	2240	2297	2354	2411	2468	2
3	2525	2581	2638	2695	2752	2809	2866	2923	2980	3037	3
4	3093	3150	3207	3264	3321	3377	3434	3491	3548	3605	4
5	3661	3718	3775	3832	3888	3945	4002	4059	4115	4172	5
6	4229	4285	4342	4399	4455	4512	4569	4625	4682	4739	6
7	4795	4852	4909	4965	5022	5078	5135	5192	5248	5305	7
8	5361	5418	5474	5531	5587	5644	5700	5757	5813	5870	8
9	5926	5983	6039	6096	6152	6209	6265	6321	6378	6434	9
	0	1	2	3	4	5	6	7	8	9	
770	6491	6547	6604	6660	6716	6773	6829	6885	6942	6998	770
1	7054	7111	7167	7223	7280	7336	7392	7449	7505	7561	1
2	7617	7674	7730	7786	7842	7898	7955	8011	8067	8123	2
3	8179	8236	8292	8348	8404	8460	8516	8573	8629	8685	3
4	8741	8797	8853	8909	8965	9021	9077	9134	9190	9246	4
5	9302	9358	9414	9470	9526	9582	9638	9694	9750	9806	5
6	9862	9918	9974								
				0030	0086	0141	0197	0253	0309	0365	6
7	890421	0477	0533	0589	0645	0700	0756	0812	0868	0924	7
8	0980	1035	1091	1147	1203	1259	1314	1370	1426	1482	8
9	1537	1593	1649	1705	1760	1816	1872	1928	1983	2039	9
	0	1	2	3	4	5	6	7	8	9	
780	2095	2150	2206	2262	2317	2373	2429	2484	2540	2595	780
1	2651	2707	2762	2818	2873	2929	2985	3040	3096	3151	1
2	3207	3262	3318	3373	3429	3484	3540	3595	3651	3706	2
3	3762	3817	3873	3928	3984	4039	4094	4150	4205	4261	3
4	4316	4371	4427	4482	4538	4593	4648	4704	4759	4814	4
5	4870	4925	4980	5036	5091	5146	5201	5257	5312	5367	5
6	5423	5478	5533	5588	5644	5699	5754	5809	5864	5920	6
7	5975	6030	6085	6140	6195	6251	6306	6361	6416	6471	7
8	6526	6581	6636	6692	6747	6802	6857	6912	6967	7022	8
9	7077	7132	7187	7242	7297	7352	7407	7462	7517	7572	9
	0	1	2	3	4	5	6	7	8	9	
790	7627	7682	7737	7792	7847	7902	7957	8012	8067	8122	790
1	8176	8231	8286	8341	8396	8451	8506	8561	8615	8670	1
2	8725	8780	8835	8890	8944	8999	9054	9109	9164	9218	2
3	9273	9328	9383	9437	9492	9547	9602	9656	9711	9766	3
4	9821	9875	9930	9985							
					0039	0094	0149	0203	0258	0312	4
5	900367	0422	0476	0531	0586	0640	0695	0749	0804	0859	5
6	0913	0968	1022	1077	1131	1186	1240	1295	1349	1404	6
7	1458	1513	1567	1622	1676	1731	1785	1840	1894	1948	7
8	2003	2057	2112	2166	2221	2275	2329	2384	2438	2492	8
9	2547	2601	2655	2710	2764	2818	2873	2927	2981	3036	9

No. 8000 L. 903090] TABLE III. [No. 8499 L. 929388

N.	0	1	2	3	4	5	6	7	8	9	N.
800	903090	3144	3199	3253	3307	3361	3416	3470	3524	3578	800
1	3633	3687	3741	3795	3849	3904	3958	4012	4066	4120	1
2	4174	4229	4283	4337	4391	4445	4499	4553	4607	4661	2
3	4716	4770	4824	4878	4932	4986	5040	5094	5148	5202	3
4	5256	5310	5364	5418	5472	5526	5580	5634	5688	5742	4
5	5796	5850	5904	5958	6012	6066	6119	6173	6227	6281	5
6	6335	6389	6443	6497	6551	6604	6658	6712	6766	6820	6
7	6874	6927	6981	7035	7089	7143	7196	7250	7304	7358	7
8	7411	7465	7519	7573	7626	7680	7734	7787	7841	7895	8
9	7949	8002	8056	8109	8163	8217	8270	8324	8378	8431	9
	0	1	2	3	4	5	6	7	8	9	
810	8485	8539	8592	8646	8699	8753	8807	8860	8914	8967	810
1	9021	9074	9128	9181	9235	9289	9342	9396	9449	9503	1
2	9556	9609	9663	9716	9770	9823	9877	9930	9984		
3	910091	0144	0197	0251	0304	0358	0411	0464	0518	0037	2
4	0624	0678	0731	0784	0838	0891	0944	0998	1051	0571	3
5	1158	1211	1264	1317	1371	1424	1477	1530	1584	1104	4
6	1690	1743	1797	1850	1903	1956	2009	2063	2116	1637	5
7	2222	2275	2328	2381	2435	2488	2541	2594	2647	2169	6
8	2753	2806	2859	2913	2966	3019	3072	3125	3178	2700	7
9	3284	3337	3390	3443	3496	3549	3602	3655	3708	3231	8
	0	1	2	3	4	5	6	7	8	9	9
820	3814	3867	3920	3973	4026	4079	4132	4184	4237	4290	820
1	4343	4396	4449	4502	4555	4608	4660	4713	4766	4819	1
2	4872	4925	4977	5030	5083	5136	5189	5241	5294	5347	2
3	5400	5453	5505	5558	5611	5664	5716	5769	5822	5875	3
4	5927	5980	6033	6085	6138	6191	6243	6296	6349	6401	4
5	6454	6507	6559	6612	6664	6717	6770	6822	6875	6927	5
6	6980	7033	7085	7138	7190	7243	7295	7348	7400	7453	6
7	7506	7558	7611	7663	7716	7768	7820	7873	7925	7978	7
8	8030	8083	8135	8188	8240	8293	8345	8397	8450	8502	8
9	8555	8607	8659	8712	8764	8816	8869	8921	8973	9026	9
	0	1	2	3	4	5	6	7	8	9	
830	9078	9130	9183	9235	9287	9340	9392	9444	9496	9549	830
1	9601	9653	9706	9758	9810	9862	9914	9967			
2	920123	0176	0228	0280	0332	0384	0436	0489	0019	0071	1
3	0645	0697	0749	0801	0853	0906	0958	1010	0541	0593	2
4	1166	1218	1270	1322	1374	1426	1478	1530	1062	1114	3
5	1686	1738	1790	1842	1894	1946	1998	2050	1582	1634	4
6	2206	2258	2310	2362	2414	2466	2518	2570	2102	2154	5
7	2725	2777	2829	2881	2933	2985	3037	3089	2622	2674	6
8	3244	3296	3348	3399	3451	3503	3555	3607	3140	3192	7
9	3762	3814	3865	3917	3969	4021	4072	4124	3658	3710	8
	0	1	2	3	4	5	6	7	4176	4228	9
840	4279	4331	4383	4434	4486	4538	4589	4641	8	9	
1	4796	4848	4899	4951	5003	5054	5106	5157	4693	4744	840
2	5312	5364	5415	5467	5518	5570	5621	5673	5209	5261	1
3	5828	5879	5931	5982	6034	6085	6137	6188	5725	5776	2
4	6342	6394	6445	6497	6548	6600	6651	6702	6239	6291	3
5	6857	6908	6959	7011	7062	7114	7165	7216	6754	6805	4
6	7370	7422	7473	7524	7576	7627	7678	7730	7268	7319	5
7	7883	7935	7986	8037	8088	8140	8191	8242	7781	7832	6
8	8396	8447	8498	8549	8601	8652	8703	8754	8293	8345	7
9	8908	8959	9010	9061	9112	9163	9215	9266	8805	8857	8
									9317	9368	9

[No. 8500 L. 929419] TABLE III. [No. 8999 L. 954194

N.	0	1	2	3	4	5	6	7	8	9	N.
850	929419	9470	9521	9572	9623	9674	9725	9776	9827	9879	850
1	9930	9981	0032	0083	0134	0185	0236	0287	0338	0389	1
2	930440	0491	0542	0592	0643	0694	0745	0796	0847	0898	2
3	0949	1000	1051	1102	1153	1203	1254	1305	1356	1407	3
4	1458	1509	1560	1610	1661	1712	1763	1814	1864	1915	4
5	1966	2017	2068	2118	2169	2220	2271	2322	2372	2423	5
6	2474	2524	2575	2626	2677	2727	2778	2829	2879	2930	6
7	2981	3031	3082	3133	3183	3234	3285	3335	3386	3437	7
8	3487	3538	3589	3639	3690	3740	3791	3841	3892	3943	8
9	3993	4044	4094	4145	4195	4246	4296	4347	4397	4448	9
	0	1	2	3	4	5	6	7	8	9	
860	4498	4549	4599	4650	4700	4751	4801	4852	4902	4953	860
1	5003	5054	5104	5154	5205	5255	5306	5356	5406	5457	1
2	5507	5558	5608	5658	5709	5759	5809	5860	5910	5960	2
3	6011	6061	6111	6162	6212	6262	6313	6363	6413	6463	3
4	6514	6564	6614	6665	6715	6765	6815	6865	6916	6966	4
5	7016	7066	7116	7167	7217	7267	7317	7367	7418	7468	5
6	7518	7568	7618	7668	7718	7769	7819	7869	7919	7969	6
7	8019	8069	8119	8169	8219	8269	8319	8370	8420	8470	7
8	8520	8570	8620	8670	8720	8770	8820	8870	8920	8970	8
9	9020	9070	9120	9170	9220	9270	9319	9369	9419	9469	9
	0	1	2	3	4	5	6	7	8	9	
870	9519	9569	9619	9669	9719	9769	9819	9869	9918	9968	870
1	940018	0068	0118	0168	0218	0267	0317	0367	0417	0467	1
2	0516	0566	0616	0666	0716	0765	0815	0865	0915	0964	2
3	1014	1064	1114	1163	1213	1263	1313	1362	1412	1462	3
4	1511	1561	1611	1660	1710	1760	1809	1859	1909	1958	4
5	2008	2058	2107	2157	2207	2256	2306	2355	2405	2455	5
6	2504	2554	2603	2653	2702	2752	2801	2851	2901	2950	6
7	3000	3049	3099	3148	3198	3247	3297	3346	3396	3445	7
8	3495	3544	3593	3643	3692	3742	3791	3841	3890	3939	8
9	3989	4038	4088	4137	4186	4236	4285	4335	4384	4433	9
	0	1	2	3	4	5	6	7	8	9	
880	4483	4532	4581	4631	4680	4729	4779	4828	4877	4927	880
1	4976	5025	5074	5124	5173	5222	5272	5321	5370	5419	1
2	5469	5518	5567	5616	5665	5715	5764	5813	5862	5912	2
3	5961	6010	6059	6108	6157	6207	6256	6305	6354	6403	3
4	6452	6501	6551	6600	6649	6698	6747	6796	6845	6894	4
5	6943	6992	7041	7090	7139	7189	7238	7287	7336	7385	5
6	7434	7483	7532	7581	7630	7679	7728	7777	7826	7875	6
7	7924	7973	8022	8070	8119	8168	8217	8266	8315	8364	7
8	8413	8462	8511	8560	8608	8657	8706	8755	8804	8853	8
9	8902	8951	8999	9048	9097	9146	9195	9244	9292	9341	9
	0	1	2	3	4	5	6	7	8	9	
890	9390	9439	9488	9536	9585	9634	9683	9731	9780	9829	890
1	9878	9926	9975	0024	0073	0121	0170	0219	0267	0316	1
2	950365	0414	0462	0511	0560	0608	0657	0706	0754	0803	2
3	0851	0900	0949	0997	1046	1095	1143	1192	1240	1289	3
4	1338	1386	1435	1483	1532	1580	1629	1677	1726	1775	4
5	1823	1872	1920	1969	2017	2066	2114	2163	2211	2259	5
6	2308	2356	2405	2453	2502	2550	2599	2647	2696	2744	6
7	2792	2841	2889	2938	2986	3034	3083	3131	3180	3228	7
8	3276	3325	3373	3421	3470	3518	3566	3615	3663	3711	8
9	3760	3808	3856	3905	3953	4001	4049	4098	4146	4194	9

No. 9000 L. 954243] TABLE III. [No. 9499 L. 977678

N.	0	1	2	3	4	5	6	7	8	9	N.
900	954243	4291	4339	4387	4435	4484	4532	4580	4628	4677	900
1	4725	4773	4821	4869	4918	4966	5014	5062	5110	5158	1
2	5207	5255	5303	5352	5399	5447	5495	5543	5592	5640	2
3	5688	5736	5784	5832	5880	5928	5976	6024	6072	6120	3
4	6168	6216	6265	6313	6361	6409	6457	6505	6553	6601	4
5	6649	6697	6745	6793	6840	6888	6936	6984	7032	7080	5
6	7128	7176	7224	7272	7320	7368	7416	7464	7512	7559	6
7	7607	7655	7703	7751	7799	7847	7894	7942	7990	8038	7
8	8086	8134	8181	8229	8277	8325	8373	8421	8468	8516	8
9	8564	8612	8659	8707	8755	8803	8850	8898	8946	8994	9
	0	1	2	3	4	5	6	7	8	9	
910	9041	9089	9137	9185	9232	9280	9328	9375	9423	9471	910
1	9518	9566	9614	9661	9709	9757	9804	9852	9900	9947	1
2	9995										
		0042	0090	0138	0185	0233	0281	0328	0376	0423	2
3	960471	0518	0566	0613	0661	0709	0756	0804	0851	0899	3
4	0946	0994	1041	1089	1136	1184	1231	1279	1326	1374	4
5	1421	1469	1516	1563	1611	1658	1706	1753	1801	1848	5
6	1895	1943	1990	2038	2085	2132	2180	2227	2275	2322	6
7	2369	2417	2464	2511	2559	2606	2653	2701	2748	2795	7
8	2843	2890	2937	2985	3032	3079	3126	3174	3221	3268	8
9	3316	3363	3410	3457	3504	3552	3599	3646	3693	3741	9
	0	1	2	3	4	5	6	7	8	9	
920	3788	3835	3882	3929	3977	4024	4071	4118	4165	4212	920
1	4260	4307	4354	4401	4448	4495	4542	4590	4637	4684	1
2	4731	4778	4825	4872	4919	4966	5013	5060	5108	5155	2
3	5202	5249	5296	5343	5390	5437	5484	5531	5578	5625	3
4	5672	5719	5766	5813	5860	5907	5954	6001	6048	6095	4
5	6142	6189	6236	6283	6329	6376	6423	6470	6517	6564	5
6	6611	6658	6705	6752	6799	6845	6892	6939	6986	7033	6
7	7080	7127	7173	7220	7267	7314	7361	7408	7454	7501	7
8	7548	7595	7642	7688	7735	7782	7829	7875	7922	7969	8
9	8016	8062	8109	8156	8203	8249	8296	8343	8389	8436	9
	0	1	2	3	4	5	6	7	8	9	
930	8483	8530	8576	8623	8670	8716	8763	8810	8856	8903	930
1	8950	8996	9043	9090	9136	9183	9229	9276	9323	9369	1
2	9416	9463	9509	9556	9602	9649	9695	9742	9789	9835	2
3	9882	9928	9975								
				0021	0068	0114	0161	0207	0254	0300	3
4	970347	0393	0440	0486	0533	0579	0626	0672	0719	0765	4
5	0812	0858	0904	0951	0997	1044	1090	1137	1183	1229	5
6	1276	1322	1369	1415	1461	1508	1554	1601	1647	1693	6
7	1740	1786	1832	1879	1925	1971	2018	2064	2110	2157	7
8	2203	2249	2295	2342	2388	2434	2481	2527	2573	2619	8
9	2666	2712	2758	2804	2851	2897	2943	2989	3035	3082	9
	0	1	2	3	4	5	6	7	8	9	
940	3128	3174	3220	3266	3313	3359	3405	3451	3497	3543	940
1	3590	3636	3682	3728	3774	3820	3866	3913	3959	4005	1
2	4051	4097	4143	4189	4235	4281	4327	4374	4420	4466	2
3	4512	4558	4604	4650	4696	4742	4788	4834	4880	4926	3
4	4972	5018	5064	5110	5156	5202	5248	5294	5340	5386	4
5	5432	5478	5524	5570	5616	5662	5707	5753	5799	5845	5
6	5891	5937	5983	6029	6075	6121	6167	6212	6258	6304	6
7	6350	6396	6442	6488	6533	6579	6625	6671	6717	6763	7
8	6808	6854	6900	6946	6992	7037	7083	7129	7175	7220	8
9	7266	7312	7358	7403	7449	7495	7541	7586	7632	7678	9

No. 9500 L. 977724] TABLE III. [No. 9999 L. 999957

N.	0	1	2	3	4	5	6	7	8	9	N.
950	977724	7769	7815	7861	7906	7952	7998	8043	8089	8135	950
1	8181	8226	8272	8317	8363	8409	8454	8500	8546	8591	1
2	8637	8683	8728	8774	8819	8865	8911	8956	9002	9047	2
3	9093	9138	9184	9230	9275	9321	9366	9412	9457	9503	3
4	9548	9594	9639	9685	9730	9776	9821	9867	9912	9958	4
5	980003	0049	0094	0140	0185	0231	0276	0322	0367	0412	5
6	0458	0503	0549	0594	0640	0685	0730	0776	0821	0867	6
7	0912	0957	1003	1048	1093	1139	1184	1229	1275	1320	7
8	1366	1411	1456	1501	1547	1592	1637	1683	1728	1773	8
9	1819	1864	1909	1954	2000	2045	2090	2135	2181	2226	9
	0	1	2	3	4	5	6	7	8	9	
960	2271	2316	2362	2407	2452	2497	2543	2588	2633	2678	960
1	2723	2769	2814	2859	2904	2949	2994	3040	3085	3130	1
2	3175	3220	3265	3310	3356	3401	3446	3491	3536	3581	2
3	3626	3671	3716	3762	3807	3852	3897	3942	3987	4032	3
4	4077	4122	4167	4212	4257	4302	4347	4392	4437	4482	4
5	4527	4572	4617	4662	4707	4752	4797	4842	4887	4932	5
6	4977	5022	5067	5112	5157	5202	5247	5292	5337	5382	6
7	5426	5471	5516	5561	5606	5651	5696	5741	5786	5830	7
8	5875	5920	5965	6010	6055	6100	6144	6189	6234	6279	8
9	6324	6369	6413	6458	6503	6548	6593	6637	6682	6727	9
	0	1	2	3	4	5	6	7	8	9	
970	6772	6817	6861	6906	6951	6996	7040	7085	7130	7175	970
1	7219	7264	7309	7353	7398	7443	7488	7532	7577	7622	1
2	7666	7711	7756	7800	7845	7890	7934	7979	8024	8068	2
3	8113	8157	8202	8247	8291	8336	8381	8425	8470	8514	3
4	8559	8604	8648	8693	8737	8782	8826	8871	8916	8960	4
5	9005	9049	9094	9138	9183	9227	9272	9316	9361	9405	5
6	9450	9494	9539	9583	9628	9672	9717	9761	9806	9850	6
7	9895	9939	9983	0028	0072	0117	0161	0206	0250	0294	7
8	990339	0383	0428	0472	0516	0561	0605	0650	0694	0738	8
9	0783	0827	0871	0916	0960	1004	1049	1093	1137	1182	9
	0	1	2	3	4	5	6	7	8	9	
980	1226	1270	1315	1359	1403	1448	1492	1536	1580	1625	980
1	1669	1713	1758	1802	1846	1890	1935	1979	2023	2067	1
2	2111	2156	2200	2244	2288	2333	2377	2421	2465	2509	2
3	2554	2598	2642	2686	2730	2774	2819	2863	2907	2951	3
4	2995	3039	3083	3127	3172	3216	3260	3304	3348	3392	4
5	3436	3480	3524	3568	3613	3657	3701	3745	3789	3833	5
6	3877	3921	3965	4009	4053	4097	4141	4185	4229	4273	6
7	4317	4361	4405	4449	4493	4537	4581	4625	4669	4713	7
8	4757	4801	4845	4889	4933	4977	5021	5065	5108	5152	8
9	5196	5240	5284	5328	5372	5416	5460	5504	5547	5591	9
	0	1	2	3	4	5	6	7	8	9	
990	5635	5679	5723	5767	5811	5854	5898	5942	5986	6030	990
1	6074	6117	6161	6205	6249	6293	6337	6380	6424	6468	1
2	6512	6555	6599	6643	6687	6731	6774	6818	6862	6906	2
3	6949	6993	7037	7080	7124	7168	7212	7255	7299	7343	3
4	7386	7430	7474	7517	7561	7605	7648	7692	7736	7779	4
5	7823	7867	7910	7954	7998	8041	8085	8129	8172	8216	5
6	8259	8303	8347	8390	8434	8477	8521	8564	8608	8652	6
7	8695	8739	8782	8826	8869	8913	8956	9000	9043	9087	7
8	9130	9174	9218	9261	9305	9348	9392	9435	9479	9522	8
9	9565	9609	9652	9696	9739	9783	9826	9870	9913	9957	9

0 DEG. TABLE IV.—LOG. SINES, ETC.

′	Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	
0	0.000000		Infinite.	0.000000		Infinite.	10.000000		10.000000	60
1	6.463726	501717	13.536274	6.463726	501717	13.536274	0		0	59
2	764756	293485	235244	764756	293485	235244	0		0	58
3	940847	208231	059153	940847	208231	059153	0		0	57
4	7.065786	161517	12.934214	7.065786	161517	12.934214	0		0	56
5	162696	131968	837304	162696	131969	837304	0		0	55
6	241877	111578	758123	241878	111578	758122	1	1	9.999999	54
7	7.308824		12.691176	7.308825		12.691175	10.000001	1	9.999999	53
8	366816	96653	633184	366817	96653	633183	1	1	9	52
9	417968	85254	582032	417970	85254	582030	1	1	9	51
10	463725	76262	536275	463727	76263	536273	2	1	8	50
11	505118	68988	494882	505120	68988	494880	2	1	8	49
12	542906	62981	457094	542909	62981	457091	3	1	7	48
13	7.577668	57936	12.422332	7.577672	57938	12.422328	10.000003	1	9.999997	47
14	609853	53641	390147	609857	53642	390143	4	1	6	46
15	639816	49938	360184	639820	49939	360180	4	1	6	45
16	667845	46714	332155	667849	46715	332151	5	1	5	44
17	694173	43881	305827	694179	43882	305821	5	1	5	43
18	718997	41372	281003	719003	41373	280997	6	1	4	42
19	7.742477	39135	12.257523	7.742484	39136	12.257516	10.000007	1	9.999993	41
20	764754	37127	235246	764761	37128	235239	7	1	3	40
21	785943	35315	214057	785951	35316	214049	8	1	2	39
22	806146	33672	193854	806155	33673	193845	9	1	1	38
23	825451	32175	174549	825460	32176	174540	10	1	0	37
24	843934	30805	156066	843944	30807	156056	11	1	89	36
25	7.861662	29547	12.138338	7.861674	29549	12.138326	10.000011	2	9.999989	35
26	878695	28388	121305	878708	28390	121292	12	2	88	34
27	895085	27317	104915	895099	27318	104901	13	2	87	33
28	910879	26323	089121	910894	26325	089106	14	2	86	32
29	926119	25399	073881	926134	25401	073866	15	2	85	31
30	940842	24538	059158	940858	24540	059142	17	2	83	30
31	7.955082	23733	12.044918	7.955100	23735	12.044900	10.000018	2	9.999982	29
32	968870	22980	031130	968889	22982	031111	19	2	81	28
33	982233	22273	017767	982253	22275	017747	20	2	80	27
34	995198	21608	004802	995219	21610	004781	21	2	79	26
35	8.007787	20981	11.992213	8.007809	20983	11.992191	23	2	77	25
36	020021	20390	979979	020045	20392	979955	24	2	76	24
37	8.031919	19831	11.968081	8.031945	19833	11.968055	10.000025	2	9.999975	23
38	043501	19302	956499	043527	19305	956473	27	2	73	22
39	054781	18801	945219	054809	18803	945191	28	2	72	21
40	065776	18325	934224	065806	18327	934194	29	2	71	20
41	076500	17872	923500	076531	17875	923469	31	2	69	19
42	086965	17441	913035	086997	17444	913003	32	2	68	18
43	8.097183	17031	11.902817	8.097217	17034	11.902783	10.000034	2	9.999966	17
44	107167	16639	892833	107202	16642	892798	36	3	64	16
45	116926	16265	883074	116963	16268	883037	37	3	63	15
46	126471	15908	873529	126510	15912	873490	39	3	61	14
47	135810	15566	864190	135851	15568	864149	41	3	59	13
48	144953	15238	855047	144996	15241	855004	42	3	58	12
49	8.153907	14924	11.846093	8.153952	14927	11.846048	10.000044	3	9.999956	11
50	162681	14622	837319	162727	14625	837273	46	3	54	10
51	171280	14333	828720	171328	14336	828672	48	3	52	9
52	179713	14054	820287	179763	14057	820237	50	3	50	8
53	187985	13786	812015	188036	13790	811964	52	3	48	7
54	196102	13529	803898	196156	13532	803844	54	3	46	6
55	8.204070	13280	11.795930	8.204126	13284	11.795874	10.000056	3	9.999944	5
56	211895	13041	788105	211953	13044	788047	58	4	42	4
57	219581	12810	780419	219641	12814	780359	60	4	40	3
58	227134	12587	772866	227195	12590	772805	62	4	38	2
59	234557	12372	765443	234621	12376	765379	64	4	36	1
60	241855	12164	758145	241921	12168	758079	66	4	34	0
	Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	′

89 DEG.

1 DEG. TABLE IV.—LOG. SINES, ETC.

′	Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	
0	8.241855	11963	11.758145	8.241921	11967	11.758079	10.000066	4	9.999934	60
1	42033	11768	50967	49102	11772	50898	68	4	32	59
2	56094	11580	43906	56165	11584	43835	71	4	29	58
3	63042	11398	36958	63115	11402	36885	73	4	27	57
4	69881	11221	30119	69956	11225	30044	75	4	25	56
5	76614	11050	23386	76691	11054	23309	78	4	22	55
6	83243	10883	16757	83323	10887	16677	80	4	20	54
7	8.289773	10723	11.710227	8.289856	10726	11.710144	10.000082	4	9.999918	53
8	96207	10565	03793	96292	10570	03708	85	4	15	52
9	8.302546	10413	11.697454	8.302634	10418	11.697366	87	4	13	51
10	08794	10266	91206	08884	10270	91116	90	4	10	50
11	14954	10122	85046	15046	10126	84954	93	4	07	49
12	21027	9982	78973	21122	9987	78878	95	4	05	48
13	8.327016	9847	11.672984	8.327114	9851	11.672886	10.000098	4	9.999902	47
14	32924	9714	67076	33025	9719	66975	10.000101	5	9.999899	46
15	38753	9586	61247	38856	9590	61144	03	5	97	45
16	44504	9460	55496	44610	9465	55390	06	5	94	44
17	50181	9338	49819	50289	9343	49711	09	5	91	43
18	55783	9219	44217	55895	9224	44105	12	5	88	42
19	8.361315	9103	11.638685	8.361430	9108	11.638570	10.000115	5	9.999885	41
20	66777	8990	33223	66895	8995	33105	18	5	82	40
21	72171	8880	27829	72292	8885	27708	21	5	79	39
22	77499	8772	22501	77622	8777	22378	24	5	76	38
23	82762	8667	17238	82889	8672	17111	27	5	73	37
24	87962	8564	12038	88092	8570	11908	30	5	70	36
25	8.393101	8464	11.606899	8.393234	8470	11.606766	10.000133	5	9.999867	35
26	98179	8366	01821	98315	8371	01685	36	5	64	34
27	8.403199	8271	11.596801	8.403338	8276	11.596662	39	5	61	33
28	08161	8177	91839	08304	8182	91696	42	5	58	32
29	13068	8086	86932	13213	8091	86787	46	5	54	31
30	17919	7996	82081	18068	8002	81932	49	5	51	30
31	8.422717	7909	11.577283	8.422869	7914	11.577131	10.000152	6	9.999848	29
32	27462	7823	72538	27618	7828	72382	56	6	44	28
33	32156	7740	67844	32315	7745	67685	59	6	41	27
34	36800	7657	63200	36962	7663	63038	62	6	38	26
35	41394	7577	58606	41560	7583	58440	66	6	34	25
36	45941	7499	54059	46110	7505	53890	69	6	31	24
37	8.450440	7422	11.549560	8.450613	7428	11.549387	10.000173	6	9.999827	23
38	54893	7346	45107	55070	7352	44930	77	6	23	22
39	59301	7273	40699	59481	7279	40519	80	6	20	21
40	63665	7200	36335	63849	7206	36151	84	6	16	20
41	67985	7129	32015	68172	7135	31828	88	6	12	19
42	72263	7060	27737	72454	7066	27546	91	6	09	18
43	8.476498	6991	11.523502	8.476693	6998	11.523307	10.000195	6	9.999805	17
44	80693	6924	19307	80892	6931	19108	99	6	01	16
45	84848	6859	15152	85050	6865	14950	10.000208	6	9.999797	15
46	88963	6794	11037	89170	6801	10830	07	7	93	14
47	93040	6731	06960	93250	6738	06750	10	7	90	13
48	97078	6669	02922	97293	6676	02707	14	7	86	12
49	8.501080	6608	11.498920	8.501298	6615	11.498702	10.000218	7	9.999782	11
50	05045	6548	94955	05267	6555	94733	22	7	78	10
51	08974	6489	91026	09200	6496	90800	26	7	74	9
52	12867	6432	87133	13098	6439	86902	31	7	69	8
53	16726	6375	83274	16961	6382	83039	35	7	65	7
54	20551	6319	79449	20790	6326	79210	39	7	61	6
55	8.524343	6264	11.475657	8.524586	6272	11.475414	10.000243	7	9.999757	5
56	28102	6211	71898	28349	6218	71651	47	7	53	4
57	31828	6158	68172	32080	6165	67920	52	7	48	3
58	35523	6106	64477	35779	6113	64221	56	7	44	2
59	39186	6055	60814	39447	6062	60553	60	7	40	1
60	42819		57181	43084		56916	65	7	35	0
′	Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	′

88 DEG.

2 DEG. TABLE IV.—LOG. SINES, ETC.

′	Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	
0	8.542819	6004	11.457181	8.543084	6012	11.456916	10.000265	7	9.999735	60
1	46422	5955	53578	46691	5962	53309	69	7	31	59
2	49995	5906	50005	50268	5914	49732	74	7	26	58
3	53539	5858	46461	53817	5866	46183	78	8	22	57
4	57054	5811	42946	57336	5819	42664	83	8	17	56
5	60540	5765	39460	60828	5773	39172	87	8	13	55
6	63999	5719	36001	64291	5727	35709	92	8	08	54
7	8.567431	5674	11.432569	8.567727	5682	11.432273	10.000296	8	9.999704	53
8	70836	5630	29164	71137	5638	28863	10.000301	8	9.999699	52
9	74214	5587	25786	74520	5595	25480	06	8	94	51
10	77566	5544	22434	77877	5552	22123	11	8	89	50
11	80892	5502	19108	81208	5510	18792	15	8	85	49
12	84193	5460	15807	84514	5468	15486	20	8	80	48
13	8.587469	5419	11.412531	8.587795	5427	11.412205	10.000325	8	9.999675	47
14	90721	5379	09279	91051	5387	08949	30	8	70	46
15	93948	5339	06052	94283	5347	05717	35	8	65	45
16	97152	5300	02848	97492	5308	02508	40	8	60	44
17	8.600332	5261	11.399668	8.600677	5270	11.399323	45	8	55	43
18	03489	5223	96511	03839	5232	96161	50	8	50	42
19	8.606628	5186	11.393377	8.606978	5194	11.393022	10.000355	9	9.999645	41
20	09734	5149	90266	10094	5158	89906	60	9	40	40
21	12823	5112	87177	13189	5121	86811	65	9	35	39
22	15891	5076	84109	16262	5085	83738	71	9	29	38
23	18937	5041	81063	19313	5050	80687	76	9	24	37
24	21962	5006	78038	22343	5015	77657	81	9	19	36
25	8.624965	4972	11.375035	8.625352	4981	11.374648	10.000386	9	9.999614	35
26	27948	4938	72052	28340	4947	71660	92	9	08	34
27	30911	4904	69089	31308	4913	68692	97	9	03	33
28	33854	4871	66146	34256	4880	65744	10.000403	9	9.999597	32
29	36776	4839	63224	37184	4848	62816	08	9	92	31
30	39680	4806	60320	40093	4816	59907	14	9	86	30
31	8.642563	4775	11.357437	8.642982	4784	11.357018	10.000419	9	9.999581	29
32	45428	4743	54572	45853	4753	54147	25	9	75	28
33	48274	4712	51726	48704	4722	51296	30	9	70	27
34	51102	4682	48898	51537	4691	48463	36	9	64	26
35	53911	4652	46089	54352	4661	45648	42	10	58	25
36	56702	4622	43298	57149	4631	42851	47	10	53	24
37	8.659475	4592	11.340525	8.659928	4602	11.340072	10.000453	10	9.999547	23
38	62230	4563	37770	62689	4573	37311	59	10	41	22
39	64968	4535	35032	65433	4544	34567	65	10	35	21
40	67689	4506	32311	68160	4517	31840	71	10	29	20
41	70393	4479	29607	70870	4488	29130	76	10	24	19
42	73080	4451	26920	73563	4461	26437	82	10	18	18
43	8.675751	4424	11.324249	8.676239	4434	11.323761	10.000488	10	9.999512	17
44	78405	4397	21595	78900	4407	21100	94	10	06	16
45	81043	4370	18957	81544	4380	18456	10.000500	10	00	15
46	83665	4344	16335	84172	4354	15828	07	10	9.999493	14
47	86272	4318	13728	86784	4328	13216	13	10	87	13
48	88863	4292	11137	89381	4303	10619	19	10	81	12
49	8.691438	4267	11.308562	8.691963	4277	11.308037	10.000525	10	9.999475	11
50	93998	4242	06002	94529	4252	05471	31	10	69	10
51	96543	4217	03457	97081	4228	02919	37	11	63	9
52	99073	4192	00927	99617	4203	00383	44	11	56	8
53	8.701589	4168	11.298411	8.702139	4179	11.297861	50	11	50	7
54	04090	4144	95910	04646	4155	95354	57	11	43	6
55	8.706577	4121	11.293423	8.707140	4132	11.292860	10.000563	11	9.999437	5
56	09049	4097	90951	09618	4108	90382	69	11	31	4
57	11507	4074	88493	12083	4085	87917	76	11	24	3
58	13952	4051	86048	14534	4062	85466	82	11	18	2
59	16383	4029	83617	16972	4040	83028	89	11	11	1
60	18800		81200	19396		80604	96		04	0
	Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	′

87 DEG.

3 Deg. TABLE IV.—Log. Sines, etc.

′	Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	
0	8.718800	4006	11.281200	8.719396	4017	11.280604	10.000596	11	9.999404	60
1	21204	3984	78796	21806	3995	78194	10.000602	11	9.999398	59
2	23595	3962	76405	24204	3974	75796	09	11	91	58
3	25972	3941	74028	26588	3952	73412	16	11	84	57
4	28337	3919	71663	28959	3930	71041	22	11	78	56
5	30688	3898	69312	31317	3909	68683	29	11	71	55
6	33027	3877	66973	33663	3889	66337	36	12	64	54
7	8.735354	3857	11.264646	8.735996	3868	11.264004	10.000643	12	9.999357	53
8	37667	3836	62333	38317	3848	61683	50	12	50	52
9	39969	3816	60031	40626	3827	59374	57	12	43	51
10	42259	3796	57741	42922	3807	57078	64	12	36	50
11	44536	3776	55464	45207	3787	54793	71	12	29	49
12	46802	3756	53198	47479	3768	52521	78	12	22	48
13	8.749055	3737	11.250945	8.749740	3749	11.250260	10.000685	12	9.999315	47
14	51297	3717	48703	51989	3729	48011	92	12	08	46
15	53528	3698	46472	54227	3710	45773	99	12	01	45
16	55747	3678	44253	56453	3692	43547	10.000706	12	9.999294	44
17	57955	3661	42045	58668	3673	41332	14	12	86	43
18	60151	3642	39849	60872	3655	39128	21	12	79	42
19	8.762337	3624	11.237663	8.763065	3636	11.236935	10.000728	12	9.999272	41
20	64511	3606	35489	65246	3618	34754	35	12	65	40
21	66675	3588	33325	67417	3600	32583	43	12	57	39
22	68828	3570	31172	69578	3583	30422	50	13	50	38
23	70970	3553	29030	71727	3565	28273	58	13	42	37
24	73101	3535	26899	73866	3548	26134	65	13	35	36
25	8.775223	3518	11.224777	8.775995	3531	11.224005	10.000773	13	9.999227	35
26	77333	3501	22667	78114	3514	21886	80	13	20	34
27	79434	3484	20566	80222	3497	19778	88	13	12	33
28	81524	3467	18476	82320	3480	17680	95	13	05	32
29	83605	3451	16395	84408	3464	15592	10.000803	13	9.999197	31
30	85675	3434	14325	86486	3447	13514	11	13	89	30
31	8.787736	3418	11.212264	8.788554	3431	11.211446	10.000819	13	9.999181	29
32	89787	3402	10213	90613	3415	09387	26	13	74	28
33	91828	3386	08172	92662	3399	07338	34	13	66	27
34	93859	3370	06141	94701	3383	05299	42	13	58	26
35	95881	3354	04119	96731	3368	03269	50	13	50	25
36	97894	3339	02106	98752	3352	01248	58	13	42	24
37	8.799897	3323	11.200103	8.800763	3337	11.199237	10.000866	13	9.999134	23
38	8.801892	3308	11.198108	02765	3322	97235	74	13	26	22
39	03876	3293	96124	04758	3307	95242	82	13	18	21
40	05852	3278	94148	06742	3292	93258	90	13	10	20
41	07819	3263	92181	08717	3277	91283	98	13	02	19
42	09777	3249	90223	10683	3262	89317	10.000906	14	9.999094	18
43	8.811726	3234	11.188274	8.812641	3248	11.187359	10.000914	14	9.999086	17
44	13667	3219	86333	14589	3233	85411	23	14	77	16
45	15599	3205	84401	16529	3219	83471	31	14	69	15
46	17522	3191	82478	18461	3205	81539	39	14	61	14
47	19436	3177	80564	20384	3191	79616	47	14	53	13
48	21343	3163	78657	22298	3177	77702	56	14	44	12
49	8.823240	3149	11.176760	8.824205	3163	11.175795	10.000964	14	9.999036	11
50	25130	3135	74870	26103	3150	73897	73	14	27	10
51	27011	3122	72989	27992	3136	72008	81	14	19	9
52	28884	3108	71116	29874	3123	70126	90	14	10	8
53	30749	3095	69251	31748	3108	68252	98	14	02	7
54	32607	3082	67393	33613	3096	66387	10.001007	14	9.998993	6
55	8.834456	3069	11.165544	8.835471	3083	11.164529	10.001016	14	9.998984	5
56	36297	3056	63703	37321	3070	62679	24	14	76	4
57	38130	3043	61870	39163	3057	60837	33	14	67	3
58	39956	3030	60044	40998	3045	59002	42	15	58	2
59	.41774	3017	58226	42825	3032	57175	50	15	50	1
60	43585		56415	44644		55356	59	15	41	0
	Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	′

86 Deg.

4 DEG. TABLE IV.—LOG. SINES, ETC.

′	Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	
0	8.843585	3005	11.156415	8.844644	3019	11.155356	10.001059	15	9.998941	60
1	45387	2992	54613	46455	3007	53545	068	15	932	59
2	47183	2980	52817	48260	2995	51740	077	15	923	58
3	48971	2967	51029	50057	2982	49943	086	15	914	57
4	50751	2955	49249	51846	2970	48154	095	15	905	56
5	52525	2943	47475	53628	2958	46372	104	15	896	55
6	54291	2931	45709	55403	2946	44597	113	15	887	54
7	8.856049	2919	11.143951	8.857171	2935	11.142829	10.001122	15	9.998878	53
8	57801	2908	42199	58932	2923	41068	131	15	869	52
9	59546	2896	40454	60686	2911	39314	140	15	860	51
10	61283	2884	38717	62433	2900	37567	149	15	851	50
11	63014	2873	36986	64173	2888	35827	159	15	841	49
12	64738	2861	35262	65906	2877	34094	168	15	832	48
13	8.866455	2850	11.133545	8.867632	2866	11.132368	10.001177	16	9.998823	47
14	68165	2839	31835	69351	2854	30649	187	16	813	46
15	69868	2828	30132	71064	2843	28936	196	16	804	45
16	71565	2817	28435	72770	2832	27230	205	16	795	44
17	73255	2806	26745	74469	2821	25531	215	16	785	43
18	74938	2795	25062	76162	2811	23838	224	16	776	42
19	8.876615	2783	11.123385	8.877849	2800	11.122151	10.001234	16	9.998766	41
20	78285	2773	21715	79529	2789	20471	243	16	757	40
21	79949	2763	20051	81202	2779	18798	253	16	747	39
22	81607	2752	18393	82869	2768	17131	262	16	738	38
23	83258	2742	16742	84530	2758	15470	272	16	728	37
24	84903	2731	15097	86185	2747	13815	282	16	718	36
25	8.886542	2721	11.113458	8.887833	2737	11.112167	10.001292	16	9.998708	35
26	88174	2711	11826	89476	2727	10524	301	16	699	34
27	89801	2700	10199	91112	2717	08888	311	16	689	33
28	91421	2690	08579	92742	2707	07258	321	16	679	32
29	93035	2680	06965	94366	2697	05634	331	17	669	31
30	94643	2670	05357	95984	2687	04016	341	17	659	30
31	8.896246	2660	11.103754	8.897596	2677	11.102404	10.001351	17	9.998649	29
32	97842	2651	02158	99203	2667	00797	361	17	639	28
33	99432	2641	00568	8.900803	2658	11.099197	371	17	629	27
34	8.901017	2631	11.098983	02398	2648	97602	381	17	619	26
35	02596	2622	97404	03987	2638	96013	391	17	609	25
36	04169	2612	95831	05570	2629	94430	401	17	599	24
37	8.905736	2603	11.094264	8.907147	2620	11.092853	10.001411	17	9.998589	23
38	07297	2593	92703	08719	2610	91281	422	17	578	22
39	08853	2584	91147	10285	2601	89715	432	17	568	21
40	10404	2575	89596	11846	2592	88154	442	17	558	20
41	11949	2566	88051	13401	2583	86599	452	17	548	19
42	13488	2556	86512	14951	2574	85049	463	17	537	18
43	8.915022	2547	11.084978	8.916495	2565	11.083505	10.001473	17	9.998527	17
44	16550	2538	83450	18034	2556	81966	484	18	516	16
45	18073	2529	81927	19568	2547	80432	494	18	506	15
46	19591	2520	80409	21096	2538	78904	505	18	495	14
47	21103	2512	78897	22619	2530	77381	515	18	485	13
48	22610	2503	77390	24136	2521	75864	526	18	474	12
49	8.924112	2494	11.075888	8.925649	2512	11.074351	10.001536	18	9.998464	11
50	25609	2486	74391	27156	2503	72844	547	18	453	10
51	27100	2477	72900	28658	2495	71342	558	18	442	9
52	28587	2469	71413	30155	2486	69845	569	18	431	8
53	30068	2460	69932	31647	2478	68353	579	18	421	7
54	31544	2452	68456	33134	2470	66866	590	18	410	6
55	8.933015	2443	11.066985	8.934616	2461	11.065384	10.001601	18	9.998399	5
56	34481	2435	65519	36093	2453	63907	612	18	388	4
57	35942	2427	64058	37565	2445	62435	623	18	377	3
58	37398	2419	62602	39032	2437	60968	634	18	366	2
59	38850	2411	61150	40494	2430	59506	645	18	355	1
60	40296		59704	41952		58048	656	18	344	0
	Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	′

85 DEG.

5 DEG. TABLE IV.—LOG. SINES, ETC.

′	Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	
0	8.940296	2403	11.059704	8.941952	2421	11.058048	10.001656	19	9.998344	60
1	41738	2394	58262	43404	2413	56596	667	19	333	59
2	43174	2387	56826	44852	2405	55148	678	19	322	58
3	44606	2379	55394	46295	2397	53705	689	19	311	57
4	46034	2371	53966	47734	2390	52266	700	19	300	56
5	47456	2363	52544	49168	2382	50832	711	19	289	55
6	48874	2355	51126	50597	2374	49403	723	19	277	54
7	8.950287	2348	11.049713	8.952021	2366	11.047979	10.001734	19	9.998266	53
8	51696	2340	48304	53441	2358	46559	745	19	255	52
9	53100	2332	46900	54856	2351	45144	757	19	243	51
10	54499	2325	45501	56267	2344	43733	768	19	232	50
11	55894	2317	44106	57674	2335	42326	780	19	220	49
12	57284	2310	42716	59075	2329	40925	791	19	209	48
13	8.958670	2302	11.041330	8.960473	2321	11.039527	10.001803	19	9.998197	47
14	60052	2295	39948	61866	2314	38134	814	19	186	46
15	61429	2288	38571	63255	2307	36745	826	19	174	45
16	62801	2280	37199	64639	2300	35361	837	19	163	44
17	64170	2273	35830	66019	2293	33981	849	19	151	43
18	65534	2266	34466	67394	2286	32606	861	19	139	42
19	8.966893	2259	11.033107	8.968766	2279	11.031234	10.001872	20	9.998128	41
20	68249	2252	31751	70133	2271	29867	884	20	116	40
21	69600	2245	30400	71496	2265	28504	896	20	104	39
22	70947	2238	29053	72855	2257	27145	908	20	092	38
23	72289	2231	27711	74209	2251	25791	920	20	080	37
24	73628	2224	26372	75560	2244	24440	932	20	068	36
25	8.974962	2217	11.025038	8.976906	2237	11.023094	10.001944	20	9.998056	35
26	76293	2210	23707	78248	2230	21752	956	20	044	34
27	77619	2203	22381	79586	2223	20414	968	20	032	33
28	78941	2197	21059	80921	2217	19079	980	20	020	32
29	80259	2190	19741	82251	2210	17749	992	20	008	31
30	81573	2183	18427	83577	2204	16423	10.002004	20	9.997996	30
31	8.982883	2177	11.017117	8.984899	2197	11.015101	10.002016	20	9.997984	29
32	84189	2170	15811	86217	2191	13783	028	20	972	28
33	85491	2163	14509	87532	2184	12468	041	20	959	27
34	86789	2157	13211	88842	2178	11158	053	20	947	26
35	88083	2150	11917	90149	2171	09851	065	21	935	25
36	89374	2144	10626	91451	2165	08549	078	21	922	24
37	8.990660	2138	11.009340	8.992750	2158	11.007250	10.002090	21	9.997910	23
38	91943	2131	08057	94045	2152	05955	103	21	897	22
39	93222	2125	06778	95337	2146	04663	115	21	885	21
40	94497	2119	05503	96624	2140	03376	128	21	872	20
41	95768	2112	04232	97908	2134	02092	140	21	860	19
42	97036	2106	02964	99188	2127	00812	153	21	847	18
43	8.998299	2100	11.001701	9.000465	2121	10.999535	10.002165	21	9.997835	17
44	99560	2094	00440	01738	2115	98262	178	21	822	16
45	9.000816	2088	10.999184	03007	2109	96993	191	21	809	15
46	02069	2082	97931	04272	2103	95728	203	21	797	14
47	03318	2076	96682	05534	2097	94466	216	21	784	13
48	04563	2070	95437	06792	2091	93208	229	21	771	12
49	9.005805	2064	10.994195	9.008047	2085	10.991953	10.002242	21	9.997758	11
50	07044	2058	92956	09298	2080	90702	255	21	745	10
51	08278	2052	91722	10546	2074	89454	268	21	733	9
52	09510	2046	90490	11790	2068	88210	281	21	719	8
53	10737	2040	89263	13031	2062	86969	294	21	706	7
54	11962	2034	88038	14268	2056	85732	307	22	693	6
55	9.013182	2029	10.986818	9.015502	2051	10.984498	10.002320	22	9.997680	5
56	14400	2023	85600	16732	2045	83268	333	22	667	4
57	15613	2017	84387	17959	2040	82041	346	22	654	3
58	16824	2012	83176	19183	2033	80817	359	22	641	2
59	18031	2006	81969	20403	2028	79597	372	22	628	1
60	19235		80765	21620		78380	386	22	614	0
	Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	′

84 DEG.

Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	
9.019235	2000	10.980765	9.021620	2023	10.978380	10.002386	22	9.997614	60
20485	1995	79565	22834	2017	77166	399	22	601	59
21632	1989	78368	24044	2011	75956	412	22	588	58
22825	1984	77175	25251	2006	74749	426	22	574	57
24016	1978	75984	26455	2000	73545	439	22	561	56
25203	1973	74797	27655	1995	72345	453	22	547	55
26386	1967	73614	28852	1990	71148	466	23	534	54
9.027567	1962	10.972433	9.030046	1985	10.969954	10.002480	23	9.997520	53
28744	1957	71256	31237	1979	68763	493	23	507	52
29918	1951	70082	32425	1974	67575	507	23	493	51
31089	1946	68911	33609	1969	66391	520	23	480	50
32257	1941	67743	34791	1964	65209	534	23	466	49
33421	1936	66579	35969	1958	64031	548	23	452	48
9.034582	1930	10.965418	9.037144	1953	10.962856	10.002561	23	9.997439	47
35741	1925	64259	38316	1948	61684	575	23	425	46
36896	1920	63104	39485	1943	60515	589	23	411	45
38048	1915	61952	40651	1938	59349	603	23	397	44
39197	1910	60803	41813	1933	58187	617	23	383	43
40342	1905	59658	42973	1928	57027	631	23	369	42
9.041485	1899	10.958515	9.044130	1923	10.955870	10.002645	23	9.997355	41
42625	1895	57375	45284	1918	54716	659	23	341	40
43762	1889	56238	46434	1913	53566	673	24	327	39
44895	1884	55105	47582	1908	52418	687	24	313	38
46026	1879	53974	48727	1903	51273	701	24	299	37
47154	1875	52846	49869	1898	50131	715	24	285	36
9.048279	1870	10.951721	9.051008	1893	10.948992	10.002729	24	9.997271	35
49400	1865	50600	52144	1889	47856	743	24	257	34
50519	1860	49481	53277	1884	46723	758	24	242	33
51635	1855	48365	54407	1879	45593	772	24	228	32
52749	1850	47251	55535	1874	44465	786	24	214	31
53859	1845	46141	56659	1870	43341	801	24	199	30
9.054966	1841	10.945034	9.057781	1865	10.942219	10.002815	24	9.997185	29
56071	1836	43929	58900	1860	41100	830	24	170	28
57172	1831	42828	60016	1855	39984	844	24	156	27
58271	1827	41729	61130	1851	38870	859	24	141	26
59367	1822	40633	62240	1846	37760	873	24	127	25
60460	1817	39540	63348	1842	36652	888	24	112	24
9.061551	1813	10.938449	9.064453	1837	10.935547	10.002902	24	9.997098	23
62639	1808	37361	65556	1833	34444	917	25	083	22
63724	1804	36276	66655	1828	33345	932	25	068	21
64806	1799	35194	67752	1824	32248	947	25	053	20
65885	1794	34115	68846	1819	31154	961	25	039	19
66962	1790	33038	69938	1815	30062	976	25	024	18
9.068036	1786	10.931964	9.071027	1810	10.928973	10.002991	25	9.997009	17
69107	1781	30893	72113	1806	27887	10.003006	25	9.996994	16
70176	1777	29824	73197	1802	26803	021	25	979	15
71242	1772	28758	74278	1797	25722	036	25	964	14
72306	1768	27694	75356	1793	24644	051	25	949	13
73366	1763	26634	76432	1789	23568	066	25	934	12
9.074424	1759	10.925576	9.077505	1784	10.922495	10.003081	25	9.996919	11
75480	1755	24520	78576	1780	21424	096	25	904	10
76533	1750	23467	79644	1776	20356	111	25	889	9
77583	1746	22417	80710	1772	19290	126	25	874	8
78631	1742	21369	81773	1767	18227	142	25	858	7
79676	1738	20324	82833	1763	17167	157	25	843	6
9.080719	1733	10.919281	9.083891	1759	10.916109	10.003172	25	9.996828	5
81759	1729	18241	84947	1755	15053	188	26	812	4
82797	1725	17203	86000	1751	14000	203	26	797	3
83832	1721	16168	87050	1747	12950	218	26	782	2
84864	1717	15136	88098	1743	11902	234	26	766	1
85894		14106	89144		10856	249	26	751	0
Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	′

7 Deg.	TABLE IV.—Log. Sines, etc.								
′	Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	′
0	9.085894	1713	10.914106	9.089144	1738	10.910856	10.003249	26	9.996751	60
1	86922	1709	13078	90187	1735	09813	265	26	735	59
2	87947	1704	12053	91228	1731	08772	280	26	720	58
3	88970	1700	11030	92266	1727	07734	296	26	704	57
4	89990	1696	10010	93302	1722	06698	312	26	688	56
5	91008	1692	08992	94336	1719	05664	327	26	673	55
6	92024	1688	07976	95367	1715	04633	343	26	657	54
7	9.093037	1684	10.906963	9.096395	1711	10.903605	10.003359	26	9.996641	53
8	94047	1680	05953	97422	1707	02578	375	26	625	52
9	95056	1676	04944	98446	1703	01554	390	26	610	51
10	96062	1673	03938	99468	1699	00532	406	26	594	50
11	97065	1668	02935	9.100487	1695	10.899513	422	27	578	49
12	98066	1665	01934	01504	1691	98496	438	27	562	48
13	9.099065	1661	10.900935	9.102519	1687	10.897481	10.003454	27	9.996546	47
14	9.100062	1657	10.899938	03532	1684	96468	470	27	530	46
15	01056	1653	98944	04542	1680	95458	486	27	514	45
16	02048	1649	97952	05550	1676	94450	502	27	498	44
17	03037	1645	96963	06556	1672	93444	518	27	482	43
18	04025	1642	95975	07559	1669	92441	535	27	465	42
19	9.105010	1638	10.894990	9.108560	1665	10.891440	10.003551	27	9.996449	41
20	05992	1634	94008	09559	1661	90441	567	27	433	40
21	06973	1630	93027	10556	1658	89444	583	27	417	39
22	07951	1627	92049	11551	1654	88449	600	27	400	38
23	08927	1623	91073	12543	1650	87457	616	27	384	37
24	09901	1619	90099	13533	1647	86467	632	27	368	36
25	9.110873	1616	10.889127	9.114521	1643	10.885479	10.003649	27	9.996351	35
26	11842	1612	88158	15507	1639	84493	665	27	335	34
27	12809	1608	87191	16491	1636	83509	682	27	318	33
28	13774	1605	86226	17472	1632	82528	698	28	302	32
29	14737	1601	85263	18452	1629	81548	715	28	285	31
30	15698	1597	84302	19429	1625	80571	731	28	269	30
31	9.116656	1594	10.883344	9.120404	1622	10.879596	10.003748	28	9.996252	29
32	17613	1590	82387	21377	1618	78623	765	28	235	28
33	18567	1587	81433	22348	1615	77652	781	28	219	27
34	19519	1583	80481	23317	1611	76683	798	28	202	26
35	20469	1580	79531	24284	1608	75716	815	28	185	25
36	21417	1576	78583	25249	1604	74751	832	28	168	24
37	9.122362	1573	10.877638	9.126211	1601	10.873789	10.003849	28	9.996151	23
38	23306	1569	76694	27172	1597	72828	866	28	134	22
39	24248	1566	75752	28130	1594	71870	883	28	117	21
40	25187	1562	74813	29087	1591	70913	900	28	100	20
41	26125	1559	73875	30041	1587	69959	917	29	083	19
42	27060	1556	72940	30994	1584	69006	934	29	066	18
43	9.127993	1552	10.872007	9.131944	1581	10.868056	10.003951	29	9.996049	17
44	28925	1549	71075	32893	1577	67107	968	29	032	16
45	29854	1545	70146	33839	1574	66161	985	29	015	15
46	30781	1542	69219	34784	1571	65216	10.004002	29	9.995998	14
47	31706	1539	68294	35726	1567	64274	020	29	980	13
48	32630	1535	67370	36667	1564	63333	037	29	963	12
49	9.133551	1532	10.866449	9.137605	1561	10.862395	10.004054	29	9.995946	11
50	34470	1529	65530	38542	1558	61458	072	29	928	10
51	35387	1525	64613	39476	1555	60524	089	29	911	9
52	36303	1522	63697	40409	1551	59591	106	29	894	8
53	37216	1519	62784	41340	1548	58660	124	29	876	7
54	38128	1516	61872	42269	1545	57731	141	29	859	6
55	9.139037	1512	10.860963	9.143196	1542	10.856804	10.004159	29	9.995841	5
56	39944	1509	60056	44121	1539	55879	177	29	823	4
57	40850	1506	59150	45044	1535	54956	194	29	806	3
58	41754	1503	58246	45966	1532	54034	212	29	788	2
59	42655	1500	57345	46885	1529	53115	229	29	771	1
60	43555		56445	47803		52197	247	29	753	0
	Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	′

8 DEG. TABLE IV.—LOG. SINES, ETC.

′	Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	
0	9.143555	1496	10.856445	9.147803	1526	10.852197	10.004247	30	9.995753	60
1	4453	1493	5547	8718	1523	1282	265	30	735	59
2	5349	1490	4651	9632	1520	0368	283	30	717	58
3	6243	1487	3757	9.150544	1517	10.849456	301	30	699	57
4	7136	1484	2864	1454	1514	8546	319	30	681	56
5	8026	1481	1974	2363	1511	7637	336	30	664	55
6	8915	1478	1085	3269	1508	6731	354	30	646	54
7	9.149802	1475	10.850198	9.154174	1505	10.845826	10.004372	30	9.995628	53
8	9.150686	1472	10.849314	5077	1502	4923	390	30	610	52
9	1569	1469	8431	5978	1499	4022	409	30	591	51
10	2451	1466	7549	6877	1496	3123	427	30	573	50
11	3330	1463	6670	7775	1493	2225	445	30	555	49
12	4208	1460	5792	8671	1490	1329	463	30	537	48
13	9.155083	1457	10.844917	9.159565	1487	10.840435	10.004481	30	9.995519	47
14	5957	1454	4043	9.160457	1484	10.839543	499	31	501	46
15	6830	1451	3170	1347	1481	8653	518	31	482	45
16	7700	1448	2300	2236	1479	7764	536	31	464	44
17	8569	1445	1431	3123	1476	6877	554	31	446	43
18	9435	1442	0565	4008	1473	5992	573	31	427	42
19	9.160301	1439	10.839699	9.164892	1470	10.835108	10.004591	31	9.995409	41
20	1164	1436	8836	5774	1467	4226	610	31	390	40
21	2025	1433	7975	6654	1464	3346	628	31	372	39
22	2885	1430	7115	7532	1461	2468	647	31	353	38
23	3743	1427	6257	8409	1458	1591	666	31	334	37
24	4600	1424	5400	9284	1455	0716	684	31	316	36
25	9.165454	1422	10.834546	9.170157	1453	10.829843	10.004703	31	9.995297	35
26	6307	1419	3693	1029	1450	8971	722	31	278	34
27	7159	1416	2841	1899	1447	8101	740	31	260	33
28	8008	1413	1992	2767	1444	7233	759	32	241	32
29	8856	1410	1144	3634	1442	6366	778	32	222	31
30	9702	1407	0298	4499	1439	5501	797	32	203	30
31	9.170547	1405	10.829453	9.175362	1436	10.824638	10.004816	32	9.995184	29
32	1389	1402	8611	6224	1433	3776	835	32	165	28
33	2230	1399	7770	7084	1431	2916	854	32	146	27
34	3070	1396	6930	7942	1428	2058	873	32	127	26
35	3908	1394	6092	8799	1425	1201	892	32	108	25
36	4744	1391	5256	9655	1423	0345	911	32	089	24
37	9.175578	1388	10.824422	9.180508	1420	10.819492	10.004930	32	9.995070	23
38	6411	1386	3589	1360	1417	8640	949	32	051	22
39	7242	1383	2758	2211	1415	7789	968	32	032	21
40	8072	1380	1928	3059	1412	6941	987	32	013	20
41	8900	1377	1100	3907	1409	6093	10.005007	32	9.994993	19
42	9726	1374	0274	4752	1407	5248	026	32	974	18
43	9.180551	1372	10.819449	9.185597	1404	10.814403	10.005045	32	9.994955	17
44	1374	1369	8626	6439	1402	3561	065	32	935	16
45	2196	1366	7804	7280	1399	2720	084	32	916	15
46	3016	1364	6984	8120	1396	1880	104	33	896	14
47	3834	1361	6166	8958	1393	1042	123	33	877	13
48	4651	1359	5349	9794	1391	0206	143	33	857	12
49	9.185466	1356	10.814534	9.190629	1389	10.809371	10.005162	33	9.994838	11
50	6280	1353	3720	1462	1386	8538	182	33	818	10
51	7092	1351	2908	2294	1384	7706	202	33	798	9
52	7903	1348	2097	3124	1381	6876	221	33	779	8
53	8712	1346	1288	3953	1379	6047	241	33	759	7
54	9519	1343	0481	4780	1376	5220	261	33	739	6
55	9.190325	1341	10.809675	9.195606	1374	10.804394	10.005281	33	9.994719	5
56	1130	1338	8870	6430	1371	3570	300	33	700	4
57	1933	1336	8067	7253	1369	2747	320	33	680	3
58	2734	1333	7266	8074	1366	1926	340	33	660	2
59	3534	1330	6466	8894	1364	1106	360	33	640	1
60	4332		5668	9713		0287	380	33	620	0
	Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	′

81 DEG.

9 Deg.	TABLE IV.—Log. Sines, etc.								
′	Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	′
0	9.194332	1328	10.805668	9.199713	1361	10.800287	10.005380	33	9.994620	60
1	5129	1326	4871	9.200529	1359	10.799471	400	33	600	59
2	5925	1323	4075	1345	1356	8655	420	33	580	58
3	6719	1321	3281	2159	1354	7841	440	34	560	57
4	7511	1318	2489	2971	1352	7029	460	34	540	56
5	8302	1316	1698	3782	1349	6218	481	34	519	55
6	9091	1313	0909	4592	1347	5408	501	34	499	54
7	9.199879	1311	10.800121	9.205400	1345	10.794600	10.005521	34	9.994479	53
8	9.200666	1308	10.799334	6207	1342	3793	541	34	459	52
9	1451	1306	8549	7013	1340	2987	562	34	438	51
10	2234	1304	7766	7817	1338	2183	582	34	418	50
11	3017	1301	6983	8619	1335	1381	603	34	397	49
12	3797	1299	6203	9420	1333	0580	623	34	377	48
13	9.204577	1296	10.795423	9.210220	1331	10.789780	10.005643	34	9.994357	47
14	5354	1294	4646	1018	1328	8982	664	34	336	46
15	6131	1292	3869	1815	1326	8185	684	34	316	45
16	6906	1289	3094	2611	1324	7389	705	34	295	44
17	7679	1287	2321	3405	1321	6595	726	34	274	43
18	8452	1285	1548	4198	1319	5802	746	35	254	42
19	9.209222	1282	10.790778	9.214989	1317	10.785011	10.005767	35	9.994233	41
20	9992	1280	0008	5780	1315	4220	788	35	212	40
21	9.210760	1278	10.789240	6568	1312	3432	809	35	191	39
22	1526	1275	8474	7356	1310	2644	829	35	171	38
23	2291	1273	7709	8142	1308	1858	850	35	150	37
24	3055	1271	6945	8926	1305	1074	871	35	129	36
25	9.213818	1268	10.786182	9.219710	1303	10.780290	10.005892	35	9.994108	35
26	4579	1266	5421	9.220492	1301	10.779508	913	35	087	34
27	5338	1264	4662	1272	1299	8728	934	35	066	33
28	6097	1261	3903	2052	1297	7948	955	35	045	32
29	6854	1259	3146	2830	1294	7170	976	35	024	31
30	7609	1257	2391	3606	1292	6394	997	35	003	30
31	9.218363	1255	10.781637	9.224382	1290	10.775618	10.006019	35	9.993981	29
32	9116	1253	0884	5156	1288	4844	040	35	960	28
33	9868	1250	0132	5929	1286	4071	061	35	939	27
34	9.220618	1248	10.779382	6700	1284	3300	082	35	918	26
35	1367	1246	8633	7471	1281	2529	104	35	896	25
36	2115	1244	7885	8239	1279	1761	125	36	875	24
37	9.222861	1242	10.777139	9.229007	1277	10.770993	10.006146	36	9.993854	23
38	3606	1239	6394	9773	1275	0227	168	36	832	22
39	4349	1237	5651	9.230539	1273	10.769461	189	36	811	21
40	5092	1235	4908	1302	1271	8698	211	36	789	20
41	5833	1233	4167	2065	1269	7935	232	36	768	19
42	6573	1231	3427	2826	1267	7174	254	36	746	18
43	9.227311	1228	10.772689	9.233586	1265	10.766414	10.006275	36	9.993725	17
44	8048	1226	1952	4345	1262	5655	297	36	703	16
45	8784	1224	1216	5103	1260	4897	319	36	681	15
46	9518	1222	0482	5859	1258	4141	340	36	660	14
47	9.230252	1220	10.769748	6614	1256	3386	362	36	638	13
48	0984	1218	9016	7368	1254	2632	384	36	616	12
49	9.231714	1216	10.768286	9.238120	1252	10.761880	10.006406	37	9.993594	11
50	2444	1214	7556	8872	1250	1128	428	37	572	10
51	3172	1212	6828	9622	1248	0378	450	37	550	9
52	3899	1209	6101	9.240371	1246	10.759629	472	37	528	8
53	4625	1207	5375	1118	1244	8882	494	37	506	7
54	5349	1205	4651	1865	1242	8135	516	37	484	6
55	9.236073	1203	10.763927	9.242610	1240	10.757390	10.006538	37	9.993462	5
56	6795	1201	3205	3354	1238	6646	560	37	440	4
57	7515	1199	2485	4097	1236	5903	582	37	418	3
58	8235	1197	1765	4839	1234	5161	604	37	396	2
59	8953	1195	1047	5579	1232	4421	626	37	374	1
60	9670		0330	6319		3681	649	37	351	0
′	Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	′

80 Deg.

DEG. TABLE IV.—LOG. SINES, ETC.

Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	
9.239670	1193	10.760330	9.246319	1230	10.753681	10.006649	37	9.993351	60
9.240386	1191	10.759614	7057	1228	2943	671	37	329	59
1101	1189	8899	7794	1226	2206	693	37	307	58
1814	1187	8186	8530	1224	1470	715	37	285	57
2526	1185	7474	9264	1222	0736	738	37	262	56
3237	1183	6763	9998	1220	0002	760	37	240	55
3947	1181	6053	9.250730	1218	10.749270	783	38	217	54
9.244656	1179	10.755344	9.251461	1217	10.748539	10.006805	38	9.993195	53
5363	1177	4637	2191	1215	7809	828	38	172	52
6069	1175	3931	2920	1213	7080	851	38	149	51
6775	1173	3225	3648	1211	6352	873	38	127	50
7478	1171	2522	4374	1209	5626	896	38	104	49
8181	1169	1819	5100	1207	4900	919	38	081	48
9.248883	1167	10.751117	9.255824	1205	10.744176	10.006941	38	9.993059	47
9583	1165	0417	6547	1203	3453	964	38	036	46
9.250282	1163	10.749718	7269	1201	2731	987	38	013	45
0980	1161	9020	7990	1200	2010	10.007010	38	9.992990	44
1677	1159	8323	8710	1198	1290	033	38	967	43
2373	1158	7627	9429	1196	0571	056	38	944	42
9.253067	1156	10.746933	9.260146	1194	10.739854	10.007079	38	9.992921	41
3761	1154	6239	0863	1192	9137	102	38	898	40
4453	1152	5547	1578	1190	8422	125	38	875	39
5144	1150	4856	2292	1189	7708	148	38	852	38
5834	1148	4166	3005	1187	6995	171	38	829	37
6523	1146	3477	3717	1185	6283	194	39	806	36
9.257211	1144	10.742789	9.264428	1183	10.735572	10.007217	39	9.992783	35
7898	1142	2102	5138	1181	4862	241	39	759	34
8583	1141	1417	5847	1179	4153	264	39	736	33
9268	1139	0732	6555	1178	3445	287	39	713	32
9951	1137	0049	7261	1176	2739	310	39	690	31
9.260633	1135	10.739367	7967	1174	2033	334	39	666	30
9.261314	1133	10.738686	9.268671	1172	10.731329	10.007357	39	9.992643	29
1994	1131	8006	9375	1170	0625	381	39	619	28
2673	1130	7327	9.270077	1169	10.729923	404	39	596	27
3351	1128	6649	0779	1167	9221	428	39	572	26
4027	1126	5973	1479	1165	8521	451	39	549	25
4703	1124	5297	2178	1164	7822	475	39	525	24
9.265377	1122	10.734623	9.272876	1162	10.727124	10.007499	39	9.992501	23
6051	1120	3949	3573	1160	6427	522	40	478	22
6723	1119	3277	4269	1158	5731	546	40	454	21
7395	1117	2605	4964	1157	5036	570	40	430	20
8065	1115	1935	5658	1155	4342	594	40	406	19
8734	1113	1266	6351	1153	3649	618	40	382	18
9.269402	1111	10.730598	9.277043	1151	10.722957	10.007641	40	9.992359	17
9.270069	1110	10.729931	7734	1150	2266	665	40	335	16
0735	1108	9265	8424	1148	1576	689	40	311	15
1400	1106	8600	9113	1147	0887	713	40	287	14
2064	1105	7936	9801	1145	0199	737	40	263	13
2726	1103	7274	9.280488	1143	10.719512	761	40	239	12
9.273388	1101	10.726612	9.281174	1141	10.718826	10.007786	40	9.992214	11
4049	1099	5951	1858	1140	8142	810	40	190	10
4708	1098	5292	2542	1138	7458	834	40	166	9
5367	1096	4633	3225	1136	6775	858	40	142	8
6024	1094	3976	3907	1135	6093	883	40	117	7
6681	1092	3319	4588	1133	5412	907	41	093	6
9.277337	1091	10.722663	9.285268	1131	10.714732	10.007931	41	9.992069	5
7991	1089	2009	5947	1130	4053	956	41	044	4
8644	1087	1356	6624	1128	3376	980	41	020	3
9297	1086	0703	7301	1126	2699	10.008004	41	9.991996	2
9948	1084	0052	7977	1125	2023	029	41	971	1
9.280599		10.719401	8652		1348	053	41	947	0
Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	′

79 DEG.

11 Deg.	TABLE IV.—Log. Sines, etc.									
′	Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	
0	9.280599	1082	10.719401	9.288652	1123	10.711348	10.008053	41	9.991947	60
1	1248	1081	8752	9326	1122	0674	078	41	922	59
2	1897	1079	8103	9999	1120	0001	103	41	897	58
3	2544	1077	7456	9.290671	1118	10.709329	127	41	873	57
4	3190	1076	6810	1342	1117	8658	152	41	848	56
5	3836	1074	6164	2013	1115	7987	177	41	823	55
6	4480	1072	5520	2682	1114	7318	201	41	799	54
7	9.285124	1071	10.714876	9.293350	1112	10.706650	10.008226	42	9.991774	53
8	5766	1069	4234	4017	1111	5983	251	42	749	52
9	6408	1067	3592	4684	1109	5316	276	42	724	51
10	7048	1066	2952	5349	1107	4651	301	42	699	50
11	7687	1064	2313	6013	1106	3987	326	42	674	49
12	8326	1063	1674	6677	1104	3323	351	42	649	48
13	9.288964	1061	10.711036	9.297339	1103	10.702661	10.008376	42	9.991624	47
14	9600	1059	0400	8001	1101	1999	401	42	599	46
15	9.290236	1058	10.709764	8662	1100	1338	426	42	574	45
16	0870	1056	9130	9322	1098	0678	451	42	549	44
17	1504	1054	8496	9980	1096	0020	476	42	524	43
18	2137	1053	7863	9.300638	1095	10.699362	502	42	498	42
19	9.292768	1051	10.707232	9.301295	1093	10.698705	10.008527	42	9.991473	41
20	3399	1050	6601	1951	1092	8049	552	42	448	40
21	4029	1048	5971	2607	1090	7393	578	42	422	39
22	4658	1046	5342	3261	1089	6739	603	42	397	38
23	5286	1045	4714	3914	1087	6086	628	43	372	37
24	5913	1043	4087	4567	1086	5433	654	43	346	36
25	9.296539	1042	10.703461	9.305218	1084	10.694782	10.008679	43	9.991321	35
26	7164	1040	2836	5869	1083	4131	705	43	295	34
27	7788	1039	2212	6519	1081	3481	730	43	270	33
28	8412	1037	1588	7168	1080	2832	756	43	244	32
29	9034	1036	0966	7815	1078	2185	782	43	218	31
30	9655	1034	0345	8463	1077	1537	807	43	193	30
31	9.300276	1032	10.699724	9.309109	1075	10.690891	10.008833	43	9.991167	29
32	0895	1031	9105	9754	1074	0246	859	43	141	28
33	1514	1029	8486	0398	1073	10.689602	885	43	115	27
34	2132	1028	7868	9.311042	1071	8958	910	43	090	26
35	2748	1026	7252	1685	1070	8315	936	43	064	25
36	3364	1025	6636	2327	1068	7673	962	43	038	24
37	9.303979	1023	10.696021	9.312967	1067	10.687033	10.008988	43	9.991012	23
38	4593	1022	5407	3608	1065	6392	10.009014	43	9.990986	22
39	5207	1020	4793	4247	1064	5753	040	43	960	21
40	5819	1019	4181	4885	1062	5115	066	44	934	20
41	6430	1017	3570	5523	1061	4477	092	44	908	19
42	7041	1016	2959	6159	1060	3841	118	44	882	18
43	9.307650	1014	10.692350	9.316795	1058	10.683205	10.009145	44	9.990855	17
44	8259	1013	1741	7430	1057	2570	171	44	829	16
45	8867	1011	1133	8064	1055	1936	197	44	803	15
46	9474	1010	0526	8697	1054	1303	223	44	777	14
47	9.310080	1008	10.689920	9329	1053	0671	250	44	750	13
48	0685	1007	9315	9961	1051	0039	276	44	724	12
49	9.311289	1005	10.688711	9.320592	1050	10.679408	10.009303	44	9.990697	11
50	1893	1004	8107	9.321222	1048	8778	329	44	671	10
51	2495	1003	7505	1851	1047	8149	356	44	644	9
52	3097	1001	6903	2479	1045	7521	382	44	618	8
53	3698	1000	6302	3106	1044	6894	409	44	591	7
54	4297	998	5703	3733	1043	6267	435	44	565	6
55	9.314897	997	10.685103	9.324358	1041	10.675642	10.009462	44	9.990538	5
56	5495	996	4505	4983	1040	5017	489	45	511	4
57	6092	994	3908	5607	1039	4393	515	45	485	3
58	6689	993	3311	6231	1037	3769	542	45	458	2
59	7284	991	2716	6853	1036	3147	569	45	431	1
60	7879		2121	7475		2525	596	45	404	0
	Cosine	D	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	′

78 Deg.

DEG. TABLE IV.—LOG. SINES, ETC.

Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	
.317879	990	10.682121	9.327474	1035	10.672526	10.009596	45	9.990404	60
8473	988	1527	8095	1033	1905	622	45	378	59
9066	987	0934	8715	1032	1285	649	45	351	58
9658	986	0344	9334	1030	0666	676	45	324	57
.320249	984	10.679751	9953	1029	0047	703	45	297	56
0840	983	9160	9.330570	1028	10.669430	730	45	270	55
1430	982	8570	1187	1026	8813	757	45	243	54
.322019	980	10.677981	9.331803	1025	10.668197	10.009785	45	9.990215	53
2607	979	7393	2418	1024	7582	812	45	188	52
3194	977	6806	3033	1023	6967	839	45	161	51
3780	976	6220	3646	1021	6354	866	45	134	50
4366	975	5634	4259	1020	5741	893	46	107	49
4950	973	5050	4871	1019	5129	921	46	079	48
.325534	972	10.674466	9.335482	1017	10.664518	10.009948	46	9.990052	47
6117	970	3883	6093	1016	3907	975	46	025	46
6700	969	3300	6702	1015	3298	10.010003	46	9.989997	45
7281	968	2719	7311	1013	2689	030	46	970	44
7862	966	2138	7919	1012	2081	058	46	942	43
8442	965	1558	8527	1011	1473	085	46	915	42
.329021	964	10.670979	9.339133	1010	10.660867	10.010113	46	9.989887	41
9599	962	0401	9739	1008	0261	140	46	860	40
.330176	961	10.669824	9.340344	1007	10.659656	168	46	832	39
0753	960	9247	0948	1006	9052	196	46	804	38
1329	958	8671	1552	1004	8448	223	46	777	37
1903	957	8097	2155	1003	7845	251	47	749	36
.332478	956	10.667522	9.342757	1002	10.657243	10.010279	47	9.989721	35
3051	954	6949	3358	1000	6642	307	47	693	34
3624	953	6376	3958	999	6042	335	47	665	33
4195	952	5805	4558	998	5442	363	47	637	32
4766	950	5234	5157	997	4843	391	47	609	31
5337	949	4663	5755	996	4245	418	47	582	30
.335906	948	10.664094	9.346353	994	10.653647	10.010447	47	9.989553	29
6475	946	3525	6949	993	3051	475	47	525	28
7043	945	2957	7545	992	2455	503	47	497	27
7610	944	2390	8141	991	1859	531	47	469	26
8176	943	1824	8735	990	1265	559	47	441	25
8742	941	1258	9329	988	0671	587	47	413	24
.339306	940	10.660694	9.349922	987	10.650078	10.010616	47	9.989384	23
9871	939	0129	9.350514	986	10.649486	644	47	356	22
.340434	937	10.659566	1106	985	8894	672	47	328	21
0996	936	9004	1697	983	8303	700	47	300	20
1558	935	8442	2287	982	7713	729	47	271	19
2119	934	7881	2876	981	7124	757	47	243	18
.342679	932	10.657321	9.353465	980	10.646535	10.010786	47	9.989214	17
3239	931	6761	4053	979	5947	814	47	186	16
3797	930	6203	4640	977	5360	843	47	157	15
4355	929	5645	5227	976	4773	872	48	128	14
4912	927	5088	5813	975	4187	900	48	100	13
5469	926	4531	6398	974	3602	929	48	071	12
.346024	925	10.653976	9.356982	973	10.643018	10.010958	48	9.989042	11
6579	924	3421	7566	971	2434	986	48	014	10
7134	922	2866	8149	970	1851	10.011015	48	9.988985	9
7687	921	2313	8731	969	1269	044	48	956	8
8240	920	1760	9313	968	0687	073	48	927	7
8792	919	1208	9893	967	0107	102	48	898	6
.349343	917	10.650657	9.360474	966	10.639526	10.011131	48	9.988869	5
9893	916	0107	1053	965	8947	160	48	840	4
.350443	915	10.649557	1632	963	8368	189	49	811	3
0992	914	9008	2210	962	7790	218	49	782	2
1540	913	8460	2787	961	7213	247	49	753	1
2088		7912	3364		6636	276		724	0
Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	/

77 DEG.

13 Deg.	TABLE IV.—Log. Sines, etc.									
′	Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	
0	9.352088	911	10.647912	9.363364	960	10.636636	10.011276	49	9.988724	60
1	2635	910	7365	3940	959	6060	305	49	695	59
2	3181	909	6819	4515	958	5485	334	49	666	58
3	3726	908	6274	5090	957	4910	364	49	636	57
4	4271	907	5729	5664	955	4336	393	49	607	56
5	4815	905	5185	6237	954	3763	422	49	578	55
6	5358	904	4642	6810	953	3190	452	49	548	54
7	9.355901	903	10.644099	9.367382	952	10.632618	10.011481	49	9.988519	53
8	6443	902	3557	7953	951	2047	511	49	489	52
9	6984	901	3016	8524	950	1476	540	49	460	51
10	7524	899	2476	9094	949	0906	570	49	430	50
11	8064	898	1936	9663	948	0337	599	49	401	49
12	8603	897	1397	9.370232	946	10.629768	629	49	371	48
13	9.359141	896	10.640859	9.370799	945	10.629201	10.011658	49	9.988342	47
14	9678	895	0322	1367	944	8633	688	50	312	46
15	0215	893	10.639785	1933	943	8067	718	50	282	45
16	0752	892	9248	2499	942	7501	748	50	252	44
17	9.361287	891	8713	3064	941	6936	777	50	223	43
18	1822	890	8178	3629	940	6371	807	50	193	42
19	9.362356	889	10.637644	9.374193	939	10.625807	10.011837	50	9.988163	41
20	2889	888	7111	4756	938	5244	867	50	133	40
21	3422	887	6578	5319	937	4681	897	50	103	39
22	3954	885	6046	5881	935	4119	927	50	073	38
23	4485	884	5515	6442	934	3558	957	50	043	37
24	5016	883	4984	7003	933	2997	987	50	013	36
25	9.365546	882	10.634454	9.377563	932	10.622437	10.012017	50	9.987983	35
26	6075	881	3925	8122	931	1878	047	50	953	34
27	6604	880	3396	8681	930	1319	078	50	922	33
28	7131	879	2869	9239	929	0761	108	50	892	32
29	7659	877	2341	9797	928	0203	138	50	862	31
30	8185	876	1815	9.380354	927	10.619646	168	50	832	30
31	9.368711	875	10.631289	9.380910	926	10.619090	10.012199	51	9.987801	29
32	9236	874	0764	1466	925	8534	229	51	771	28
33	9761	873	0239	2020	924	7980	260	51	740	27
34	0285	872	10.629715	2575	923	7425	290	51	710	26
35	0808	871	9192	3129	922	6871	321	51	679	25
36	9.371330	870	8670	3682	921	6318	351	51	649	24
37	9.371852	869	10.628148	9.384234	920	10.615766	10.012382	51	9.987618	23
38	2373	867	7627	4786	919	5214	412	51	588	22
39	2894	866	7106	5337	918	4663	443	51	557	21
40	3414	865	6586	5888	917	4112	474	51	526	20
41	3933	864	6067	6438	915	3562	504	51	496	19
42	4452	863	5548	6987	914	3013	535	51	465	18
43	9.374970	862	10.625030	9.387536	913	10.612464	10.012566	51	9.987434	17
44	5487	861	4513	8084	912	1916	597	52	403	16
45	6003	860	3997	8631	911	1369	628	52	372	15
46	6519	859	3481	9178	910	0822	659	52	341	14
47	7035	858	2965	9724	909	0276	690	52	310	13
48	7549	857	2451	9.390270	908	10.609730	721	52	279	12
49	9.378063	856	10.621937	9.390815	907	10.609185	10.012752	52	9.987248	11
50	8577	854	1423	1360	906	8640	783	52	217	10
51	9089	853	0911	1903	905	8097	814	52	186	9
52	9601	852	0399	2447	904	7553	845	52	155	8
53	9.380113	851	10.619887	2989	903	7011	876	52	124	7
54	0624	850	9376	3531	902	6469	908	52	092	6
55	9.381134	849	10.618866	9.394073	901	10.605927	10.012939	52	9.987061	5
56	1643	848	8357	4614	900	5386	970	52	030	4
57	2152	847	7848	5154	899	4846	10.013002	52	9.986998	3
58	2661	846	7339	5694	898	4306	033	52	967	2
59	3168	845	6832	6233	897	3767	064	52	936	1
60	3675		6325	6771		3229	096	52	904	0
	Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	′

76 Deg.

DEG. TABLE IV.—LOG. SINES, ETC.

Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	
9.383675	844	10.616325	9.396771	896	10.603229	10.018096	52	9.986904	60
4182	843	5818	7309	896	2691	127	53	873	59
4687	843	5313	7846	895	2154	159	53	841	58
5192	842	4808	8383	894	1617	191	53	809	57
5697	841	4303	8919	893	1081	222	53	778	56
6201	840	3799	9455	892	0545	254	53	746	55
6704	839	3296	9990	891	0010	286	53	714	54
9.387207	838	10.612793	9.400524	890	10.599476	10.018317	53	9.986683	53
7709	837	2291	1058	889	8942	349	53	651	52
8210	836	1790	1591	888	8409	381	53	619	51
8711	835	1289	2124	887	7876	413	53	587	50
9211	834	0789	2656	886	7344	445	53	555	49
9711	833	0289	3187	885	6813	477	53	523	48
9.390210	832	10.609790	9.403718	884	10.596282	10.018509	53	9.986491	47
0708	831	9292	4249	883	5751	541	53	459	46
1206	830	8794	4778	882	5222	573	53	427	45
1703	828	8297	5308	881	4692	605	53	395	44
2199	827	7801	5836	880	4164	637	54	363	43
2695	826	7305	6364	879	3636	669	54	331	42
9.393191	825	10.606809	9.406892	878	10.593108	10.018701	54	9.986299	41
3685	824	6315	7419	877	2581	734	54	266	40
4179	823	5821	7945	876	2055	766	54	234	39
4673	822	5327	8471	875	1529	798	54	202	38
5166	821	4834	8997	874	1003	831	54	169	37
5658	820	4342	9521	874	0479	863	54	137	36
9.396150	819	10.603850	9.410045	873	10.589955	10.018896	54	9.986104	35
6641	818	3359	0569	872	9431	928	54	072	34
7132	817	2868	1092	871	8908	961	54	039	33
7621	817	2379	1615	870	8385	993	54	007	32
8111	816	1889	2137	869	7863	10.014026	54	9.985974	31
8600	815	1400	2658	868	7342	058	54	942	30
9.399088	814	10.600912	9.413179	867	10.586821	10.014091	55	9.985909	29
9575	813	0425	3699	866	6301	124	55	876	28
9.400062	812	10.599938	4219	865	5781	157	55	843	27
0549	811	9451	4738	864	5262	189	55	811	26
1035	810	8965	5257	864	4743	222	55	778	25
1520	809	8480	5775	863	4225	255	55	745	24
9.402005	808	10.597995	9.416293	862	10.583707	10.014288	55	9.985712	23
2489	807	7511	6810	861	3190	321	55	679	22
2972	806	7028	7326	860	2674	354	55	646	21
3455	805	6545	7842	859	2158	387	55	613	20
3938	804	6062	8358	858	1642	420	55	580	19
4420	803	5580	8873	857	1127	453	55	547	18
9.404901	802	10.595099	9.419387	856	10.580613	10.014486	55	9.985514	17
5382	801	4618	9901	855	0099	520	55	480	16
5862	800	4138	9.420415	855	10.579585	553	55	447	15
6341	799	3659	0927	854	9073	586	56	414	14
6820	798	3180	1440	853	8560	620	56	380	13
7299	797	2701	1952	852	8048	653	56	347	12
9.407777	796	10.592223	9.422463	851	10.577537	10.014686	56	8.985314	11
8254	795	1746	2974	850	7026	720	56	280	10
8731	794	1269	3484	849	6516	753	56	247	9
9207	794	0793	3993	848	6007	787	56	213	8
9682	793	0318	4503	848	5497	820	56	180	7
9.410157	792	10.589843	5011	847	4989	854	56	146	6
9.410632	791	10.589368	9.425519	846	10.574481	10.014887	56	9.985113	5
1106	790	8894	6027	845	3973	921	56	079	4
1579	789	8421	6534	844	3466	955	56	045	3
2052	788	7948	7041	843	2959	989	56	011	2
2524	787	7476	7547	843	2453	10.015022	56	9.984978	1
2996	786	7004	8052	843	1948	056	56	944	0
Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	′

75 DEG.

15 Deg. TABLE IV.—Log. Sines, etc.

′	Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	
0	9.412996	785	10.587004	9.428052	842	10.571948	10.015056	57	9.984944	60
1	3467	784	6533	8557	841	1443	090	57	910	59
2	3938	783	6062	9062	840	0938	124	57	876	58
3	4408	783	5592	9566	839	0434	158	57	842	57
4	4878	782	5122	9.430070	838	10.569930	192	57	808	56
5	5347	781	4653	0573	838	9427	226	57	774	55
6	5815	780	4185	1075	837	8925	260	57	740	54
7	9.416283	779	10.583717	9.431577	836	10.568423	10.015294	57	9.984706	53
8	6751	778	3249	2079	835	7921	328	57	672	52
9	7217	777	2783	2580	834	7420	363	57	637	51
10	7684	776	2316	3080	833	6920	397	57	603	50
11	8150	775	1850	3580	832	6420	431	57	569	49
12	8615	774	1385	4080	832	5920	465	57	535	48
13	9.419079	773	10.580921	9.434579	831	10.565421	10.015500	57	9.984500	47
14	9544	773	0456	5078	830	4922	534	57	466	46
15	9.420007	772	10.579993	5576	829	4424	568	57	432	45
16	0470	771	9530	6073	828	3927	603	58	397	44
17	0933	770	9067	6570	828	3430	637	58	363	43
18	1395	769	8605	7067	827	2933	672	58	328	42
19	9.421857	768	10.578143	9.437563	826	10.562437	10.015706	58	9.984294	41
20	2318	767	7682	8059	825	1941	741	58	259	40
21	2778	767	7222	8554	824	1446	776	58	224	39
22	3238	766	6762	9048	823	0952	810	58	190	38
23	3697	765	6303	9543	823	0457	845	58	155	37
24	4156	764	5844	9.440036	822	10.559964	880	58	120	36
25	9.424615	763	10.575385	9.440529	821	10.559471	10.015915	58	9.984085	35
26	5073	762	4927	1022	820	8978	950	58	050	34
27	5530	761	4470	1514	819	8486	985	58	015	33
28	5987	760	4013	2006	819	7994	10.016019	58	9.983981	32
29	6443	760	3557	2497	818	7503	054	58	946	31
30	6899	759	3101	2988	817	7012	089	58	911	30
31	9.427354	758	10.572646	9.443479	816	10.556521	10.016125	58	9.983875	29
32	7809	757	2191	3968	816	6032	160	59	840	28
33	8263	756	1737	4458	815	5542	195	59	805	27
34	8717	755	1283	4947	814	5053	230	59	770	26
35	9170	754	0830	5435	813	4565	265	59	735	25
36	9623	753	0377	5923	812	4077	300	59	700	24
37	9.430075	752	10.569925	9.446411	812	10.553589	10.016336	59	9.983664	23
38	0527	752	9473	6898	811	3102	371	59	629	22
39	0978	751	9022	7384	810	2616	406	59	594	21
40	1429	750	8571	7870	809	2130	442	59	558	20
41	1879	749	8121	8356	809	1644	477	59	523	19
42	2329	749	7671	8841	808	1159	513	59	487	18
43	9.432778	748	10.567222	9.449326	807	10.550674	10.016548	59	9.983452	17
44	3226	747	6774	9810	806	0190	584	59	416	16
45	3675	746	6325	9.450294	806	10.549706	619	59	381	15
46	4122	745	5878	0777	805	9223	655	59	345	14
47	4569	744	5431	1260	804	8740	691	59	309	13
48	5016	744	4984	1743	803	8257	727	60	273	12
49	9.435462	743	10.564538	9.452225	802	10.547775	10.016762	60	9.983238	11
50	5908	742	4092	2706	802	7294	798	60	202	10
51	6353	741	3647	3187	801	6813	834	60	166	9
52	6798	740	3202	3668	800	6332	870	60	130	8
53	7242	740	2758	4148	799	5852	906	60	094	7
54	7686	739	2314	4628	799	5372	942	60	058	6
55	9.438129	738	10.561871	9.455107	798	10.544893	10.016978	60	9.983022	5
56	8572	737	1428	5586	797	4414	10.017014	60	9.982986	4
57	9014	736	0986	6064	796	3936	050	60	950	3
58	9456	736	0544	6542	796	3458	086	60	914	2
59	9897	735	0103	7019	795	2981	122	60	878	1
60	9.440338		10.559662	7496		2504	158	60	842	0
	Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	′

DEG. TABLE IV.—LOG. SINES, ETC.

Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	
9.440338	734	10.559662	9.457496	794	10.542504	10.017158	60	9.982842	60
0778	733	9222	7973	793	2027	195	60	805	59
1218	732	8782	8449	793	1551	231	61	769	58
1658	731	8342	8925	792	1075	267	61	733	57
2096	731	7904	9400	791	0600	304	61	696	56
2535	730	7465	9875	790	0125	340	61	660	55
2973	729	7027	9.460349	790	10.539651	376	61	624	54
9.443410	728	10.556590	9.460823	789	10.539177	10.017413	61	9.982587	53
3847	727	6153	1297	788	8703	449	61	551	52
4284	727	5716	1770	788	8230	486	61	514	51
4720	726	5280	2242	787	7758	523	61	477	50
5155	725	4845	2714	786	7286	559	61	441	49
5590	724	4410	3186	785	6814	596	61	404	48
9.446025	723	10.553975	9.463658	785	10.536342	10.017633	61	9.982367	47
6459	723	3541	4129	784	5871	669	61	331	46
6893	722	3107	4599	783	5401	706	61	294	45
7326	721	2674	5069	783	4931	743	61	257	44
7759	720	2241	5539	782	4461	780	62	220	43
8191	720	1809	6008	781	3992	817	62	183	42
9.448623	719	10.551377	9.466476	780	10.533524	10.017854	62	9.982146	41
9054	718	0946	6945	780	3055	891	62	109	40
9485	717	0515	7413	779	2587	928	62	072	39
9915	716	0085	7880	778	2120	965	62	035	38
9.450345	716	10.549655	8347	778	1653	10.018002	62	9.981998	37
0775	715	9225	8814	777	1186	039	62	961	36
9.451204	714	10.548796	9.469280	776	10.530720	10.018076	62	9.981924	35
1632	713	8368	9746	775	0254	114	62	886	34
2060	713	7940	9.470211	775	10.529789	151	62	849	33
2488	712	7512	0676	774	9324	188	62	812	32
2915	711	7085	1141	773	8859	226	62	774	31
3342	710	6658	1605	773	8395	263	62	737	30
9.453768	710	10.546232	9.472068	772	10.527932	10.018301	63	9.981699	29
4194	709	5806	2532	771	7468	338	63	662	28
4619	708	5381	2995	771	7005	375	63	625	27
5044	707	4956	3457	770	6543	413	63	587	26
5469	707	4531	3919	769	6081	451	63	549	25
5893	706	4107	4381	769	5619	488	63	512	24
9.456316	705	10.543684	9.474842	768	10.525158	10.018526	63	9.981474	23
6739	704	3261	5303	767	4697	564	63	436	22
7162	704	2838	5763	767	4237	601	63	399	21
7584	703	2416	6223	766	3777	639	63	361	20
8006	702	1994	6683	765	3317	677	63	323	19
8427	701	1573	7142	765	2858	715	63	285	18
9.458848	701	10.541152	9.477601	764	10.522399	10.018753	63	9.981247	17
9268	700	0732	8059	763	1941	791	63	209	16
9688	699	0312	8517	763	1483	829	63	171	15
9.460108	698	10.539892	8975	762	1025	867	64	133	14
0527	698	9473	9432	761	0568	905	64	095	13
0946	697	9054	9889	761	0111	943	64	057	12
9.461364	696	10.538636	9.480345	760	10.519655	10.018981	64	9.981019	11
1782	695	8218	0801	759	9199	10.019019	64	9.980981	10
2199	695	7801	1257	759	8743	058	64	942	9
2616	694	7384	1712	758	8288	096	64	904	8
3032	693	6968	2167	757	7833	134	64	866	7
3448	693	6552	2621	757	7379	173	64	827	6
9.463864	692	10.536136	9.483075	756	10.516925	10.019211	64	9.980789	5
4279	691	5721	3529	755	6471	250	64	750	4
4694	690	5306	3982	755	6018	288	64	712	3
5108	690	4892	4435	754	5565	327	64	673	2
5522	689	4478	4887	753	5113	365	64	635	1
5935		4065	5339		4661	404	64	596	0
Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	′

73 DEG.

17 Deg.	TABLE IV.—Log. Sines, etc.								
′	Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	′
0	9.465935	688	10.534065	9.485339	753	10.514661	10.019404	64	9.980596	60
1	6348	688	3652	5791	752	4209	442	64	558	59
2	6761	687	3239	6242	751	3758	481	65	519	58
3	7173	686	2827	6693	751	3307	520	65	480	57
4	7585	685	2415	7143	750	2857	558	65	442	56
5	7996	685	2004	7593	749	2407	597	65	403	55
6	8407	684	1593	8043	749	1957	636	65	364	54
7	9.468817	683	10.531183	9.488492	748	10.511508	10.019675	65	9.980325	53
8	9227	683	0773	8941	747	1059	714	65	286	52
9	9637	682	0363	9390	747	0610	753	65	247	51
10	9.470046	681	10.529954	9838	746	0162	792	65	208	50
11	0455	680	9545	9.490286	746	10.509714	831	65	169	49
12	0863	680	9137	0733	745	9267	870	65	130	48
13	9.471271	679	10.528729	9.491180	744	10.508820	10.019909	65	9.980091	47
14	1679	678	8321	1627	744	8373	948	65	052	46
15	2086	678	7914	2073	743	7927	988	65	012	45
16	2492	677	7508	2519	743	7481	10.020027	65	9.979973	44
17	2898	676	7102	2965	742	7035	066	66	934	43
18	3304	676	6696	3410	741	6590	105	66	895	42
19	9.473710	675	10.526290	9.493854	740	10.506146	10.020145	66	9.979855	41
20	4115	674	5885	4299	740	5701	184	66	816	40
21	4519	674	5481	4743	740	5257	224	66	776	39
22	4923	673	5077	5186	739	4814	263	66	737	38
23	5327	672	4673	5630	738	4370	303	66	697	37
24	5730	672	4270	6073	737	3927	342	66	658	36
25	9.476133	671	10.523867	9.496515	737	10.503485	10.020382	66	9.979618	35
26	6536	670	3464	6957	736	3043	421	66	579	34
27	6938	669	3062	7399	736	2601	461	66	539	33
28	7340	669	2660	7841	735	2159	501	66	499	32
29	7741	668	2259	8282	734	1718	541	66	459	31
30	8142	667	1858	8722	734	1278	580	66	420	30
31	9.478542	667	10.521458	9.499163	733	10.500837	10.020620	66	9.979380	29
32	8942	666	1058	9603	733	0397	660	66	340	28
33	9342	665	0658	9.500042	732	10.499958	700	67	300	27
34	9741	665	0259	0481	731	9519	740	67	260	26
35	9.480140	664	10.519860	0920	731	9080	780	67	220	25
36	0539	663	9461	1359	730	8641	820	67	180	24
37	9.480937	663	10.519063	9.501797	730	10.498203	10.020860	67	9.979140	23
38	1334	662	8666	2235	729	7765	900	67	100	22
39	1731	661	8269	2672	728	7328	941	67	059	21
40	2128	661	7872	3109	728	6891	981	67	019	20
41	2525	660	7475	3546	727	6454	10.021021	67	9.978979	19
42	2921	659	7079	3982	727	6018	061	67	939	18
43	9.483316	659	10.516684	9.504418	726	10.495582	10.021102	67	9.978898	17
44	3712	658	6288	4854	725	5146	142	67	858	16
45	4107	657	5893	5289	725	4711	183	67	817	15
46	4501	657	5499	5724	724	4276	223	67	777	14
47	4895	656	5105	6159	724	3841	264	67	736	13
48	5289	655	4711	6593	723	3407	304	68	696	12
49	9.485682	655	10.514318	9.507027	722	10.492973	10.021345	68	9.978655	11
50	6075	654	3925	7460	722	2540	385	68	615	10
51	6467	653	3533	7893	721	2107	426	68	574	9
52	6860	653	3140	8326	721	1674	467	68	533	8
53	7251	652	2749	8759	720	1241	507	68	493	7
54	7643	651	2357	9191	719	0809	548	68	452	6
55	9.488034	651	10.511966	9.509622	719	10.490378	10.021589	68	9.978411	5
56	8424	650	1576	9.510054	718	10.489946	630	68	370	4
57	8814	650	1186	0485	718	9515	671	68	329	3
58	9204	649	0796	0916	717	9084	712	68	288	2
59	9593	648	0407	1346	716	8654	753	68	247	1
60	9982		0018	1776		8224	794	68	206	0
	Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	′

DEG. TABLE IV.—LOG. SINES, ETC.

Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	
).489982	648	10.510018	9.511776	716	10.488224	10.021794	68	9.978206	60
).490371	648	10.509629	2206	716	7794	835	68	165	59
0759	647	9241	2635	715	7365	876	68	124	58
1147	646	8853	3064	714	6936	917	69	083	57
1535	646	8465	3493	714	6507	958	69	042	56
1922	645	8078	3921	713	6079	999	69	001	55
2308	644	7692	4349	713	5651	10.022041	69	9.977959	54
).492695	644	10.507305	9.514777	712	10.485223	10.022082	69	9.977918	53
3081	643	6919	5204	712	4796	123	69	877	52
3466	642	6534	5631	711	4369	165	69	835	51
3851	642	6149	6057	710	3943	206	69	794	50
4236	641	5764	6484	710	3516	248	69	752	49
4621	641	5379	6910	709	3090	289	69	711	48
).495005	640	10.504995	9.517335	709	10.482665	10.022331	69	9.977669	47
5388	639	4612	7761	708	2239	372	69	628	46
5772	639	4228	8185	708	1815	414	69	586	45
6154	638	3846	8610	707	1390	456	70	544	44
6537	637	3463	9034	706	0966	497	70	503	43
6919	637	3081	9458	706	0542	539	70	461	42
).497301	636	10.502699	9.519882	705	10.480118	10.022581	70	9.977419	41
7682	636	2318	9.520305	705	10.479695	623	70	377	40
8064	635	1936	0728	704	9272	665	70	335	39
8444	634	1556	1151	703	8849	707	70	293	38
8825	634	1175	1573	703	8427	749	70	251	37
9204	633	0796	1995	703	8005	791	70	209	36
).499584	632	10.500416	9.522417	702	10.477583	10.022833	70	9.977167	35
9963	632	0037	2838	702	7162	875	70	125	34
).500342	631	10.499658	3259	701	6741	917	70	083	33
0721	631	9279	3680	701	6320	959	70	041	32
1099	630	8901	4100	700	5900	10.023001	70	9.976999	31
1476	629	8524	4520	699	5480	043	70	957	30
).501854	629	10.498146	9.524939	699	10.475061	10.023086	70	9.976914	29
2231	628	7769	5359	698	4641	128	71	872	28
2607	628	7393	5778	698	4222	170	71	830	27
2984	627	7016	6197	697	3803	213	71	787	26
3360	626	6640	6615	697	3385	255	71	745	25
3735	626	6265	7033	696	2967	298	71	702	24
).504110	625	10.495890	9.527451	696	10.472549	10.023340	71	9.976660	23
4485	625	5515	7868	695	2132	383	71	617	22
4860	624	5140	8285	695	1715	426	71	574	21
5234	623	4766	8702	694	1298	468	71	532	20
5608	623	4392	9119	693	0881	511	71	489	19
5981	622	4019	9535	693	0465	554	71	446	18
).506354	622	10.493646	9.529950	693	10.470050	10.023596	71	9.976404	17
6727	621	3273	9.530366	692	10.469634	639	71	361	16
7099	620	2901	0781	691	9219	682	71	318	15
7471	620	2529	1196	691	8804	725	71	275	14
7843	619	2157	1611	690	8389	768	72	232	13
8214	619	1786	2025	690	7975	811	72	189	12
).508585	618	10.491415	9.532439	689	10.467561	10.023854	72	9.976146	11
8956	618	1044	2853	689	7147	897	72	103	10
9326	617	0674	3266	688	6734	940	72	060	9
9696	616	0304	3679	688	6321	983	72	017	8
).510065	616	10.489935	4092	687	5908	10.024026	72	9.975974	7
0434	615	9566	4504	687	5496	070	72	930	6
).510803	615	10.489197	9.534916	686	10.465084	10.024113	72	9.975887	5
1172	614	8828	5328	686	4672	156	72	844	4
1540	613	8460	5739	685	4261	200	72	800	3
1907	613	8093	6150	685	3850	243	72	757	2
2275	613	7725	6561	685	3439	286	72	714	1
2642	612	7358	6972	684	3028	330	72	670	0
Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	′

71 DEG.

19 Deg.	TABLE IV.—Log. Sines, etc.								
′	Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	
0	9.512642	612	10.487358	9.536972	684	10.463028	10.024330	73	9.975670	60
1	3009	611	6991	7352	683	2618	373	73	627	59
2	3375	611	6625	7792	683	2208	417	73	583	58
3	3741	610	6259	8202	682	1798	461	73	539	57
4	4107	609	5893	8611	682	1389	504	73	496	56
5	4472	609	5528	9020	681	0980	548	73	452	55
6	4837	608	5163	9429	681	0571	592	73	408	54
7	9.515202	608	10.484798	9.539837	680	10.460163	10.024635	73	9.975365	53
8	5566	607	4434	9.540245	680	10.459755	679	73	321	52
9	5930	607	4070	0653	679	9347	723	73	277	51
10	6294	606	3706	1061	679	8939	767	73	233	50
11	6657	605	3343	1468	678	8532	811	73	189	49
12	7020	605	2980	1875	678	8125	855	73	145	48
13	9.517382	604	10.482618	9.542281	677	10.457719	10.024899	73	9.975101	47
14	7745	604	2255	2688	677	7312	943	73	057	46
15	8107	603	1893	3094	676	6906	987	73	013	45
16	8468	603	1532	3499	676	6501	10.025031	74	9.974969	44
17	8829	602	1171	3905	675	6095	075	74	925	43
18	9190	601	0810	4310	675	5690	120	74	880	42
19	9.519551	601	10.480449	9.544715	674	10.455285	10.025164	74	9.974836	41
20	9911	600	0089	5119	674	4881	208	74	792	40
21	9.520271	600	10.479729	5524	673	4476	252	74	748	39
22	0631	599	9369	5928	673	4072	297	74	703	38
23	0990	599	9010	6331	672	3669	341	74	659	37
24	1349	598	8651	6735	672	3265	386	74	614	36
25	9.521707	598	10.478293	9.547138	671	10.452862	10.025430	74	9.974570	35
26	2066	597	7934	7540	671	2460	475	74	525	34
27	2424	596	7576	7943	670	2057	519	74	481	33
28	2781	596	7219	8345	670	1655	564	74	436	32
29	3138	595	6862	8747	669	1253	609	74	391	31
30	3495	595	6505	9149	669	0851	653	75	347	30
31	9.523852	594	10.476148	9.549550	668	10.450450	10.025698	75	9.974302	29
32	4208	594	5792	9951	668	0049	743	75	257	28
33	4564	593	5436	9.550352	667	10.449648	788	75	212	27
34	4920	593	5080	0752	667	9248	833	75	167	26
35	5275	592	4725	1152	666	8848	878	75	122	25
36	5630	591	4370	1552	666	8448	923	75	077	24
37	9.525984	591	10.474016	9.551952	665	10.448048	10.025968	75	9.974032	23
38	6339	590	3661	2351	665	7649	10.026013	75	9.973987	22
39	6693	590	3307	2750	665	7250	058	75	942	21
40	7046	589	2954	3149	664	6851	103	75	897	20
41	7400	589	2600	3548	664	6452	148	75	852	19
42	7753	588	2247	3946	663	6054	193	75	807	18
43	9.528105	588	10.471895	9.554344	663	10.445656	10.026239	75	9.973761	17
44	8458	587	1542	4741	662	5259	284	76	716	16
45	8810	587	1190	5139	662	4861	329	76	671	15
46	9161	586	0839	5536	661	4464	375	76	625	14
47	9513	586	0487	5933	661	4067	420	76	580	13
48	9864	585	0136	6329	660	3671	465	76	535	12
49	9.530215	585	10.469785	9.556725	660	10.443275	10.026511	76	9.973489	11
50	0565	584	9435	7121	659	2879	556	76	444	10
51	0915	584	9085	7517	659	2483	602	76	398	9
52	1265	583	8735	7913	659	2087	648	76	352	8
53	1614	582	8386	8308	658	1692	693	76	307	7
54	1963	582	8037	8702	658	1298	739	76	261	6
55	9.532312	581	10.467688	9.559097	657	10.440903	10.026785	76	9.973215	5
56	2661	581	7339	9491	657	0509	831	76	169	4
57	3009	580	6991	9885	656	0115	876	76	124	3
58	3357	580	6643	9.560279	656	10.439721	922	76	078	2
59	3704	579	6296	0673	655	9327	968	76	032	1
60	4052		5948	1066		9934	10.027014	77	9.972986	0
	Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	′

DEG. TABLE IV.—LOG. SINES, ETC.

Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	
.534052	578	10.465948	9.561066	655	10.438934	10.027014	77	9.972986	60
4399	577	5601	1459	654	8541	060	77	940	59
4745	577	5255	1851	654	8149	106	77	894	58
5092	577	4908	2244	653	7756	152	77	848	57
5438	576	4562	2636	653	7364	198	77	802	56
5783	576	4217	3028	653	6972	245	77	755	55
6129	575	3871	3419	652	6581	291	77	709	54
.536474	574	10.463526	9.563811	652	10.436189	10.027337	77	9.972663	53
6818	574	3182	4202	651	5798	383	77	617	52
7163	573	2837	4592	651	5408	430	77	570	51
7507	573	2493	4983	650	5017	476	77	524	50
7851	572	2149	5373	650	4627	522	77	478	49
8194	572	1806	5763	649	4237	569	78	431	48
.538538	571	10.461462	9.566153	649	10.433847	10.027615	78	9.972385	47
8880	571	1120	6542	649	3458	662	78	338	46
9223	570	0777	6932	648	3068	709	78	291	45
9565	570	0435	7320	648	2680	755	78	245	44
9907	569	0093	7709	647	2291	802	78	198	43
.540249	569	10.459751	8098	647	1902	849	78	151	42
.540590	568	10.459410	9.568486	646	10.431514	10.027895	78	9.972105	41
0931	568	9069	8873	646	1127	942	78	058	40
1272	567	8728	9261	645	0739	989	78	011	39
1613	567	8387	9648	645	0352	10.028036	78	9.971964	38
1953	566	8047	9.570035	645	10.429965	083	78	917	37
2293	566	7707	0422	644	9578	130	78	870	36
.542632	565	10.457368	9.570809	644	10.429191	10.028177	78	9.971823	35
2971	565	7029	1195	643	8805	224	78	776	34
3310	565	6690	1581	643	8419	271	79	729	33
3649	564	6351	1967	642	8033	318	79	682	32
3987	564	6013	2352	642	7648	365	79	635	31
4325	563	5675	2738	642	7262	412	79	588	30
.544663	563	10.455337	9.573123	641	10.426877	10.028460	79	9.971540	29
5000	562	5000	3507	641	6493	507	79	493	28
5338	562	4662	3892	640	6108	554	79	446	27
5674	561	4326	4276	640	5724	602	79	398	26
6011	561	3989	4660	639	5340	649	79	351	25
6347	560	3653	5044	639	4956	697	79	303	24
.546683	559	10.453317	9.575427	639	10.424573	10.028744	79	9.971256	23
7019	559	2981	5810	638	4190	792	79	208	22
7354	558	2646	6193	638	3807	839	79	161	21
7689	558	2311	6576	637	3424	887	79	113	20
8024	557	1976	6958	637	3042	934	80	066	19
8359	557	1641	7341	636	2659	982	80	018	18
.548693	556	10.451307	9.577723	636	10.422277	10.029030	80	9.970970	17
9027	556	0973	8104	636	1896	078	80	922	16
9360	555	0640	8486	635	1514	126	80	874	15
9693	555	0307	8867	635	1133	173	80	827	14
.550026	554	10.449974	9248	634	0752	221	80	779	13
0359	554	9641	9629	634	0371	269	80	731	12
.550692	553	10.449308	9.580009	634	10.419991	10.029317	80	9.970683	11
1024	553	8976	0389	633	9611	365	80	635	10
1356	552	8644	0769	633	9231	414	80	586	9
1687	552	8313	1149	632	8851	462	80	538	8
2018	552	7982	1528	632	8472	510	80	490	7
2349	551	7651	1907	632	8093	558	80	442	6
.552680	551	10.447320	9.582286	631	10.417714	10.029606	80	9.970394	5
3010	550	6990	2665	631	7335	655	81	345	4
3341	550	6659	3043	630	6957	703	81	297	3
3670	549	6330	3422	630	6578	751	81	249	2
4000	549	6000	3800	629	6200	800	81	200	1
4329		5671	4177		5823	848		152	0
Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	′

69 DEG.

21 DEG. TABLE IV.—LOG. SINES, ETC.

′	Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	
0	9.554329	548	10.445671	9.584177	629	10.415823	10.029848	81	9.970152	60
1	4658	548	5342	4555	629	5445	897	81	103	59
2	4987	547	5013	4932	628	5068	945	81	055	58
3	5315	547	4685	5309	628	4691	994	81	006	57
4	5643	546	4357	5686	627	4314	10.030043	81	9.969957	56
5	5971	546	4029	6062	627	3938	091	81	909	55
6	6299	545	3701	6439	627	3561	140	81	860	54
7	9.556626	545	10.443374	9.586815	626	10.413185	10.030189	81	9.969811	53
8	6953	544	3047	7190	626	2810	238	81	762	52
9	7280	544	2720	7566	625	2434	286	81	714	51
10	7606	543	2394	7941	625	2059	335	81	665	50
11	7932	543	2068	8316	625	1684	384	82	616	49
12	8258	543	1742	8691	624	1309	433	82	567	48
13	9.558583	542	10.441417	9.589066	624	10.410934	10.030482	82	9.969518	47
14	8909	542	1091	9440	623	0560	531	82	469	46
15	9234	541	0766	9814	623	0186	580	82	420	45
16	9558	541	0442	9.590188	623	10.409812	630	82	370	44
17	9883	540	0117	0562	622	9438	679	82	321	43
18	9.560207	540	10.439793	0935	622	9065	728	82	272	42
19	9.560531	539	10.439469	9.591308	622	10.408692	10.030777	82	9.969223	41
20	0855	539	9145	1681	621	8319	827	82	173	40
21	1178	538	8822	2054	621	7946	876	82	124	39
22	1501	538	8499	2426	620	7574	925	82	075	38
23	1824	537	8176	2798	620	7202	975	82	025	37
24	2146	537	7854	3171	619	6829	10.031024	82	9.968976	36
25	9.562468	536	10.437532	9.593542	619	10.406458	10.031074	83	9.968926	35
26	2790	536	7210	3914	618	6086	123	83	877	34
27	3112	536	6888	4285	618	5715	173	83	827	33
28	3433	535	6567	4656	618	5344	223	83	777	32
29	3755	535	6245	5027	617	4973	272	83	728	31
30	4075	534	5925	5398	617	4602	322	83	678	30
31	9.564396	534	10.435604	9.595768	617	10.404232	10.031372	83	9.968628	29
32	4716	533	5284	6138	616	3862	422	83	578	28
33	5036	533	4964	6508	616	3492	472	83	528	27
34	5356	532	4644	6878	616	3122	521	83	479	26
35	5676	532	4324	7247	615	2753	571	83	429	25
36	5995	531	4005	7616	615	2384	621	83	379	24
37	9.566314	531	10.433686	9.597985	615	10.402015	10.031671	83	9.968329	23
38	6632	531	3368	8354	614	1646	722	83	278	22
39	6951	530	3049	8722	614	1278	772	84	228	21
40	7269	530	2731	9091	613	0909	822	84	178	20
41	7587	529	2413	9459	613	0541	872	84	128	19
42	7904	529	2096	9827	613	0173	922	84	078	18
43	9.568222	528	10.431778	9.600194	612	10.399806	10.031973	84	9.968027	17
44	8539	528	1461	0562	612	9438	10.032023	84	9.967977	16
45	8856	528	1144	0929	611	9071	073	84	927	15
46	9172	527	0828	1296	611	8704	124	84	876	14
47	9488	527	0512	1662	611	8338	174	84	826	13
48	9804	526	0196	2029	610	7971	225	84	775	12
49	9.570120	526	10.429880	9.602395	610	10.397605	10.032275	84	9.967725	11
50	0435	525	9565	2761	610	7239	326	84	674	10
51	0751	525	9249	3127	609	6873	376	84	624	9
52	1066	524	8934	3493	609	6507	427	84	573	8
53	1380	524	8620	3858	609	6142	478	84	522	7
54	1695	523	8305	4223	608	5777	529	85	471	6
55	9.572009	523	10.427991	9.604588	608	10.395412	10.032579	85	9.967421	5
56	2323	523	7677	4953	607	5047	630	85	370	4
57	2636	522	7364	5317	607	4683	681	85	319	3
58	2950	522	7050	5682	607	4318	732	85	268	2
59	3263	521	6737	6046	607	3954	783	85	217	1
60	3575		6425	6410	606	3590	834	85	166	0
	Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	′

68 DEG.

Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	
.573575	521	10.426425	9.606410	606	10.393590	10.032834	85	9.967166	60
3888	520	6112	6773	606	3227	885	85	115	59
4200	520	5800	7137	605	2863	936	85	064	58
4512	519	5488	7500	605	2500	987	85	013	57
4824	519	5176	7863	604	2137	10.033039	85	9.966961	56
5136	519	4864	8225	604	1775	090	85	910	55
5447	518	4553	8588	604	1412	141	85	859	54
.575758	518	10.424242	9.608950	603	10.391050	10.033192	85	9.966808	53
6069	517	3931	9312	603	0688	244	86	756	52
6379	517	3621	9674	603	0326	295	86	705	51
6689	516	3311	9.610036	602	10.389964	347	86	653	50
6999	516	3001	0397	602	9603	398	86	602	49
7309	516	2691	0759	602	9241	450	86	550	48
.577618	515	10.422382	9.611120	601	10.388880	10.033501	86	9.966499	47
7927	515	2073	1480	601	8520	553	86	447	46
8236	514	1764	1841	601	8159	605	86	395	45
8545	514	1455	2201	600	7799	656	86	344	44
8853	513	1147	2561	600	7439	708	86	292	43
9162	513	0838	2921	600	7079	760	86	240	42
.579470	513	10.420530	9.613281	599	10.386719	10.033812	86	9.966188	41
9777	512	0223	3641	599	6359	864	86	136	40
.580085	512	10.419915	4000	598	6000	915	87	085	39
0392	511	9608	4359	598	5641	967	87	033	38
0699	511	9301	4718	598	5282	10.034019	87	9.965981	37
1005	511	8995	5077	597	4923	072	87	928	36
.581312	510	10.418688	9.615435	597	10.384565	10.034124	87	9.965876	35
1618	510	8382	5793	597	4207	176	87	824	34
1924	509	8076	6151	596	3849	228	87	772	33
2229	509	7771	6509	596	3491	280	87	720	32
2535	509	7465	6867	596	3133	332	87	668	31
2840	508	7160	7224	595	2776	385	87	615	30
.583145	508	10.416855	9.617582	595	10.382418	10.034437	87	9.965563	29
3449	507	6551	7939	595	2061	489	87	511	28
3754	507	6246	8295	594	1705	542	87	458	27
4058	506	5942	8652	594	1348	594	87	406	26
4361	506	5639	9008	594	0992	647	88	353	25
4665	506	5335	9364	593	0636	699	88	301	24
.584968	505	10.415032	9.619721	593	10.380279	10.034752	88	9.965248	23
5272	505	4728	9.620076	593	10.379924	805	88	195	22
5574	504	4426	0432	592	9568	857	88	143	21
5877	504	4123	0787	592	9213	910	88	090	20
6179	503	3821	1142	592	8858	963	88	037	19
6482	503	3518	1497	591	8503	10.035016	88	9.964984	18
.586783	503	10.413217	9.621852	591	10.378148	10.035069	88	9.964931	17
7085	502	2915	2207	590	7793	121	88	879	16
7386	502	2614	2561	590	7439	174	88	826	15
7688	501	2312	2915	590	7085	227	88	773	14
7989	501	2011	3269	589	6731	281	88	719	13
8289	501	1711	3623	589	6377	334	89	666	12
.588590	500	10.411410	9.623976	589	10.376024	10.035387	89	9.964613	11
8890	500	1110	4330	588	5670	440	89	560	10
9190	499	0810	4683	588	5317	493	89	507	9
9489	499	0511	5036	588	4964	546	89	454	8
9789	499	0211	5388	587	4612	600	89	400	7
.590088	498	10.409912	5741	587	4259	653	89	347	6
.590387	498	10.409613	9.626093	587	10.373907	10.035706	89	9.964294	5
0686	497	9314	6445	586	3555	760	89	240	4
0984	497	9016	6797	586	3203	813	89	187	3
1282	497	8718	7149	586	2851	867	89	133	2
1580	496	8420	7501	585	2499	920	89	080	1
1878		8122	7852		2148	974		026	0
Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	/

23 DEG. TABLE IV.—LOG. SINES, ETC.

′	Sine	D.	Cosec.	Tang	D.	Cotang.	Secant	D.	Cosine	
0	9.591878	496	10.408122	9.627852	585	10.372148	10.035974	89	9.964026	60
1	2176	495	7824	8203	585	1797	6028	89	3972	59
2	2473	495	7527	8554	585	1446	6081	89	3919	58
3	2770	495	7230	8905	584	1095	6135	90	3865	57
4	3067	494	6933	9255	584	0745	6189	90	3811	56
5	3363	494	6637	9606	583	0394	6243	90	3757	55
6	3659	493	6341	9956	583	0044	6296	90	3704	54
7	9.593955	493	10.406045	9.630306	583	10.369694	10.036350	90	9.963650	53
8	4251	493	5749	0656	583	9344	6404	90	3596	52
9	4547	492	5453	1005	582	8995	6458	90	3542	51
10	4842	492	5158	1355	582	8645	6512	90	3488	50
11	5137	491	4863	1704	582	8296	6566	90	3434	49
12	5432	491	4568	2053	581	7947	6621	90	3379	48
13	9.595727	491	10.404273	9.632401	581	10.367599	10.036675	90	9.963325	47
14	6021	490	3979	2750	581	7250	6729	90	3271	46
15	6315	490	3685	3098	580	6902	6783	90	3217	45
16	6609	489	3391	3447	580	6553	6837	90	3163	44
17	6903	489	3097	3795	580	6205	6892	90	3108	43
18	7196	489	2804	4143	579	5857	6946	91	3054	42
19	9.597490	488	10.402510	9.634490	579	10.365510	10.037001	91	9.962999	41
20	7783	488	2217	4838	579	5162	7055	91	2945	40
21	8075	487	1925	5185	578	4815	7110	91	2890	39
22	8368	487	1632	5532	578	4468	7164	91	2836	38
23	8660	487	1340	5879	578	4121	7219	91	2781	37
24	8952	486	1048	6226	577	3774	7273	91	2727	36
25	9.599244	486	10.400756	9.636572	577	10.363428	10.037328	91	9.962672	35
26	9536	485	0464	6919	577	3081	7383	91	2617	34
27	9827	485	0173	7265	577	2735	7438	91	2562	33
28	9.600118	485	10.399882	7611	576	2389	7492	91	2508	32
29	0409	484	9591	7956	576	2044	7547	91	2453	31
30	0700	484	9300	8302	576	1698	7602	91	2398	30
31	9.600990	484	10.399010	9.638647	575	10.361353	10.037657	92	9.962343	29
32	1280	483	8720	8992	575	1008	7712	92	2288	28
33	1570	483	8430	9337	575	0663	7767	92	2233	27
34	1860	482	8140	9682	574	0318	7822	92	2178	26
35	2150	482	7850	9.640027	574	10.359973	7877	92	2123	25
36	2439	482	7561	0371	574	9629	7933	92	2067	24
37	9.602728	481	10.397272	9.640716	573	10.359284	10.037988	92	9.962012	23
38	3017	481	6983	1060	573	8940	8043	92	1957	22
39	3305	481	6695	1404	573	8596	8098	92	1902	21
40	3594	480	6406	1747	572	8253	8154	92	1846	20
41	3882	480	6118	2091	572	7909	8209	92	1791	19
42	4170	479	5830	2434	572	7566	8265	92	1735	18
43	9.604457	479	10.395543	9.642777	572	10.357223	10.038320	92	9.961680	17
44	4745	479	5255	3120	571	6880	8376	93	1624	16
45	5032	478	4968	3463	571	6537	8431	93	1569	15
46	5319	478	4681	3806	571	6194	8487	93	1513	14
47	5606	478	4394	4148	570	5852	8542	93	1458	13
48	5892	477	4108	4490	570	5510	8598	93	1402	12
49	9.606179	477	10.393821	9.644832	570	10.355168	10.038654	93	9.961346	11
50	6465	476	3535	5174	569	4826	8710	93	1290	10
51	6751	476	3249	5516	569	4484	8765	93	1235	9
52	7036	476	2964	5857	569	4143	8821	93	1179	8
53	7322	475	2678	6199	569	3801	8877	93	1123	7
54	7607	475	2393	6540	568	3460	8933	93	1067	6
55	9.607892	474	10.392108	9.646881	568	10.353119	10.038989	93	9.961011	5
56	8177	474	1823	7222	568	2778	9045	93	0955	4
57	8461	474	1539	7562	567	2438	9101	93	0899	3
58	8745	473	1255	7903	567	2097	9157	94	0843	2
59	9029	473	0971	8243	567	1757	9214	94	0786	1
60	9313		0687	8583		1417	9270		0730	0
	Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	′

66 DEG.

)EG. TABLE IV.—Log. Sines, etc.

Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	
.609313	473	10.390687	9.648583	566	10.351417	10.039270	94	9.960730	60
9597	472	0403	8923	566	1077	9326	94	0674	59
9880	472	0120	9263	566	0737	9382	94	0618	58
.610164	472	10.389836	9602	566	0398	9439	94	0561	57
0447	471	9553	9942	565	0058	9495	94	0505	56
0729	471	9271	9.650281	565	10.349719	9552	94	0448	55
1012	470	8988	0620	565	9380	9608	94	0392	54
.611294	470	10.388706	9.650959	564	10.349041	10.039665	94	9.960335	53
1576	470	8424	1297	564	8703	9721	94	0279	52
1858	469	8142	1636	564	8364	9778	94	0222	51
2140	469	7860	1974	563	8026	9835	94	0165	50
2421	469	7579	2312	563	7688	9891	95	0109	49
2702	468	7298	2650	563	7350	9948	95	0052	48
.612983	468	10.387017	9.652988	563	10.347012	10.040005	95	9.959995	47
3264	467	6736	3326	562	6674	0062	95	9938	46
3545	467	6455	3663	562	6337	0118	95	9882	45
3825	467	6175	4000	562	6000	0175	95	9825	44
4105	466	5895	4337	561	5663	0232	95	9768	43
4385	466	5615	4674	561	5326	0289	95	9711	42
.614665	466	10.385335	9.655011	561	10.344989	10.040346	95	9.959654	41
4944	465	5056	5348	561	4652	0404	95	9596	40
5223	465	4777	5684	560	4316	0461	95	9539	39
5502	465	4498	6020	560	3980	0518	95	9482	38
5781	464	4219	6356	560	3644	0575	95	9425	37
6060	464	3940	6692	559	3308	0632	95	9368	36
.616338	464	10.383662	9.657028	559	10.342972	10.040690	96	9.959310	35
6616	463	3384	7364	559	2636	0747	96	9253	34
6894	463	3106	7699	559	2301	0805	96	9195	33
7172	462	2828	8034	558	1966	0862	96	9138	32
7450	462	2550	8369	558	1631	0919	96	9081	31
7727	462	2273	8704	558	1296	0977	96	9023	30
.618004	461	10.381996	9.659039	558	10.340961	10.041035	96	9.958965	29
8281	461	1719	9373	557	0627	1092	96	8908	28
8558	461	1442	9708	557	0292	1150	96	8850	27
8834	460	1166	9.660042	557	10.339958	1208	96	8792	26
9110	460	0890	0376	557	9624	1266	96	8734	25
9386	460	0614	0710	556	9290	1323	96	8677	24
.619662	459	10.380338	9.661043	556	10.338957	10.041381	96	9.958619	23
9938	459	0062	1377	556	8623	1439	96	8561	22
.620213	459	10.379787	1710	555	8290	1497	97	8503	21
0488	458	9512	2043	555	7957	1555	97	8445	20
0763	458	9237	2376	555	7624	1613	97	8387	19
1038	457	8962	2709	554	7291	1671	97	8329	18
.621313	457	10.378687	9.663042	554	10.336958	10.041729	97	9.958271	17
1587	457	8413	3375	554	6625	1787	97	8213	16
1861	456	8139	3707	554	6293	1846	97	8154	15
2135	456	7865	4039	553	5961	1904	97	8096	14
2409	456	7591	4371	553	5629	1962	97	8038	13
2682	455	7318	4703	553	5297	2021	97	7979	12
.622956	455	10.377044	9.665035	553	10.334965	10.042079	97	9.957921	11
3229	455	6771	5366	552	4634	2137	97	7863	10
3502	454	6498	5697	552	4303	2196	97	7804	9
3774	454	6226	6029	552	3971	2254	97	7746	8
4047	454	5953	6360	551	3640	2313	98	7687	7
4319	453	5681	6691	551	3309	2372	98	7628	6
.624591	453	10.375409	9.667021	551	10.332979	10.042430	98	9.957570	5
4863	453	5137	7352	551	2648	2489	98	7511	4
5135	452	4865	7682	550	2318	2548	98	7452	3
5406	452	4594	8013	550	1987	2607	98	7393	2
5677	452	4323	8343	550	1657	2665	98	7335	1
5948	452	4052	8672	550	1328	2724	98	7276	0
Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	/

25 DEG. TABLE IV.—LOG. SINES, ETC.

′	Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	
0	9.625948	451	10.374052	9.668673	550	10.331327	10.042724	98	9.957276	60
1	6219	451	3781	9002	549	0998	2783	98	7217	59
2	6490	451	3510	9332	549	0668	2842	98	7158	58
3	6760	450	3240	9661	549	0339	2901	98	7099	57
4	7030	450	2970	9991	548	0009	2960	98	7040	56
5	7300	450	2700	9.670320	548	10.329680	3019	98	6981	55
6	7570	449	2430	0649	548	9351	3079	99	6921	54
7	9.627840	449	10.372160	9.670977	548	10.329023	10.043138	99	9.956862	53
8	8109	449	1891	1306	547	8694	3197	99	6803	52
9	8378	448	1622	1634	547	8366	3256	99	6744	51
10	8647	448	1353	1963	547	8037	3316	99	6684	50
11	8916	447	1084	2291	547	7709	3375	99	6625	49
12	9185	447	0815	2619	546	7381	3434	99	6566	48
13	9.629453	447	10.370547	9.672947	546	10.327053	10.043494	99	9.956506	47
14	9721	446	0279	3274	546	6726	3553	99	6447	46
15	9989	446	0011	3602	546	6398	3613	99	6387	45
16	9.630257	446	10.369743	3929	545	6071	3673	99	6327	44
17	0524	446	9476	4257	545	5743	3732	99	6268	43
18	0792	445	9208	4584	545	5416	3792	100	6208	42
19	9.631059	445	10.368941	9.674910	544	10.325090	10.043852	100	9.956148	41
20	1326	445	8674	5237	544	4763	3911	100	6089	40
21	1593	444	8407	5564	544	4436	3971	100	6029	39
22	1859	444	8141	5890	544	4110	4031	100	5969	38
23	2125	444	7875	6216	543	3784	4091	100	5909	37
24	2392	443	7608	6543	543	3457	4151	100	5849	36
25	9.632658	443	10.367342	9.676869	543	10.323131	10.044211	100	9.955789	35
26	2923	443	7077	7194	543	2806	4271	100	5729	34
27	3189	442	6811	7520	542	2480	4331	100	5669	33
28	3454	442	6546	7846	542	2154	4391	100	5609	32
29	3719	442	6281	8171	542	1829	4452	100	5548	31
30	3984	441	6016	8496	542	1504	4512	100	5488	30
31	9.634249	441	10.365751	9.678821	541	10.321179	10.044572	101	9.955428	29
32	4514	440	5486	9146	541	0854	4632	101	5368	28
33	4778	440	5222	9471	541	0529	4693	101	5307	27
34	5042	440	4958	9795	541	0205	4753	101	5247	26
35	5306	439	4694	9.680120	540	10.319880	4814	101	5186	25
36	5570	439	4430	0444	540	9556	4874	101	5126	24
37	9.635834	439	10.364166	9.680768	540	10.319232	10.044935	101	9.955065	23
38	6097	438	3903	1092	540	8908	4995	101	5005	22
39	6360	438	3640	1416	539	8584	5056	101	4944	21
40	6623	438	3377	1740	539	8260	5117	101	4883	20
41	6886	437	3114	2063	539	7937	5177	101	4823	19
42	7148	437	2852	2387	539	7613	5238	101	4762	18
43	9.637411	437	10.362589	9.682710	538	10.317290	10.045299	101	9.954701	17
44	7673	437	2327	3033	538	6967	5360	101	4640	16
45	7935	436	2065	3356	538	6644	5421	101	4579	15
46	8197	436	1803	3679	538	6321	5482	102	4518	14
47	8458	436	1542	4001	537	5999	5543	102	4457	13
48	8720	435	1280	4324	537	5676	5604	102	4396	12
49	9.638981	435	10.361019	9.684646	537	10.315354	10.045665	102	9.954335	11
50	9242	435	0758	4968	537	5032	5726	102	4274	10
51	9503	434	0497	5290	536	4710	5787	102	4213	9
52	9764	434	0236	5612	536	4388	5848	102	4152	8
53	9.640024	434	10.359976	5934	536	4066	5910	102	4090	7
54	0284	433	9716	6255	536	3745	5971	102	4029	6
55	9.640544	433	10.359456	9.686577	535	10.313423	10.046032	102	9.953968	5
56	0804	433	9196	6898	535	3102	6094	102	3906	4
57	1064	432	8936	7219	535	2781	6155	102	3845	3
58	1324	432	8676	7540	535	2460	6217	102	3783	2
59	1584	432	8416	7861	534	2139	6278	102	3722	1
60	1842		8158	8182		1818	6340	103	3660	0
	Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	′

64 DEG.

DEG. TABLE IV.—LOG. SINES, ETC.

Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	
9.641842	431	10.358158	9.688182	534	10.311818	10.046340	103	9.953660	60
2101	431	7899	8502	534	1498	6401	103	3599	59
2360	431	7640	8823	534	1177	6463	103	3537	58
2618	430	7382	9143	533	0857	6525	103	3475	57
2877	430	7123	9463	533	0537	6587	103	3413	56
3135	430	6865	9783	533	0217	6648	103	3352	55
3393	430	6607	9.690103	533	10.309897	6710	103	3290	54
9.643650	429	10.356350	9.690423	533	10.309577	10.046772	103	9.953228	53
3908	429	6092	0742	532	9258	6834	103	3166	52
4165	429	5835	1062	532	8938	6896	103	3104	51
4423	428	5577	1381	532	8619	6958	103	3042	50
4680	428	5320	1700	531	8300	7020	104	2980	49
4936	428	5064	2019	531	7981	7082	104	2918	48
9.645193	427	10.354807	9.692338	531	10.307662	10.047145	104	9.952855	47
5450	427	4550	2656	531	7344	7207	104	2793	46
5706	427	4294	2975	531	7025	7269	104	2731	45
5962	426	4038	3293	530	6707	7331	104	2669	44
6218	426	3782	3612	530	6388	7394	104	2606	43
6474	426	3526	3930	530	6070	7456	104	2544	42
9.646729	425	10.353271	9.694248	530	10.305752	10.047519	104	9.952481	41
6984	425	3016	4566	529	5434	7581	104	2419	40
7240	425	2760	4883	529	5117	7644	104	2356	39
7494	424	2506	5201	529	4799	7706	104	2294	38
7749	424	2251	5518	529	4482	7769	104	2231	37
8004	424	1996	5836	529	4164	7832	105	2168	36
9.648258	424	10.351742	9.696153	528	10.303847	10.047894	105	9.952106	35
8512	423	1488	6470	528	3530	7957	105	2043	34
8766	423	1234	6787	528	3213	8020	105	1980	33
9020	423	0980	7103	528	2897	8083	105	1917	32
9274	422	0726	7420	527	2580	8146	105	1854	31
9527	422	0473	7736	527	2264	8209	105	1791	30
9.649781	422	10.350219	9.698053	527	10.301947	10.048272	105	9.951728	29
9.650034	422	10.349966	8369	527	1631	8335	105	1665	28
0287	421	9713	8685	526	1315	8398	105	1602	27
0539	421	9461	9001	526	0999	8461	105	1539	26
0792	421	9208	9316	526	0684	8524	105	1476	25
1044	420	8956	9632	526	0368	8588	105	1412	24
9.651297	420	10.348703	9.699947	526	10.300053	10.048651	106	9.951349	23
1549	420	8451	9.700263	525	10.299737	8714	106	1286	22
1800	419	8200	0578	525	9422	8778	106	1222	21
2052	419	7948	0893	525	9107	8841	106	1159	20
2304	419	7696	1208	524	8792	8904	106	1096	19
2555	418	7445	1523	524	8477	8968	106	1032	18
9.652806	418	10.347194	9.701837	524	10.298163	10.049032	106	9.950968	17
3057	418	6943	2152	524	7848	9095	106	0905	16
3308	418	6692	2466	524	7534	9159	106	0841	15
3558	417	6442	2780	523	7220	9222	106	0778	14
3808	417	6192	3095	523	6905	9286	106	0714	13
4059	417	5941	3409	523	6591	9350	106	0650	12
9.654309	416	10.345691	9.703723	523	10.296277	10.049414	106	9.950586	11
4558	416	5442	4036	522	5964	9478	107	0522	10
4808	416	5192	4350	522	5650	9542	107	0458	9
5058	416	4942	4663	522	5337	9606	107	0394	8
5307	415	4693	4977	522	5023	9670	107	0330	7
5556	415	4444	5290	522	4710	9734	107	0266	6
9.655805	415	10.344195	9.705603	521	10.294397	10.049798	107	9.950202	5
6054	414	3946	5916	521	4084	9862	107	0138	4
6302	414	3698	6228	521	3772	9926	107	0074	3
6551	414	3449	6541	521	3459	9990	107	0010	2
6799	413	3201	6854	521	3146	10.050055	107	9.949945	1
7047		2953	7166		2834	0119	107	9881	0
Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	′

63 DEG.

27 Deg.	TABLE IV.—Log. Sines, etc.									
′	Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	
0	9.657047	413	10.342953	9.707166	520	10.292834	10.050119	107	9.949881	60
1	7295	413	2705	7478	520	2522	0184	107	9816	59
2	7542	412	2458	7790	520	2210	0248	107	9752	58
3	7790	412	2210	8102	520	1898	0312	108	9688	57
4	8037	412	1963	8414	519	1586	0377	108	9623	56
5	8284	412	1716	8726	519	1274	0442	108	9558	55
6	8531	411	1469	9037	519	0963	0506	108	9494	54
7	9.658778	411	10.341222	9.709349	519	10.290651	10.050571	108	9.949429	53
8	9025	411	0975	9660	519	0340	0636	108	9364	52
9	9271	410	0729	9971	518	0029	0700	108	9300	51
10	9517	410	0483	9.710282	518	10.289718	0765	108	9235	50
11	9763	410	0237	0593	518	9407	0830	108	9170	49
12	9.660009	409	10.339991	0904	518	9096	0895	108	9105	48
13	9.660255	409	10.339745	9.711215	518	10.288785	10.050960	108	9.949040	47
14	0501	409	9499	1525	517	8475	1025	108	8975	46
15	0746	409	9254	1836	517	8164	1090	108	8910	45
16	0991	408	9009	2146	517	7854	1155	108	8845	44
17	1236	408	8764	2456	517	7544	1220	109	8780	43
18	1481	408	8519	2766	516	7234	1285	109	8715	42
19	9.661726	407	10.338274	9.713076	516	10.286924	10.051350	109	9.948650	41
20	1970	407	8030	3386	516	6614	1416	109	8584	40
21	2214	407	7786	3696	516	6304	1481	109	8519	39
22	2459	407	7541	4005	516	5995	1546	109	8454	38
23	2703	406	7297	4314	515	5686	1612	109	8388	37
24	2946	406	7054	4624	515	5376	1677	109	8323	36
25	9.663190	406	10.336810	9.714933	515	10.285067	10.051743	109	9.948257	35
26	3433	405	6567	5242	515	4758	1808	109	8192	34
27	3677	405	6323	5551	514	4449	1874	109	8126	33
28	3920	405	6080	5860	514	4140	1940	109	8060	32
29	4163	405	5837	6168	514	3832	2005	110	7995	31
30	4406	404	5594	6477	514	3523	2071	110	7929	30
31	9.664648	404	10.335352	9.716785	514	10.283215	10.052137	110	9.947863	29
32	4891	404	5109	7093	513	2907	2203	110	7797	28
33	5133	403	4867	7401	513	2599	2269	110	7731	27
34	5375	403	4625	7709	513	2291	2335	110	7665	26
35	5617	403	4383	8017	513	1983	2400	110	7600	25
36	5859	402	4141	8325	513	1675	2467	110	7533	24
37	9.666100	402	10.333900	9.718633	512	10.281367	10.052533	110	9.947467	23
38	6342	402	3658	8940	512	1060	2599	110	7401	22
39	6583	402	3417	9248	512	0752	2665	110	7335	21
40	6824	401	3176	9555	512	0445	2731	110	7269	20
41	7065	401	2935	9862	512	0138	2797	110	7203	19
42	7305	401	2695	9.720169	511	10.279831	2864	111	7136	18
43	9.667546	401	10.332454	9.720476	511	10.279524	10.052930	111	9.947070	17
44	7786	400	2214	0783	511	9217	2996	111	7004	16
45	8027	400	1973	1089	511	8911	3063	111	6937	15
46	8267	400	1733	1396	511	8604	3129	111	6871	14
47	8506	399	1494	1702	510	8298	3196	111	6804	13
48	8746	399	1254	2009	510	7991	3262	111	6738	12
49	9.668986	399	10.331014	9.722315	510	10.277685	10.053329	111	9.946671	11
50	9225	399	0775	2621	510	7379	3396	111	6604	10
51	9464	398	0536	2927	510	7073	3462	111	6538	9
52	9703	398	0297	3232	509	6768	3529	111	6471	8
53	9942	398	0058	3538	509	6462	3596	111	6404	7
54	9.670181	397	10.329819	3844	509	6156	3663	111	6337	6
55	9.670419	397	10.329581	9.724149	509	10.275851	10.053730	112	9.946270	5
56	0658	397	9342	4454	509	5546	3797	112	6203	4
57	0896	397	9104	4759	508	5241	3864	112	6136	3
58	1134	396	8866	5065	508	4935	3931	112	6069	2
59	1372	396	8628	5369	508	4631	3998	112	6002	1
60	1609		8391	5674	508	4326	4065	112	5935	0
	Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	′

62 Deg.

DEG. TABLE IV.—LOG. SINES, ETC.

Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	
.671609		10.328391	9.725674		10.274326	10.054065		9.945935	60
1847	396	8153	5979	508	4021	4132	112	5868	59
2084	395	7916	6284	508	3716	4200	112	5800	58
2321	395	7679	6588	507	3412	4267	112	5733	57
2558	395	7442	6892	507	3108	4334	112	5666	56
2795	395	7205	7197	507	2803	4402	112	5598	55
3032	394	6968	7501	507	2499	4469	112	5531	54
.673268	394	10.326732	9.727805	507	10.272195	10.054536	113	9.945464	53
3505	394	6495	8109	506	1891	4604	113	5396	52
3741	393	6259	8412	506	1588	4672	113	5328	51
3977	393	6023	8716	506	1284	4739	113	5261	50
4213	393	5787	9020	506	0980	4807	113	5193	49
4448	392	5552	9323	506	0677	4875	113	5125	48
.674684	392	10.325316	9.729626	505	10.270374	10.054942	113	9.945058	47
4919	392	5081	9929	505	0071	5010	113	4990	46
5155	392	4845	9.730233	505	10.269767	5078	113	4922	45
5390	391	4610	0535	505	9465	5146	113	4854	44
5624	391	4376	0838	504	9162	5214	113	4786	43
5859	391	4141	1141	504	8859	5282	113	4718	42
.676094	391	10.323906	9.731444	504	10.268556	10.055350	113	9.944650	41
6328	390	3672	1746	504	8254	5418	114	4582	40
6562	390	3438	2048	504	7952	5486	114	4514	39
6796	390	3204	2351	503	7649	5554	114	4446	38
7030	390	2970	2653	503	7347	5623	114	4377	37
7264	389	2736	2955	503	7045	5691	114	4309	36
.677498	389	10.322502	9.733257	503	10.266743	10.055759	114	9.944241	35
7731	389	2269	3558	503	6442	5828	114	4172	34
7964	388	2036	3860	502	6140	5896	114	4104	33
8197	388	1803	4162	502	5838	5964	114	4036	32
8430	388	1570	4463	502	5537	6033	114	3967	31
8663	388	1337	4764	502	5236	6101	114	3899	30
.678895	387	10.321105	9.735066	502	10.264934	10.056170	114	9.943830	29
9128	387	0872	5367	502	4633	6239	114	3761	28
9360	387	0640	5668	501	4332	6307	115	3693	27
9592	387	0408	5969	501	4031	6376	115	3624	26
9824	386	0176	6269	501	3731	6445	115	3555	25
.680056	386	10.319944	6570	501	3430	6514	115	3486	24
.680288	386	10.319712	9.736871	501	10.263129	10.056583	115	9.943417	23
0519	385	9481	7171	500	2829	6652	115	3348	22
0750	385	9250	7471	500	2529	6721	115	3279	21
0982	385	9018	7771	500	2229	6790	115	3210	20
1213	385	8787	8071	500	1929	6859	115	3141	19
1443	384	8557	8371	500	1629	6928	115	3072	18
.681674	384	10.318326	9.738671	499	10.261329	10.056997	115	9.943003	17
1905	384	8095	8971	499	1029	7066	115	2934	16
2135	384	7865	9271	499	0729	7136	115	2864	15
2365	383	7635	9570	499	0430	7205	115	2795	14
2595	383	7405	9870	499	0130	7274	116	2726	13
2825	383	7175	9.740169	499	10.259831	7344	116	2656	12
.683055	383	10.316945	9.740468	498	10.259532	10.057413	116	9.942587	11
3284	382	6716	0767	498	9233	7483	116	2517	10
3514	382	6486	1066	498	8934	7552	116	2448	9
3743	382	6257	1365	498	8635	7622	116	2378	8
3972	382	6028	1664	498	8336	7692	116	2308	7
4201	381	5799	1962	497	8038	7761	116	2239	6
.684430	381	10.315570	9.742261	497	10.257739	10.057831	116	9.942169	5
4658	381	5342	2559	497	7441	7901	116	2099	4
4887	380	5113	2858	497	7142	7971	116	2029	3
5115	380	4885	3156	497	6844	8041	116	1959	2
5343	380	4657	3454	497	6546	8111	116	1889	1
5571	380	4429	3752	497	6248	8181	117	1819	0
Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	′

61 DEG.

29 DEG. TABLE IV.—LOG. SINES, ETC.

′	Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	
0	9.685572	380	10.314429	9.743752	496	10.256248	10.058181	117	9.941819	60
1	5799	379	4201	4050	496	5950	8251	117	1749	59
2	6027	379	3973	4348	496	5652	8321	117	1679	58
3	6254	379	3746	4645	496	5355	8391	117	1609	57
4	6482	379	3518	4943	496	5057	8461	117	1539	56
5	6709	378	3291	5240	495	4760	8531	117	1469	55
6	6936	378	3064	5538	495	4462	8602	117	1398	54
7	9.687163	378	10.312837	9.745835	495	10.254165	10.058672	117	9.941328	53
8	7389	378	2611	6132	495	3868	8742	117	1258	52
9	7616	377	2384	6429	495	3571	8813	117	1187	51
10	7843	377	2157	6726	495	3274	8883	117	1117	50
11	8069	377	1931	7023	494	2977	8954	118	1046	49
12	8295	377	1705	7319	494	2681	9025	118	0975	48
13	9.688521	376	10.311479	9.747616	494	10.252384	10.059095	118	9.940905	47
14	8747	376	1253	7913	494	2087	9166	118	0834	46
15	8972	376	1028	8209	494	1791	9237	118	0763	45
16	9198	376	0802	8505	493	1495	9307	118	0693	44
17	9423	375	0577	8801	493	1199	9378	118	0622	43
18	9648	375	0352	9097	493	0903	9449	118	0551	42
19	9.689873	375	10.310127	9.749393	493	10.250607	10.059520	118	9.940480	41
20	9.690098	375	10.309902	9689	493	0311	9591	118	0409	40
21	0323	374	9677	9985	493	0015	9662	118	0338	39
22	0548	374	9452	9.750281	492	10.249719	9733	118	0267	38
23	0772	374	9228	0576	492	9424	9804	118	0196	37
24	0996	374	9004	0872	492	9128	9875	119	0125	36
25	9.691220	373	10.308780	9.751167	492	10.248833	10.059946	119	9.940054	35
26	1444	373	8556	1462	492	8538	10.060018	119	9.939982	34
27	1668	373	8332	1757	492	8243	0089	119	9911	33
28	1892	373	8108	2052	491	7948	0160	119	9840	32
29	2115	372	7885	2347	491	7653	0232	119	9768	31
30	2339	372	7661	2642	491	7358	0303	119	9697	30
31	9.692562	372	10.307438	9.752937	491	10.247063	10.060375	119	9.939625	29
32	2785	371	7215	3231	491	6769	0446	119	9554	28
33	3008	371	6992	3526	491	6474	0518	119	9482	27
34	3231	371	6769	3820	490	6180	0590	119	9410	26
35	3453	371	6547	4115	490	5885	0661	119	9339	25
36	3676	370	6324	4409	490	5591	0733	120	9267	24
37	9.693898	370	10.306102	9.754703	490	10.245297	10.060805	120	9.939195	23
38	4120	370	5880	4997	490	5003	0877	120	9123	22
39	4342	370	5658	5291	490	4709	0948	120	9052	21
40	4564	369	5436	5585	489	4415	1020	120	8980	20
41	4786	369	5214	5878	489	4122	1092	120	8908	19
42	5007	369	4993	6172	489	3828	1164	120	8836	18
43	9.695229	369	10.304771	9.756465	489	10.243535	10.061237	120	9.938763	17
44	5450	368	4550	6759	489	3241	1309	120	8691	16
45	5671	368	4329	7052	489	2948	1381	120	8619	15
46	5892	368	4108	7345	488	2655	1453	120	8547	14
47	6113	368	3887	7638	488	2362	1525	120	8475	13
48	6334	367	3666	7931	488	2069	1598	121	8402	12
49	9.696554	367	10.303446	9.758224	488	10.241776	10.061670	121	9.938330	11
50	6775	367	3225	8517	488	1483	1742	121	8258	10
51	6995	367	3005	8810	488	1190	1815	121	8185	9
52	7215	366	2785	9102	487	0898	1887	121	8113	8
53	7435	366	2565	9395	487	0605	1960	121	8040	7
54	7654	366	2346	9687	487	0313	2033	121	7967	6
55	9.697874	366	10.302126	9.759979	487	10.240021	10.062105	121	9.937895	5
56	8094	366	1906	9.760272	487	10.239728	2178	121	7822	4
57	8313	365	1687	0564	487	9436	2251	121	7749	3
58	8532	365	1468	0856	487	9144	2324	121	7676	2
59	8751	365	1249	1148	486	8852	2396	121	7604	1
60	8970	365	1030	1439	486	8561	2469	121	7531	0
	Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	′

60 DEG.

DEG. TABLE IV.—LOG. SINES, ETC.

Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	
9.698970	364	10.301030	9.761439	486	10.238561	10.062469	121	9.937531	60
9189	364	0811	1731	486	8269	2542	122	7458	59
9407	364	0593	2023	486	7977	2615	122	7385	58
9626	364	0374	2314	486	7686	2688	122	7312	57
9844	363	0156	2606	485	7394	2762	122	7238	56
9.700062	363	10.299938	2897	485	7103	2835	122	7165	55
0280	363	9720	3188	485	6812	2908	122	7092	54
9.700498	363	10.299502	9.763479	485	10.236521	10.062981	122	9.937019	53
0716	363	9284	3770	485	6230	3054	122	6946	52
0933	362	9067	4061	485	5939	3128	122	6872	51
1151	362	8849	4352	484	5648	3201	122	6799	50
1368	362	8632	4643	484	5357	3275	122	6725	49
1585	362	8415	4933	484	5067	3348	122	6652	48
9.701802	361	10.298198	9.765224	484	10.234776	10.063422	123	9.936578	47
2019	361	7981	5514	484	4486	3495	123	6505	46
2236	361	7764	5805	484	4195	3569	123	6431	45
2452	361	7548	6095	484	3905	3643	123	6357	44
2669	360	7331	6385	483	3615	3716	123	6284	43
2885	360	7115	6675	483	3325	3790	123	6210	42
9.703101	360	10.296899	9.766965	483	10.233035	10.063864	123	9.936136	41
3317	360	6683	7255	483	2745	3938	123	6062	40
3533	359	6467	7545	483	2455	4012	123	5988	39
3749	359	6251	7834	483	2166	4086	123	5914	38
3964	359	6036	8124	482	1876	4160	123	5840	37
4179	359	5821	8413	482	1587	4234	123	5766	36
9.704395	359	10.295605	9.768703	482	10.231297	10.064308	124	9.935692	35
4610	358	5390	8992	482	1008	4382	124	5618	34
4825	358	5175	9281	482	0719	4457	124	5543	33
5040	358	4960	9570	482	0430	4531	124	5469	32
5254	358	4746	9860	481	0140	4605	124	539[illegible]	31
5469	357	4531	9.770148	481	10.229852	4680	124	5320	30
9.705683	357	10.294317	9.770437	481	10.229563	10.064754	124	9.935246	29
5898	357	4102	0726	481	9274	4829	124	5171	28
6112	357	3888	1015	481	8985	4903	124	5097	27
6326	356	3674	1303	481	8697	4978	124	5022	26
6539	356	3461	1592	481	8408	5052	124	4948	25
6753	356	3247	1880	480	8120	5127	124	4873	24
9.706967	356	10.293033	9.772168	480	10.227832	10.065202	125	9.934798	23
7180	355	2820	2457	480	7543	5277	125	4723	22
7393	355	2607	2745	480	7255	5351	125	4649	21
7606	355	2394	3033	480	6967	5426	125	4574	20
7819	355	2181	3321	480	6679	5501	125	4499	19
8032	354	1968	3608	479	6392	5576	125	4424	18
9.708245	354	10.291755	9.773896	479	10.226104	10.065651	125	9.934349	17
8458	354	1542	4184	479	5816	5726	125	4274	16
8670	354	1330	4471	479	5529	5801	125	4199	15
8882	353	1118	4759	479	5241	5877	125	4123	14
9094	353	0906	5046	479	4954	5952	125	4048	13
9306	353	0694	5333	479	4667	10.066027	125	3973	12
9.709518	353	10.290482	9.775621	478	10.224379	10.066102	126	9.933898	11
9730	353	0270	5908	478	4092	6178	126	3822	10
9941	352	0059	6195	478	3805	6253	126	3747	9
9.710153	352	10.289847	6482	478	3518	6329	126	3671	8
0364	352	9636	6769	478	3231	6404	126	3596	7
0575	352	9425	7055	478	2945	6480	126	3520	6
9.710786	351	10.289214	9.777342	478	10.222658	10.066555	126	9.933445	5
0997	351	9003	7628	477	2372	6631	126	3369	4
1208	351	8792	7915	477	2085	6707	126	3293	3
1419	351	8581	8201	477	1799	6783	126	3217	2
1629	351	8371	8487	477	1513	6859	126	3141	1
1839	350	8161	8774	477	1226	6934	126	3066	0
Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	′

31 DEG. TABLE IV.—LOG. SINES, ETC.

′	Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	
0	9.711839	350	10.288161	9.778774	477	10.221226	10.066934	126	9.933066	60
1	2050	350	7950	9060	477	0940	10.067010	127	9.932990	59
2	2260	350	7740	9346	476	0654	086	127	914	58
3	2469	349	7531	9632	476	0368	162	127	838	57
4	2679	349	7321	9918	476	0082	238	127	762	56
5	2889	349	7111	9.780203	476	10.219797	315	127	685	55
6	3098	349	6902	0489	476	9511	391	127	609	54
7	9.713308	349	10.286692	9.780775	476	10.219225	10.067467	127	9.932533	53
8	3517	348	6483	1060	476	8940	543	127	457	52
9	3726	348	6274	1346	475	8654	620	127	380	51
10	3935	348	6065	1631	475	8369	696	127	304	50
11	4144	348	5856	1916	475	8084	772	127	228	49
12	4352	347	5648	2201	475	7799	849	127	151	48
13	9.714561	347	10.285439	9.782486	475	10.217514	10.067925	128	9.932075	47
14	4769	347	5231	2771	475	7229	10.068002	128	9.931998	46
15	4978	347	5022	3056	475	6944	079	128	921	45
16	5186	347	4814	3341	475	6659	155	128	845	44
17	5394	346	4606	3626	474	6374	232	128	768	43
18	5602	346	4398	3910	474	6090	309	128	691	42
19	9.715809	346	10.284191	9.784195	474	10.215805	10.068386	128	9.931614	41
20	6017	346	3983	4479	474	5521	463	128	537	40
21	6224	345	3776	4764	474	5236	540	128	460	39
22	6432	345	3568	5048	474	4952	617	128	383	38
23	6639	345	3361	5332	473	4668	694	128	306	37
24	6846	345	3154	5616	473	4384	771	128	229	36
25	9.717053	345	10.282947	9.785900	473	10.214100	10.068848	129	9.931152	35
26	7259	344	2741	6184	473	3816	925	129	075	34
27	7466	344	2534	6468	473	3532	10.069002	129	9.930998	33
28	7673	344	2327	6752	473	3248	079	129	921	32
29	7879	344	2121	7036	473	2964	157	129	843	31
30	8085	343	1915	7319	472	2681	234	129	766	30
31	9.718291	343	10.281709	9.787603	472	10.212397	10.069312	129	9.930688	29
32	8497	343	1503	7886	472	2114	389	129	611	28
33	8703	343	1297	8170	472	1830	467	129	533	27
34	8909	343	1091	8453	472	1547	544	129	456	26
35	9114	342	0886	8736	472	1264	622	129	378	25
36	9320	342	0680	9019	472	0981	700	129	300	24
37	9.719525	342	10.280475	9.789302	471	10.210698	10.069777	130	9.930223	23
38	9730	342	0270	9585	471	0415	855	130	145	22
39	9935	341	0065	9868	471	0132	933	130	067	21
40	9.720140	341	10.279860	9.790151	471	10.209849	10.070011	130	9.929989	20
41	0345	341	9655	0433	471	9567	089	130	911	19
42	0549	341	9451	0716	471	9284	167	130	833	18
43	9.720754	340	10.279246	9.790999	471	10.209001	10.070245	130	9.929755	17
44	0958	340	9042	1281	471	8719	323	130	677	16
45	1162	340	8838	1563	470	8437	401	130	599	15
46	1366	340	8634	1846	470	8154	479	130	521	14
47	1570	340	8430	2128	470	7872	558	130	442	13
48	1774	339	8226	2410	470	7590	636	130	364	12
49	9.721978	339	10.278022	9.792692	470	10.207308	10.070714	131	9.929286	11
50	2181	339	7819	2974	470	7026	793	131	207	10
51	2385	339	7615	3256	470	6744	871	131	129	9
52	2588	339	7412	3538	469	6462	950	131	050	8
53	2791	338	7209	3819	469	6181	10.071028	131	9.928972	7
54	2994	338	7006	4101	469	5899	107	131	893	6
55	9.723197	338	10.276803	9.794383	469	10.205617	10.071185	131	9.928815	5
56	3400	338	6600	4664	469	5336	264	131	736	4
57	3603	337	6397	4945	469	5055	343	131	657	3
58	3805	337	6195	5227	469	4773	422	131	578	2
59	4007	337	5993	5508	469	4492	501	131	499	1
60	4210	337	5790	5789	468	4211	580	131	420	0
	Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	′

58 DEG.

DEG. TABLE IV.—LOG. SINES, ETC.

Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	
.724210		10.275790	9.795789		10.204211	10.071580		9.928420	60
4412	337	5588	6070	468	3930	658	132	342	59
4614	337	5386	6351	468	3649	737	132	263	58
4816	336	5184	6632	468	3368	817	132	183	57
5017	336	4983	6913	468	3087	896	132	104	56
5219	336	4781	7194	468	2806	975	132	025	55
5420	335	4580	7475	468	2525	10.072054	132	9.927946	54
.725622	335	10.274378	9.797755	468	10.202245	10.072133	132	9.927867	53
5823	335	4177	8036	467	1964	213	132	787	52
6024	335	3976	8316	467	1684	292	132	708	51
6225	335	3775	8596	467	1404	371	132	629	50
6426	334	3574	8877	467	1123	451	132	549	49
6626	334	3374	9157	467	0843	530	133	470	48
.726827	334	10.273173	9.799437	467	10.200563	10.072610	133	9.927390	47
7027	334	2973	9717	467	0283	690	133	310	46
7228	334	2772	9997	466	0003	769	133	231	45
7428	333	2572	9.800277	466	10.199723	849	133	151	44
7628	333	2372	0557	466	9443	929	133	071	43
7828	333	2172	0836	466	9164	10.073009	133	9.926991	42
.728027	333	10.271973	9.801116	466	10.198884	10.073089	133	9.926911	41
8227	333	1773	1396	466	8604	169	133	831	40
8427	332	1573	1675	466	8325	249	133	751	39
8626	332	1374	1955	466	8045	329	133	671	38
8825	332	1175	2234	465	7766	409	133	591	37
9024	332	0976	2513	465	7487	489	133	511	36
.729223	331	10.270777	9.802792	465	10.197208	10.073569	134	9.926431	35
9422	331	0578	3072	465	6928	649	134	351	34
9621	331	0379	3351	465	6649	730	134	270	33
9820	331	0180	3630	465	6370	810	134	190	32
.730018	330	10.269982	3908	465	6092	890	134	110	31
0216	330	9784	4187	465	5813	971	134	029	30
.730415	330	10.269585	9.804466	464	10.195534	10.074051	134	9.925949	29
0613	330	9387	4745	464	5255	132	134	868	28
0811	330	9189	5023	464	4977	212	134	788	27
1009	329	8991	5302	464	4698	293	134	707	26
1206	329	8794	5580	464	4420	374	134	626	25
1404	329	8596	5859	464	4141	455	134	545	24
.731602	329	10.268398	9.806137	464	10.193863	10.074535	135	9.925465	23
1799	329	8201	6415	463	3585	616	135	384	22
1996	328	8004	6693	463	3307	697	135	303	21
2193	328	7807	6971	463	3029	778	135	222	20
2390	328	7610	7249	463	2751	859	135	141	19
2587	328	7413	7527	463	2473	940	135	060	18
.732784	328	10.267216	9.807805	463	10.192195	10.075021	135	9.924979	17
2980	327	7020	8083	463	1917	103	135	897	16
3177	327	6823	8361	463	1639	184	135	816	15
3373	327	6627	8638	462	1362	265	135	735	14
3569	327	6431	8916	462	1084	346	136	654	13
3765	327	6235	9193	462	0807	428	136	572	12
.733961	326	10.266039	9.809471	462	10.190529	10.075509	136	9.924491	11
4157	326	5843	9748	462	0252	591	136	409	10
4353	326	5647	9.810025	462	10.189975	672	136	328	9
4549	326	5451	0302	462	9698	754	136	246	8
4744	325	5256	0580	462	9420	836	136	164	7
4939	325	5061	0857	462	9143	917	136	083	6
.735135	325	10.264865	9.811134	461	10.188866	10.075999	136	9.924001	5
5330	325	4670	1410	461	8590	10.076081	136	9.923919	4
5525	325	4475	1687	461	8313	163	136	837	3
5719	324	4281	1964	461	8036	245	136	755	2
5914	324	4086	2241	461	7759	327	137	673	1
6109	324	3891	2517	461	7483	409	137	591	0
Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	/

33 DEG. TABLE IV.—LOG. SINES, ETC.

′	Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	
0	9.736109	324	10.263891	9.812517	461	10.187483	10.076409	137	9.923591	60
1	6303	324	3697	2794	461	7206	6491	137	3509	59
2	6498	324	3502	3070	461	6930	6573	137	3427	58
3	6692	323	3308	3347	460	6653	6655	137	3345	57
4	6886	323	3114	3623	460	6377	6737	137	3263	56
5	7080	323	2920	3899	460	6101	6819	137	3181	55
6	7274	323	2726	4175	460	5825	6902	137	3098	54
7	9.737467	323	10.262533	9.814452	460	10.185548	10.076984	137	9.923016	53
8	7661	322	2339	4728	460	5272	7067	137	2933	52
9	7855	322	2145	5004	460	4996	7149	137	2851	51
10	8048	322	1952	5279	460	4721	7232	138	2768	50
11	8241	322	1759	5555	459	4445	7314	138	2686	49
12	8434	322	1566	5831	459	4169	7397	138	2603	48
13	9.738627	321	10.261373	9.816107	459	10.183893	10.077480	138	9.922520	47
14	8820	321	1180	6382	459	3618	7562	138	2438	46
15	9013	321	0987	6658	459	3342	7645	138	2355	45
16	9206	321	0794	6933	459	3067	7728	138	2272	44
17	9398	321	0602	7209	459	2791	7811	138	2189	43
18	9590	320	0410	7484	459	2516	7894	138	2106	42
19	9.739783	320	10.260217	9.817759	459	10.182241	10.077977	138	9.922023	41
20	9975	320	0025	8035	458	1965	8060	138	1940	40
21	9.740167	320	10.259833	8310	458	1690	8143	139	1857	39
22	0359	320	9641	8585	458	1415	8226	139	1774	38
23	0550	319	9450	8860	458	1140	8309	139	1691	37
24	0742	319	9258	9135	458	0865	8393	139	1607	36
25	9.740934	319	10.259066	9.819410	458	10.180590	10.078476	139	9.921524	35
26	1125	319	8875	9684	458	0316	8559	139	1441	34
27	1316	319	8684	9959	458	0041	8643	139	1357	33
28	1508	318	8492	9.820234	458	10.179766	8726	139	1274	32
29	1699	318	8301	0508	458	9492	8810	139	1190	31
30	1889	318	8111	0783	457	9217	8893	139	1107	30
31	9.742080	318	10.257920	9.821057	457	10.178943	10.078977	139	9.921023	29
32	2271	318	7729	1332	457	8668	9061	139	0939	28
33	2462	318	7538	1606	457	8394	9144	140	0856	27
34	2652	317	7348	1880	457	8120	9228	140	0772	26
35	2842	317	7158	2154	457	7846	9312	140	0688	25
36	3033	317	6967	2429	457	7571	9396	140	0604	24
37	9.743223	317	10.256777	9.822703	457	10.177297	10.079480	140	9.920520	23
38	3413	316	6587	2977	456	7023	9564	140	0436	22
39	3602	316	6398	3250	456	6750	9648	140	0352	21
40	3792	316	6208	3524	456	6476	9732	140	0268	20
41	3982	316	6018	3798	456	6202	9816	140	0184	19
42	4171	316	5829	4072	456	5928	9901	140	0099	18
43	9.744361	315	10.255639	9.824345	456	10.175655	10.079985	140	9.920015	17
44	4550	315	5450	4619	456	5381	10.080069	140	9.919931	16
45	4739	315	5261	4893	456	5107	0154	141	9846	15
46	4928	315	5072	5166	456	4834	0238	141	9762	14
47	5117	315	4883	5439	456	4561	0323	141	9677	13
48	5306	315	4694	5713	455	4287	0407	141	9593	12
49	9.745494	314	10.254506	9.825986	455	10.174014	10.080492	141	9.919508	11
50	5683	314	4317	6259	455	3741	0576	141	9424	10
51	5871	314	4129	6532	455	3468	0661	141	9339	9
52	6059	314	3941	6805	455	3195	0746	141	9254	8
53	6248	313	3752	7078	455	2922	0831	141	9169	7
54	6436	313	3564	7351	455	2649	0915	141	9085	6
55	9.746624	313	10.253376	9.827624	455	10.172376	10.081000	141	9.919000	5
56	6812	313	3188	7897	454	2103	1085	142	8915	4
57	6999	313	3001	8170	454	1830	1170	142	8830	3
58	7187	312	2813	8442	454	1558	1255	142	8745	2
59	7374	312	2626	8715	454	1285	1341	142	8659	1
60	7562	312	2438	8987	454	1013	1426	142	8574	0
	Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	′

56 DEG.

DEG. TABLE IV.—LOG. SINES, ETC.

Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	
.747562	312	10.252438	9.828987	454	10.171013	10.081426	142	9.918574	60
7749	312	2251	9260	454	0740	1511	142	8489	59
7936	312	2064	9532	454	0468	1596	142	8404	58
8123	311	1877	9805	454	0195	1682	142	8318	57
8310	311	1690	9.830077	454	10.169923	1767	142	8233	56
8497	311	1503	0349	453	9651	1853	142	8147	55
8683	311	1317	0621	453	9379	1938	142	8062	54
.748870	311	10.251130	9.830893	453	10.169107	10.082024	143	9.917976	53
9056	310	0944	1165	453	8835	2109	143	7891	52
9243	310	0757	1437	453	8563	2195	143	7805	51
9429	310	0571	1709	453	8291	2281	143	7719	50
9615	310	0385	1981	453	8019	2366	143	7634	49
9801	310	0199	2253	453	7747	2452	143	7548	48
.749987	309	10.250013	9.832525	453	10.167475	10.082538	143	9.917462	47
.750172	309	10.249828	2796	453	7204	2624	143	7376	46
0358	309	9642	3068	452	6932	2710	143	7290	45
0543	309	9457	3339	452	6661	2796	143	7204	44
0729	309	9271	3611	452	6389	2882	144	7118	43
0914	308	9086	3882	452	6118	2968	144	7032	42
.751099	308	10.248901	9.834154	452	10.165846	10.083054	144	9.916946	41
1284	308	8716	4425	452	5575	3141	144	6859	40
1469	308	8531	4696	452	5304	3227	144	6773	39
1654	308	8346	4967	452	5033	3313	144	6687	38
1839	308	8161	5238	452	4762	3400	144	6600	37
2023	307	7977	5509	452	4491	3486	144	6514	36
.752208	307	10.247792	9.835780	451	10.164220	10.083573	144	9.916427	35
2392	307	7608	6051	451	3949	3659	144	6341	34
2576	307	7424	6322	451	3678	3746	144	6254	33
2760	307	7240	6593	451	3407	3833	144	6167	32
2944	306	7056	6864	451	3136	3919	145	6081	31
3128	306	6872	7134	451	2866	4006	145	5994	30
.753312	306	10.246688	9.837405	451	10.162595	10.084093	145	9.915907	29
3495	306	6505	7675	451	2325	4180	145	5820	28
3679	306	6321	7946	451	2054	4267	145	5733	27
3862	305	6138	8216	451	1784	4354	145	5646	26
4046	305	5954	8487	450	1513	4441	145	5559	25
4229	305	5771	8757	450	1243	4528	145	5472	24
.754412	305	10.245588	9.839027	450	10.160973	10.084615	145	9.915385	23
4595	305	5405	9297	450	0703	4703	145	5297	22
4778	304	5222	9568	450	0432	4790	145	5210	21
4960	304	5040	9838	450	0162	4877	145	5123	20
5143	304	4857	9.840108	450	10.159892	4965	146	5035	19
5326	304	4674	0378	450	9622	5052	146	4948	18
.755508	304	10.244492	9.840647	450	10.159353	10.085140	146	9.914860	17
5690	304	4310	0917	449	9083	5227	146	4773	16
5872	303	4128	1187	449	8813	5315	146	4685	15
6054	303	3946	1457	449	8543	5402	146	4598	14
6236	303	3764	1726	449	8274	5490	146	4510	13
6418	303	3582	1996	449	8004	5578	146	4422	12
.756600	303	10.243400	9.842266	449	10.157734	10.085666	146	9.914334	11
6782	302	3218	2535	449	7465	5754	147	4246	10
6963	302	3037	2805	449	7195	5842	147	4158	9
7144	302	2856	3074	449	6926	5930	147	4070	8
7326	302	2674	3343	449	6657	6018	147	3982	7
7507	302	2493	3612	449	6388	6106	147	3894	6
.757688	301	10.242312	9.843882	448	10.156118	10.086194	147	9.913806	5
7869	301	2131	4151	448	5849	6282	147	3718	4
8050	301	1950	4420	448	5580	6370	147	3630	3
8230	301	1770	4689	448	5311	6459	147	3541	2
8411	301	1589	4958	448	5042	6547	147	3453	1
8591		1409	5227		4773	6635		3365	0
Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	/

35 Deg.	TABLE IV.—Log. Sines, etc.									
′	Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	
0	9.758591	301	10.241409	9.845227	448	10.154773	10.086635	147	9.913365	60
1	8772	300	1228	5496	448	4504	6724	147	3276	59
2	8952	300	1048	5764	448	4236	6813	148	3187	58
3	9132	300	0868	6033	448	3967	6901	148	3099	57
4	9312	300	0688	6302	448	3698	6990	148	3010	56
5	9492	300	0508	6570	447	3430	7078	148	2922	55
6	9672	299	0328	6839	447	3161	7167	148	2833	54
7	9.759852	299	10.240148	9.847107	447	10.152893	10.087256	148	9.912744	53
8	9.760031	299	10.239969	7376	447	2624	7345	148	2655	52
9	0211	299	9789	7644	447	2356	7434	148	2566	51
10	0390	299	9610	7913	447	2087	7523	148	2477	50
11	0569	298	9431	8181	447	1819	7612	148	2388	49
12	0748	298	9252	8449	447	1551	7701	149	2299	48
13	9.760927	298	10.239073	9.848717	447	10.151283	10.087790	149	9.912210	47
14	1106	298	8894	8986	447	1014	7879	149	2121	46
15	1285	298	8715	9254	447	0746	7969	149	2031	45
16	1464	298	8536	9522	447	0478	8058	149	1942	44
17	1642	297	8358	9790	446	0210	8147	149	1853	43
18	1821	297	8179	9.850058	446	10.149942	8237	149	1763	42
19	9.761999	297	10.238001	9.850325	446	10.149675	10.088326	149	9.911674	41
20	2177	297	7823	0593	446	9407	8416	149	1584	40
21	2356	297	7644	0861	446	9139	8505	149	1495	39
22	2534	296	7466	1129	446	8871	8595	149	1405	38
23	2712	296	7288	1396	446	8604	8685	150	1315	37
24	2889	296	7111	1664	446	8336	8774	150	1226	36
25	9.763067	296	10.236933	9.851931	446	10.148069	10.088864	150	9.911136	35
26	3245	296	6755	2199	446	7801	8954	150	1046	34
27	3422	296	6578	2466	446	7534	9044	150	0956	33
28	3600	295	6400	2733	445	7267	9134	150	0866	32
29	3777	295	6223	3001	445	6999	9224	150	0776	31
30	3954	295	6046	3268	445	6732	9314	150	0686	30
31	9.764131	295	10.235869	9.853535	445	10.146465	10.089404	150	9.910596	29
32	4308	295	5692	3802	445	6198	9494	150	0506	28
33	4485	294	5515	4069	445	5931	9585	150	0415	27
34	4662	294	5338	4336	445	5664	9675	150	0325	26
35	4838	294	5162	4603	445	5397	9765	151	0235	25
36	5015	294	4985	4870	445	5130	9856	151	0144	24
37	9.765191	294	10.234809	9.855137	445	10.144863	10.089946	151	9.910054	23
38	5367	294	4633	5404	445	4596	10.090037	151	9.909963	22
39	5544	293	4456	5671	444	4329	0127	151	9873	21
40	5720	293	4280	5938	444	4062	0218	151	9782	20
41	5896	293	4104	6204	444	3796	0309	151	9691	19
42	6072	293	3928	6471	444	3529	0399	151	9601	18
43	9.766247	293	10.233753	9.856737	444	10.143263	10.090490	151	9.909510	17
44	6423	293	3577	7004	444	2996	0581	151	9419	16
45	6598	292	3402	7270	444	2730	0672	152	9328	15
46	6774	292	3226	7537	444	2463	0763	152	9237	14
47	6949	292	3051	7803	444	2197	0854	152	9146	13
48	7124	292	2876	8069	444	1931	0945	152	9055	12
49	9.767300	292	10.232700	9.858336	444	10.141664	10.091036	152	9.908964	11
50	7475	291	2525	8602	443	1398	1127	152	8873	10
51	7649	291	2351	8868	443	1132	1219	152	8781	9
52	7824	291	2176	9134	443	0866	1310	152	8690	8
53	7999	291	2001	9400	443	0600	1401	152	8599	7
54	8173	291	1827	9666	443	0334	1493	152	8507	6
55	9.768348	290	10.231652	9.859932	443	10.140068	10.091584	153	9.908416	5
56	8522	290	1478	9.860198	443	10.139802	1676	153	8324	4
57	8697	290	1303	0464	443	9536	1767	153	8233	3
58	8871	290	1129	0730	443	9270	1859	153	8141	2
59	9045	290	0955	0995	443	9005	1951	153	8049	1
60	9219		0781	1261		8739	2042	153	7958	0
	Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	′

DEG. TABLE IV.—LOG. SINES, ETC.

Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	
.769219	290	10.230781	9.861261	443	10.138739	10.092042	153	9.907958	60
9393	289	0607	1527	443	8473	2134	153	7866	59
9566	289	0434	1792	442	8208	2226	153	7774	58
9740	289	0260	2058	442	7942	2318	153	7682	57
9913	289	0087	2323	442	7677	2410	153	7590	56
.770087	289	10.229913	2589	442	7411	2502	153	7498	55
0260	288	9740	2854	442	7146	2594	153	7406	54
.770433	288	10.229567	9.863119	442	10.136881	10.092686	154	9.907314	53
0606	288	9394	3385	442	6615	2778	154	7222	52
0779	288	9221	3650	442	6350	2871	154	7129	51
0952	288	9048	3915	442	6085	2963	154	7037	50
1125	288	8875	4180	442	5820	3055	154	6945	49
1298	287	8702	4445	442	5555	3148	154	6852	48
.771470	287	10.228530	9.864710	442	10.135290	10.093240	154	9.906760	47
1643	287	8357	4975	441	5025	3333	154	6667	46
1815	287	8185	5240	441	4760	3425	154	6575	45
1987	287	8013	5505	441	4495	3518	154	6482	44
2159	287	7841	5770	441	4230	3611	155	6389	43
2331	286	7669	6035	441	3965	3704	155	6296	42
.772503	286	10.227497	9.866300	441	10.133700	10.093796	155	9.906204	41
2675	286	7325	6564	441	3436	3889	155	6111	40
2847	286	7153	6829	441	3171	3982	155	6018	39
3018	286	6982	7094	441	2906	4075	155	5925	38
3190	286	6810	7358	441	2642	4168	155	5832	37
3361	285	6639	7623	441	2377	4261	155	5739	36
.773533	285	10.226467	9.867887	441	10.132113	10.094355	155	9.905645	35
3704	285	6296	8152	440	1848	4448	155	5552	34
3875	285	6125	8416	440	1584	4541	155	5459	33
4046	285	5954	8680	440	1320	4634	156	5366	32
4217	285	5783	8945	440	1055	4728	156	5272	31
4388	284	5612	9209	440	0791	4821	156	5179	30
.774558	284	10.225442	9.869473	440	10.130527	10.094915	156	9.905085	29
4729	284	5271	9737	440	0263	5008	156	4992	28
4899	284	5101	9.870001	440	10.129999	5102	156	4898	27
5070	284	4930	0265	440	9735	5196	156	4804	26
5240	284	4760	0529	440	9471	5289	156	4711	25
5410	283	4590	0793	440	9207	5383	156	4617	24
.775580	283	10.224420	9.871057	440	10.128943	10.095477	156	9.904523	23
5750	283	4250	1321	440	8679	5571	156	4429	22
5920	283	4080	1585	440	8415	5665	157	4335	21
6090	283	3910	1849	439	8151	5759	157	4241	20
6259	283	3741	2112	439	7888	5853	157	4147	19
6429	282	3571	2376	439	7624	5947	157	4053	18
.776598	282	10.223402	9.872640	439	10.127360	10.096041	157	9.903959	17
6768	282	3232	2903	439	7097	6136	157	3864	16
6937	282	3063	3167	439	6833	6230	157	3770	15
7106	282	2894	3430	439	6570	6324	157	3676	14
7275	281	2725	3694	439	6306	6419	157	3581	13
7444	281	2556	3957	439	6043	6513	157	3487	12
.777613	281	10.222387	9.874220	439	10.125780	10.096608	157	9.903892	11
7781	281	2219	4484	439	5516	6702	158	3298	10
7950	281	2050	4747	439	5253	6797	158	3203	9
8119	281	1881	5010	439	4990	6892	158	3108	8
8287	280	1713	5273	438	4727	6986	158	3014	7
8455	280	1545	5536	438	4464	7081	158	2919	6
.778624	280	10.221376	9.875800	438	10.124200	10.097176	158	9.902824	5
8792	280	1208	6063	438	3937	7271	158	2729	4
8960	280	1040	6326	438	3674	7366	158	2634	3
9128	280	0872	6589	438	3411	7461	158	2539	2
9295	279	0705	6851	438	3149	7556	159	2444	1
9463		0537	7114		2886	7651	159	2349	0
Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	′

37 DEG. TABLE IV.—LOG. SINES, ETC.

′	Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	
0	9.779463	279	10.220537	9.877114	438	10.122886	10.097651	159	9.902349	60
1	9631	279	0369	7377	438	2623	7747	159	2253	59
2	9798	279	0202	7640	438	2360	7842	159	2158	58
3	9966	279	0034	7903	438	2097	7937	159	2063	57
4	9.780133	279	10.219867	8165	438	1835	8033	159	1967	56
5	0300	278	9700	8428	438	1572	8128	159	1872	55
6	0467	278	9533	8691	438	1309	8224	159	1776	54
7	9.780634	278	10.219366	9.878953	437	10.121047	10.098319	159	9.901681	53
8	0801	278	9199	9216	437	0784	8415	159	1585	52
9	0968	278	9032	9478	437	0522	8510	159	1490	51
10	1134	278	8866	9741	437	0259	8606	160	1394	50
11	1301	277	8699	9.880003	437	9997	8702	160	1298	49
12	1468	277	8532	0265	437	9735	8798	160	1202	48
13	9.781634	277	10.218366	9.880528	437	10.119472	10.098894	160	9.901106	47
14	1800	277	8200	0790	437	9210	8990	160	1010	46
15	1966	277	8034	1052	437	8948	9086	160	0914	45
16	2132	277	7868	1314	437	8686	9182	160	0818	44
17	2298	276	7702	1576	437	8424	9278	160	0722	43
18	2464	276	7536	1839	437	8161	9374	160	0626	42
19	9.782630	276	10.217370	9.882101	437	10.117899	10.099471	160	9.900529	41
20	2796	276	7204	2363	436	7637	9567	161	0433	40
21	2961	276	7039	2625	436	7375	9663	161	0337	39
22	3127	276	6873	2887	436	7113	9760	161	0240	38
23	3292	275	6708	3148	436	6852	9856	161	0144	37
24	3458	275	6542	3410	436	6590	9953	161	0047	36
25	9.783623	275	10.216377	9.883672	436	10.116328	10.100049	161	9.899951	35
26	3788	275	6212	3934	436	6066	0146	161	9854	34
27	3953	275	6047	4196	436	5804	0243	161	9757	33
28	4118	275	5882	4457	436	5543	0340	161	9660	32
29	4282	274	5718	4719	436	5281	0436	161	9564	31
30	4447	274	5553	4980	436	5020	0533	162	9467	30
31	9.784612	274	10.215388	9.885242	436	10.114758	10.100630	162	9.899370	29
32	4776	274	5224	5503	436	4497	0727	162	9273	28
33	4941	274	5059	5765	436	4235	0824	162	9176	27
34	5105	274	4895	6026	436	3974	0922	162	9078	26
35	5269	273	4731	6288	436	3712	1019	162	8981	25
36	5433	273	4567	6549	435	3451	1116	162	8884	24
37	9.785597	273	10.214403	9.886810	435	10.113190	10.101213	162	9.898787	23
38	5761	273	4239	7072	435	2928	1311	162	8659	22
39	5925	273	4075	7333	435	2667	1408	162	8592	21
40	6089	273	3911	7594	435	2406	1506	162	8494	20
41	6252	272	3748	7855	435	2145	1603	163	8397	19
42	6416	272	3584	8116	435	1884	1701	163	8299	18
43	9.786579	272	10.213421	9.888377	435	10.111623	10.101798	163	9.898202	17
44	6742	272	3258	8639	435	1361	1896	163	8104	16
45	6906	272	3094	8900	435	1100	1994	163	8006	15
46	7069	272	2931	9160	435	0840	2092	163	7908	14
47	7232	271	2768	9421	435	0579	2190	163	7810	13
48	7395	271	2605	9682	435	0318	2288	163	7712	12
49	9.787557	271	10.212443	9.889943	435	10.110057	10.102386	163	9.897614	11
50	7720	271	2280	9.890204	434	10.109796	2484	163	7516	10
51	7883	271	2117	0465	434	9535	2582	163	7418	9
52	8045	271	1955	0725	434	9275	2680	164	7320	8
53	8208	271	1792	0986	434	9014	2778	164	7222	7
54	8370	270	1630	1247	434	8753	2877	164	7123	6
55	9.788532	270	10.211468	9.891507	434	10.108493	10.102975	164	9.897025	5
56	8694	270	1306	1768	434	8232	3074	164	6926	4
57	8856	270	1144	2028	434	7972	3172	164	6828	3
58	9018	270	0982	2289	434	7711	3271	164	6729	2
59	9180	270	0820	2549	434	7451	3369	164	6631	1
60	9342		0658	2810		7190	3468	164	6532	0
	Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	′

52 DEG.

DEG. TABLE IV.—LOG. SINES, ETC.

Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	
9.789342	269	10.210658	9.892810	434	10.107190	10.103468	164	9.896532	60
9504	269	0496	3070	434	6930	3567	165	6433	59
9665	269	0335	3331	434	6669	3665	165	6335	58
9827	269	0173	3591	434	6409	3764	165	6236	57
9988	269	0012	3851	434	6149	3863	165	6137	56
9.790149	269	10.209851	4111	434	5889	3962	165	6038	55
0310	268	9690	4371	434	5629	4061	165	5939	54
9.790471	268	10.209529	9.894632	433	10.105368	10.104160	165	9.895840	53
0632	268	9368	4892	433	5108	4259	165	5741	52
0793	268	9207	5152	433	4848	4359	165	5641	51
0954	268	9046	5412	433	4588	4458	165	5542	50
1115	268	8885	5672	433	4328	4557	165	5443	49
1275	267	8725	5932	433	4068	4657	166	5343	48
9.791436	267	10.208564	9.896192	433	10.103808	10.104756	166	9.895244	47
1596	267	8404	6452	433	3548	4855	166	5145	46
1757	267	8243	6712	433	3288	4955	166	5045	45
1917	267	8083	6971	433	3029	5055	166	4945	44
2077	267	7923	7231	433	2769	5154	166	4846	43
2237	266	7763	7491	433	2509	5254	166	4746	42
9.792397	266	10.207603	9.897751	433	10.102249	10.105354	166	9.894646	41
2557	266	7443	8010	433	1990	5454	166	4546	40
2716	266	7284	8270	433	1730	5554	166	4446	39
2876	266	7124	8530	433	1470	5654	167	4346	38
3035	266	6965	8789	433	1211	5754	167	4246	37
3195	265	6805	9049	432	0951	5854	167	4146	36
9.793354	265	10.206646	9.899308	432	10.100692	10.105954	167	9.894046	35
3514	265	6486	9568	432	0432	6054	167	3946	34
3673	265	6327	9827	432	0173	6154	167	3846	33
3832	265	6168	9.900086	432	10.099914	6255	167	3745	32
3991	265	6009	0346	432	9654	6355	167	3645	31
4150	264	5850	0605	432	9395	6456	167	3544	30
9.794308	264	10.205692	9.900864	432	10.099136	10.106556	167	9.893444	29
4467	264	5533	1124	432	8876	6657	168	3343	28
4626	264	5374	1383	432	8617	6757	168	3243	27
4784	264	5216	1642	432	8358	6858	168	3142	26
4942	264	5058	1901	432	8099	6959	168	3041	25
5101	264	4899	2160	432	7840	7060	168	2940	24
9.795259	263	10.204741	9.902419	432	10.097581	10.107161	168	9.892839	23
5417	263	4583	2679	432	7321	7261	168	2739	22
5575	263	4425	2938	432	7062	7362	168	2638	21
5733	263	4267	3197	432	6803	7464	168	2536	20
5891	263	4109	3455	431	6545	7565	168	2435	19
6049	263	3951	3714	431	6286	7666	169	2334	18
9.796206	263	10.203794	9.903973	431	10.096027	10.107767	169	9.892233	17
6364	262	3636	4232	431	5768	7868	169	2132	16
6521	262	3479	4491	431	5509	7970	169	2030	15
6679	262	3321	4750	431	5250	8071	169	1929	14
6836	262	3164	5008	431	4992	8173	169	1827	13
6993	262	3007	5267	431	4733	8274	169	1726	12
9.797150	261	10.202850	9.905526	431	10.094474	10.108376	169	9.891624	11
7307	261	2693	5784	431	4216	8477	170	1523	10
7464	261	2536	6043	431	3957	8579	170	1421	9
7621	261	2379	6302	431	3698	8681	170	1319	8
7777	261	2223	6560	431	3440	8783	170	1217	7
7934	261	2066	6819	431	3181	8885	170	1115	6
9.798091	261	10.201909	9.907077	431	10.092923	10.108987	170	9.891013	5
8247	261	1753	7336	431	2664	9089	170	0911	4
8403	260	1597	7594	431	2406	9191	170	0809	3
8560	260	1440	7852	431	2148	9293	170	0707	2
8716	260	1284	8111	430	1889	9395	170	0605	1
8872	260	1128	8369	430	1631	9497	170	0503	0
Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	′

39 DEG. TABLE IV.—LOG. SINES, ETC.

′	Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	
0	9.798872	260	10.201128	9.908369	430	10.091631	10.109497	170	9.890503	60
1	9028	260	0972	8628	430	1372	9600	171	0400	59
2	9184	260	0816	8886	430	1114	9702	171	0298	58
3	9339	259	0661	9144	430	0856	9805	171	0195	57
4	9495	259	0505	9402	430	0598	9907	171	0093	56
5	9651	259	0349	9660	430	0340	10.110010	171	9.889990	55
6	9806	259	0194	9918	430	0082	0112	171	9888	54
7	9.799962	259	10.200038	9.910177	430	10.089823	10.110215	171	9.889785	53
8	9.800117	259	10.199883	0435	430	9565	0318	171	9682	52
9	0272	258	9728	0693	430	9307	0421	171	9579	51
10	0427	258	9573	0951	430	9049	0523	171	9477	50
11	0582	258	9418	1209	430	8791	0626	172	9374	49
12	0737	258	9263	1467	430	8533	0729	172	9271	48
13	9.800892	258	10.199108	9.911724	430	10.088276	10.110832	172	9.889168	47
14	1047	258	8953	1982	430	8018	0936	172	9064	46
15	1201	258	8799	2240	430	7760	1039	172	8961	45
16	1356	257	8644	2498	430	7502	1142	172	8858	44
17	1511	257	8489	2756	430	7244	1245	172	8755	43
18	1665	257	8335	3014	429	6986	1349	172	8651	42
19	9.801819	257	10.198181	9.913271	429	10.086729	10.111452	172	9.888548	41
20	1973	257	8027	3529	429	6471	1556	173	8444	40
21	2128	257	7872	3787	429	6213	1659	173	8341	39
22	2282	256	7718	4044	429	5956	1763	173	8237	38
23	2436	256	7564	4302	429	5698	1866	173	8134	37
24	2589	256	7411	4560	429	5440	1970	173	8030	36
25	9.802743	256	10.197257	9.914817	429	10.085183	10.112074	173	9.887926	35
26	2897	256	7103	5075	429	4925	2178	173	7822	34
27	3050	256	6950	5332	429	4668	2282	173	7718	33
28	3204	256	6796	5590	429	4410	2386	173	7614	32
29	3357	255	6643	5847	429	4153	2490	173	7510	31
30	3511	255	6489	6104	429	3896	2594	173	7406	30
31	9.803664	255	10.196336	9.916362	429	10.083638	10.112698	174	9.887302	29
32	3817	255	6183	6619	429	3381	2802	174	7198	28
33	3970	255	6030	6877	429	3123	2907	174	7093	27
34	4123	255	5877	7134	429	2866	3011	174	6989	26
35	4276	254	5724	7391	429	2609	3115	174	6885	25
36	4428	254	5572	7648	429	2352	3220	174	6780	24
37	9.804581	254	10.195419	9.917905	429	10.082095	10.113324	174	9.886676	23
38	4734	254	5266	8163	428	1837	3429	174	6571	22
39	4886	254	5114	8420	428	1580	3534	174	6466	21
40	5039	254	4961	8677	428	1323	3638	174	6362	20
41	5191	254	4809	8934	428	1066	3743	175	6257	19
42	5343	253	4657	9191	428	0809	3848	175	6152	18
43	9.805495	253	10.194505	9.919448	428	10.080552	10.113953	175	9.886047	17
44	5647	253	4353	9705	428	0295	4058	175	5942	16
45	5799	253	4201	9962	428	0038	4163	175	5837	15
46	5951	253	4049	9.920219	428	10.079781	4268	175	5732	14
47	6103	253	3897	0476	428	9524	4373	175	5627	13
48	6254	253	3746	0733	428	9267	4478	175	5522	12
49	9.806406	252	10.193594	9.920990	428	10.079010	10.114584	175	8.885416	11
50	6557	252	3443	1247	428	8753	4689	175	5311	10
51	6709	252	3291	1503	428	8497	4795	176	5205	9
52	6860	252	3140	1760	428	8240	4900	176	5100	8
53	7011	252	2989	2017	428	7983	5006	176	4994	7
54	7163	252	2837	2274	428	7726	5111	176	4889	6
55	9.807314	252	10.192686	9.922530	428	10.077470	10.115217	176	9.884783	5
56	7465	251	2535	2787	428	7213	5323	176	4677	4
57	7615	251	2385	3044	428	6956	5428	176	4572	3
58	7766	251	2234	3300	428	6700	5534	176	4466	2
59	7917	251	2083	3557	428	6443	5640	176	4360	1
60	8067	251	1933	3813	427	6187	5746	176	4254	0
	Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	′

50 DEG.

DEG. TABLE IV.—LOG. SINES, ETC.

Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	
9.808067	251	10.191933	9.923813	427	10.076187	10.115746	177	9.884254	60
8218	251	1782	4070	427	5930	5852	177	4148	59
8368	251	1632	4327	427	5673	5958	177	4042	58
8519	250	1481	4583	427	5417	6064	177	3936	57
8669	250	1331	4840	427	5160	6171	177	3829	56
8819	250	1181	5096	427	4904	6277	177	3723	55
8969	250	1031	5352	427	4648	6383	177	3617	54
9.809119	250	10.190881	9.925609	427	10.074391	10.116490	177	9.883510	53
9269	250	0731	5865	427	4135	6596	177	3404	52
9419	249	0581	6122	427	3878	6703	178	3297	51
9569	249	0431	6378	427	3622	6809	178	3191	50
9718	249	0282	6634	427	3366	6916	178	3084	49
9868	249	0132	6890	427	3110	7023	178	2977	48
9.810017	249	10.189983	9.927147	427	10.072853	10.117129	178	9.882871	47
0167	249	9833	7403	427	2597	7236	178	2764	46
0316	248	9684	7659	427	2341	7343	178	2657	45
0465	248	9535	7915	427	2085	7450	178	2550	44
0614	248	9386	8171	427	1829	7557	178	2443	43
0763	248	9237	8427	427	1573	7664	178	2336	42
9.810912	248	10.189088	9.928683	427	10.071317	10.117771	179	9.882229	41
1061	248	8939	8940	427	1060	7879	179	2121	40
1210	248	8790	9196	427	0804	7986	179	2014	39
1358	247	8642	9452	427	0548	8093	179	1907	38
1507	247	8493	9708	427	0292	8201	179	1799	37
1655	247	8345	9964	426	0036	8308	179	1692	36
9.811804	247	10.188196	9.930220	426	10.069780	10.118416	179	9.881584	35
1952	247	8048	0475	426	10.069525	8523	179	1477	34
2100	247	7900	0731	426	9269	8631	179	1369	33
2248	247	7752	0987	426	9013	8739	179	1261	32
2396	246	7604	1243	426	8757	8847	180	1153	31
2544	246	7456	1499	426	8501	8954	180	1046	30
9.812692	246	10.187308	9.931755	426	10.068245	10.119062	180	9.880938	29
2840	246	7160	2010	426	7990	9170	180	0830	28
2988	246	7012	2266	426	7734	9278	180	0722	27
3135	246	6865	2522	426	7478	9387	180	0613	26
3283	246	6717	2778	426	7222	9495	180	0505	25
3430	245	6570	3033	426	6967	9603	180	0397	24
9.813578	245	10.186422	9.933289	426	10.066711	10.119711	180	9.880289	23
3725	245	6275	3545	426	6455	9820	181	0180	22
3872	245	6128	3800	426	6200	9928	181	0072	21
4019	245	5981	4056	426	5944	10.120037	181	9.879963	20
4166	245	5834	4311	426	5689	0145	181	9855	19
4313	245	5687	4567	426	5433	0254	181	9746	18
9.814460	244	10.185540	9.934823	426	10.065177	10.120363	181	9.879637	17
4607	244	5393	5078	426	4922	0471	181	9529	16
4753	244	5247	5333	426	4667	0580	181	9420	15
4900	244	5100	5589	426	4411	0689	181	9311	14
5046	244	4954	5844	426	4156	0798	181	9202	13
5193	244	4807	6100	426	3900	0907	182	9093	12
9.815339	244	10.184661	9.936355	426	10.063645	10.121016	182	9.878984	11
5485	243	4515	6610	426	3390	1125	182	8875	10
5632	243	4368	6866	425	3134	1234	182	8766	9
5778	243	4222	7121	425	2879	1344	182	8656	8
5924	243	4076	7376	425	2624	1453	182	8547	7
6069	243	3931	7632	425	2368	1562	182	8438	6
9.816215	243	10.183785	9.937887	425	10.062113	10.121672	182	9.878328	5
6361	243	3639	8142	425	1858	1781	183	8219	4
6507	242	3493	8398	425	1602	1891	183	8109	3
6652	242	3348	8653	425	1347	2001	183	7999	2
6798	242	3202	8908	425	1092	2110	183	7890	1
6943	242	3057	9163	425	0837	2220	183	7780	0
Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	′

41 DEG. TABLE IV.—LOG. SINES, ETC.

′	Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	
0	9.816943	242	10.183057	9.939163	425	10.060837	10.122220	183	9.877780	60
1	7088	242	2912	9418	425	0582	2330	183	7670	59
2	7233	242	2767	9673	425	0327	2440	183	7560	58
3	7379	242	2621	9928	425	0072	2550	183	7450	57
4	7524	241	2476	9.940183	425	10.059817	2660	183	7340	56
5	7668	241	2332	0438	425	9562	2770	184	7230	55
6	7813	241	2187	0694	425	9306	2880	184	7120	54
7	9.817958	241	10.182042	9.940949	425	10.059051	10.122990	184	9.877010	53
8	8103	241	1897	1204	425	8796	3101	184	6899	52
9	8247	241	1753	1458	425	8542	3211	184	6789	51
10	8392	241	1608	1714	425	8286	3322	184	6678	50
11	8536	240	1464	1968	425	8032	3432	184	6568	49
12	8681	240	1319	2223	425	7777	3543	184	6457	48
13	9.818825	240	10.181175	9.942478	425	10.057522	10.123653	184	9.876347	47
14	8969	240	1031	2733	425	7267	3764	185	6236	46
15	9113	240	0887	2988	425	7012	3875	185	6125	45
16	9257	240	0743	3243	425	6757	3986	185	6014	44
17	9401	240	0599	3498	425	6502	4096	185	5904	43
18	9545	239	0455	3752	425	6248	4207	185	5793	42
19	9.819689	239	10.180311	9.944007	425	10.055993	10.124318	185	9.875682	41
20	9832	239	0168	4262	425	5738	4429	185	5571	40
21	9976	239	0024	4517	425	5483	4541	185	5459	39
22	9.820120	239	10.179880	4771	424	5229	4652	185	5348	38
23	0263	239	9737	5026	424	4974	4763	185	5237	37
24	0406	239	9594	5281	424	4719	4874	186	5126	36
25	9.820550	238	10.179450	9.945535	424	10.054465	10.124986	186	9.875014	35
26	0693	238	9307	5790	424	4210	5097	186	4903	34
27	0836	238	9164	6045	424	3955	5209	186	4791	33
28	0979	238	9021	6299	424	3701	5320	186	4680	32
29	1122	238	8878	6554	424	3446	5432	186	4568	31
30	1265	238	8735	6808	424	3192	5544	186	4456	30
31	9.821407	238	10.178593	9.947063	424	10.052937	10.125656	186	9.874344	29
32	1550	238	8450	7318	424	2682	5768	187	4232	28
33	1693	238	8307	7572	424	2428	5879	187	4121	27
34	1835	237	8165	7826	424	2174	5991	187	4009	26
35	1977	237	8023	8081	424	1919	6104	187	3896	25
36	2120	237	7880	8336	424	1664	6216	187	3784	24
37	9.822262	237	10.177738	9.948590	424	10.051410	10.126328	187	9.873672	23
38	2404	237	7596	8844	424	1156	6440	187	3560	22
39	2546	237	7454	9099	424	0901	6552	187	3448	21
40	2688	236	7312	9353	424	0647	6665	187	3335	20
41	2830	236	7170	9607	424	0393	6777	187	3223	19
42	2972	236	7028	9862	424	0138	6890	187	3110	18
43	9.823114	236	10.176886	9.950116	424	10.049884	10.127002	188	9.872998	17
44	3255	236	6745	0370	424	9630	7115	188	2885	16
45	3397	236	6603	0625	424	9375	7228	188	2772	15
46	3539	236	6461	0879	424	9121	7341	188	2659	14
47	3680	235	6320	1133	424	8867	7453	188	2547	13
48	3821	235	6179	1388	424	8612	7566	188	2434	12
49	9.823963	235	10.176037	9.951642	424	10.048358	10.127679	188	9.872321	11
50	4104	235	5896	1896	424	8104	7792	188	2208	10
51	4245	235	5755	2150	424	7850	7905	188	2095	9
52	4386	235	5614	2405	424	7595	8019	189	1981	8
53	4527	235	5473	2659	424	7341	8132	189	1868	7
54	4668	235	5332	2913	424	7087	8245	189	1755	6
55	9.824808	234	10.175192	9.253167	424	10.046833	10.128359	189	9.871641	5
56	4949	234	5051	3421	423	6579	8472	189	1528	4
57	5090	234	4910	3675	423	6325	8586	189	1414	3
58	5230	234	4770	3929	423	6071	8699	189	1301	2
59	5371	234	4629	4183	423	5817	8813	189	1187	1
60	5511	234	4489	4437	423	5563	8927	189	1073	0
	Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	′

48 DEG.

DEG. TABLE IV.—LOG. SINES, ETC.

Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	
9.825511	234	10.174489	9.954437	423	10.045563	10.128927	190	9.871073	60
5651	233	4349	4691	423	5309	9040	190	0960	59
5791	233	4209	4945	423	5055	9154	190	0846	58
5931	233	4069	5200	423	4800	9268	190	0732	57
6071	233	3929	5454	423	4546	9382	190	0618	56
6211	233	3789	5707	423	4293	9496	190	0504	55
6351	233	3649	5961	423	4039	9610	190	0390	54
9.826491	233	10.173509	9.956215	423	10.043785	10.129724	190	9.870276	53
6631	233	3369	6469	423	3531	9839	190	0161	52
6770	232	3230	6723	423	3277	9953	191	0047	51
6910	232	3090	6977	423	3023	10.130067	191	9.869933	50
7049	232	2951	7231	423	2769	0182	191	9818	49
7189	232	2811	7485	423	2515	0296	191	9704	48
9.827328	232	10.172672	9.957739	423	10.042261	10.130411	191	9.869589	47
7467	232	2533	7993	423	2007	0526	191	9474	46
7606	232	2394	8246	423	1754	0640	191	9360	45
7745	232	2255	8500	423	1500	0755	191	9245	44
7884	231	2116	8754	423	1246	0870	191	9130	43
8023	231	1977	9008	423	0992	0985	192	9015	42
9.828162	231	10.171838	9.959262	423	10.040738	10.131100	192	9.868900	41
8301	231	1699	9516	423	0484	1215	192	8785	40
8439	231	1561	9769	423	0231	1330	192	8670	39
8578	231	1422	9.960023	423	10.039977	1445	192	8555	38
8716	231	1284	0277	423	9723	1560	192	8440	37
8855	230	1145	0531	423	9469	1676	192	8324	36
9.828993	230	10.171007	9.960784	423	10.039216	10.131791	192	9.868209	35
9131	230	0869	1038	423	8962	1907	192	8093	34
9269	230	0731	1291	423	8709	2022	193	7978	33
9407	230	0593	1545	423	8455	2138	193	7862	32
9545	230	0455	1799	423	8201	2253	193	7747	31
9683	230	0317	2052	423	7948	2369	193	7631	30
9.829821	229	10.170179	9.962306	423	10.037694	10.132485	193	9.867515	29
9959	229	0041	2560	423	7440	2601	193	7399	28
9.830097	229	10.169903	2813	423	7187	2717	193	7283	27
0234	229	9766	3067	423	6933	2833	193	7167	26
0372	229	9628	3320	423	6680	2949	193	7051	25
0509	229	9491	3574	423	6426	3065	194	6935	24
9.830646	229	10.169354	9.963827	423	10.036173	10.133181	194	9.866819	23
0784	229	9216	4081	423	5919	3297	194	6703	22
0921	228	9079	4335	423	5665	3414	194	6586	21
1058	228	8942	4588	422	5412	3530	194	6470	20
1195	228	8805	4842	422	5158	3647	194	6353	19
1332	228	8668	5095	422	4905	3763	194	6237	18
9.831469	228	10.168531	9.965349	422	10.034651	10.133880	194	9.866120	17
1606	228	8394	5602	422	4398	3996	195	6004	16
1742	228	8258	5855	422	4145	4113	195	5887	15
1879	228	8121	6109	422	3891	4230	195	5770	14
2015	227	7985	6362	422	3638	4347	195	5653	13
2152	227	7848	6616	422	3384	4464	195	5536	12
9.832288	227	10.167712	9.966869	422	10.033131	10.134581	195	9.865419	11
2425	227	7575	7123	422	2877	4698	195	5302	10
2561	227	7439	7376	422	2624	4815	195	5185	9
2697	227	7303	7629	422	2371	4932	195	5068	8
2833	227	7167	7883	422	2117	5050	195	4950	7
2969	226	7031	8136	422	1864	5167	196	4833	6
9.833105	226	10.166895	9.968389	422	10.031611	10.135284	196	9.864716	5
3241	226	6759	8643	422	1357	5402	196	4598	4
3377	226	6623	8896	422	1104	5519	196	4481	3
3512	226	6488	9149	422	0851	5637	196	4363	2
3648	226	6352	9403	422	0597	5755	196	4245	1
3783		6217	9656	422	0344	5873	196	4127	0
Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	′

43 DEG. TABLE IV.—LOG. SINES, ETC.

′	Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosin
0	9.833783	226	10.166217	9.969656	422	10.030344	10.135873	196	9.8641
1	3919	225	6081	9909	422	0091	5990	196	4(
2	4054	225	5946	9.970162	422	10.029838	6108	197	3{
3	4189	225	5811	0416	422	9584	6226	197	3{
4	4325	225	5675	0669	422	9331	6344	197	3{
5	4460	225	5540	0922	422	9078	6462	197	35
6	4595	225	5405	1175	422	8825	6581	197	34
7	9.834730	225	10.165270	9.971429	422	10.028571	10.136699	197	9.8633
8	4865	225	5135	1682	422	8318	6817	197	31
9	4999	224	5001	1935	422	8065	6936	197	30
10	5134	224	4866	2188	422	7812	7054	198	29
11	5269	224	4731	2441	422	7559	7173	198	28
12	5403	224	4597	2694	422	7306	7291	198	27
13	9.835538	224	10.164462	9.972948	422	10.027052	10.137410	198	9.8626
14	5672	224	4328	3201	422	6799	7529	198	24
15	5807	224	4193	3454	422	6546	7647	198	23
16	5941	224	4059	3707	422	6293	7766	198	22
17	6075	224	3925	3960	422	6040	7885	198	21
18	6209	223	3791	4213	422	5787	8004	198	19
19	9.836343	223	10.163657	9.974466	422	10.025534	10.138123	198	9.8618
20	6477	223	3523	4719	422	5281	8242	198	17
21	6611	223	3389	4973	422	5027	8362	199	16
22	6745	223	3255	5226	422	4774	8481	199	15
23	6878	223	3122	5479	422	4521	8600	199	14
24	7012	222	2988	5732	422	4268	8720	199	12
25	9.837146	222	10.162854	9.975985	422	10.024015	10.138839	199	9.8611
26	7279	222	2721	6238	422	3762	8959	199	10
27	7412	222	2588	6491	422	3509	9078	199	09
28	7546	222	2454	6744	422	3256	9198	199	08
29	7679	222	2321	6997	422	3003	9318	199	06
30	7812	222	2188	7250	422	2750	9438	200	05
31	9.837945	222	10.162055	9.977503	422	10.022497	10.139558	200	9.8604
32	8078	221	1922	7756	422	2244	9678	200	03
33	8211	221	1789	8009	422	1991	9798	200	02
34	8344	221	1656	8262	422	1738	9918	200	00
35	8477	221	1523	8515	422	1485	10.140038	200	9.8599
36	8610	221	1390	8768	422	1232	0158	200	98
37	9.838742	221	10.161258	9.979021	422	10.020979	10.140279	200	9.8597
38	8875	221	1125	9274	422	0726	0399	201	96
39	9007	221	0993	9527	422	0473	0520	201	94
40	9140	220	0860	9780	422	0220	0640	201	93
41	9272	220	0728	9.980033	422	10.019967	0761	201	92
42	9404	220	0596	0286	422	9714	0881	201	91
43	9.839536	220	10.160464	9.980538	422	10.019462	10.141002	201	9.8589
44	9668	220	0332	0791	421	9209	1123	201	88
45	9800	220	0200	1044	421	8956	1244	201	87
46	9932	220	0068	1297	421	8703	1365	202	86
47	9.840064	219	10.159936	1550	421	8450	1486	202	85
48	0196	219	9804	1803	421	8197	1607	202	84
49	9.840328	219	10.159672	9.982056	421	10.017944	10.141728	202	9.858
50	0459	219	9541	2309	421	7691	1849	202	82
51	0591	219	9409	2562	421	7438	1971	202	81
52	0722	219	9278	2814	421	7186	2092	202	79
53	0854	219	9146	3067	421	6933	2214	202	78
54	0985	219	9015	3320	421	6680	2335	203	77
55	9.841116	218	10.158884	9.983573	421	10.016427	10.142457	203	9.857
56	1247	218	8753	3826	421	6174	2578	203	75
57	1378	218	8622	4079	421	5921	2700	203	74
58	1509	218	8491	4331	421	5669	2822	203	73
59	1640	218	8360	4584	421	5416	2944	203	71
60	1771	218	8229	4837	421	5163	3066	203	7(
	Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Si

Sine	D.	Cosec.	Tang.	D.	Cotang.	Secant	D.	Cosine	
.841771	218	10.158229	9.984837	421	10.015163	10.143066	203	9.856934	60
1902	218	8098	5090	421	4910	3188	203	6812	59
2033	218	7967	5343	421	4657	3310	204	6690	58
2163	217	7837	5596	421	4404	3432	204	6568	57
2294	217	7706	5848	421	4152	3554	204	6446	56
2424	217	7576	6101	421	3899	3677	204	6323	55
2555	217	7445	6354	421	3646	3799	204	6201	54
.842685	217	10.157315	9.986607	421	10.013393	10.143922	204	9.856078	53
2815	217	7185	6860	421	3140	4044	204	5956	52
2946	217	7054	7112	421	2888	4167	204	5833	51
3076	217	6924	7365	421	2635	4289	205	5711	50
3206	216	6794	7618	421	2382	4412	205	5588	49
3336	216	6664	7871	421	2129	4535	205	5465	48
.843466	216	10.156534	9.988123	421	10.011877	10.144658	205	9.855342	47
3595	216	6405	8376	421	1624	4781	205	5219	46
3725	216	6275	8629	421	1371	4904	205	5096	45
3855	216	6145	8882	421	1118	5027	205	4973	44
3984	216	6016	9134	421	0866	5150	205	4850	43
4114	215	5886	9387	421	0613	5273	206	4727	42
.844243	215	10.155757	9.989640	421	10.010360	10.145397	206	9.854603	41
4372	215	5628	9893	421	0107	5520	206	4480	40
4502	215	5498	9.990145	421	10.009855	5644	206	4356	39
4631	215	5369	0398	421	9602	5767	206	4233	38
4760	215	5240	0651	421	9349	5891	206	4109	37
4889	215	5111	0903	421	9097	6014	206	3986	36
.845018	215	10.154982	9.991156	421	10.008844	10.146138	206	9.853862	35
5147	215	4853	1409	421	8591	6262	206	3738	34
5276	214	4724	1662	421	8338	6386	206	3614	33
5405	214	4595	1914	421	8086	6510	207	3490	32
5533	214	4467	2167	421	7833	6634	207	3366	31
5662	214	4338	2420	421	7580	6758	207	3242	30
.845790	214	10.154210	9.992672	421	10.007328	10.146882	207	9.853118	29
5919	214	4081	2925	421	7075	7006	207	2994	28
6047	214	3953	3178	421	6822	7131	207	2869	27
6175	214	3825	3430	421	6570	7255	207	2745	26
6304	214	3696	3683	421	6317	7380	207	2620	25
6432	213	3568	3936	421	6064	7504	207	2496	24
.846560	213	10.153440	9.994189	421	10.005811	10.147629	208	9.852371	23
6688	213	3312	4441	421	5559	7753	208	2247	22
6816	213	3184	4694	421	5306	7878	208	2122	21
6944	213	3056	4947	421	5053	8003	208	1997	20
7071	213	2929	5199	421	4801	8128	208	1872	19
7199	213	2801	5452	421	4548	8253	208	1747	18
.847327	213	10.152673	9.995705	421	10.004295	10.148378	208	9.851622	17
7454	212	2546	5957	421	4043	8503	208	1497	16
7582	212	2418	6210	421	3790	8628	209	1372	15
7709	212	2291	6463	421	3537	8754	209	1246	14
7836	212	2164	6715	421	3285	8879	209	1121	13
7964	212	2036	6968	421	3032	9004	209	0996	12
9.848091	212	10.151909	9.997221	421	10.002779	10.149130	209	9.850870	11
8218	212	1782	7473	421	2527	9255	209	0745	10
8345	212	1655	7726	421	2274	9381	209	0619	9
8472	211	1528	7979	421	2021	9507	210	0493	8
8599	211	1401	8231	421	1769	9632	210	0368	7
8726	211	1274	8484	421	1516	9758	210	0242	6
9.848852	211	10.151148	9.998737	421	10.001263	10.149884	210	9.850116	5
8979	211	1021	8989	421	1011	10.150010	210	9.849990	4
9106	211	0894	9242	421	0758	0136	210	9864	3
9232	211	0768	9495	421	0505	0262	210	9738	2
9359	211	0641	9747	421	0253	0389	210	9611	1
9485	211	0515	10.00000	421	0000	0515	210	9485	0
Cosine	D.	Secant	Cotang.	D.	Tang.	Cosec.	D.	Sine	′

TABLE V.

Useful Factors, extended to several places of decimals, in which π represents the circumference of a circle, whose diameter is 1.

Then

π = 3.14159 26535 89793 23846 26433 83279 50288 41971
69399 37510 58209 74944 59230 78164 06286 20899
86280 34825 34211 70679 82148 08651 32723 06647
09384 46 +

2π = 6.28318 53071 79586 47692 52867 66559

4π = 12.56637 06143 59172 95385 05735 33118

$\frac{1}{2}\pi$ = 1.57079 63267 94896 61923 13216 91639

$\frac{1}{4}\pi$ = 0.78539 81633 97448 30961 56608 45819

$\frac{4}{3}\pi$ = 4.18879 02047 86390 98461 68578 44372

$\frac{1}{6}\pi$ = 0.52359 87755 98298 87307 71072 30546

$\frac{1}{8}\pi$ = 0.39269 90816 98724 15480 78304 22909

$\frac{1}{12}\pi$ = 0.26179 93877 99149 43653 85536 15273

$\frac{1}{360}\pi$ = 0.00872 66462 59971 64788 46184 53842

$\frac{1}{\pi}$ = 0.31830 98861 83790 67153 77675 26745 0

$\frac{2}{\pi}$ = 0.63661 97723 67581 34307 55350 53490

$\frac{4}{\pi}$ = 1.27323 95447 35162 68615 10701 06980

$\frac{1}{4\pi}$ = 0.07957 74715 45947 66788 44418 81686 2

$\sqrt{2}$ = 1.41421 35623 73095 04880 16887 24209 7

$\sqrt{\frac{1}{2}}$ = 0.70710 67811 86547 52440 08443 62104

$\pi\sqrt{2}$ = 4.44288 29381 58366 24701 58809 90060 5

$\pi\sqrt{\frac{1}{2}}$ = 2.22144 14690 79183 12350 79404 95030 2

$\frac{1}{\pi}\sqrt{\frac{1}{2}}$ = 0.22507 90790 39276 51738 87997 751

$\frac{1}{\pi}\sqrt{2}$ = 0.45015 81580 78553 03477 75995 955

$\sqrt{\pi}$ = 1.77245 38509 05516 02729 81674 83341

$\frac{1}{2}\sqrt{\pi}$ = 0.88622 69254 52758 01364 90837 41670

$2\sqrt{\pi}$ = 3.54490 77018 11032 05459 63349 66682

$\sqrt{\frac{\pi}{2}}$ = 1.25331 41373 15500 25120 78826 42402

$\sqrt{\frac{2}{\pi}} =$ 0.79788 45608 02865 35587 98921 19868

$\sqrt{\frac{1}{\pi}} =$ 0.56418 95835 47756 28694 80794 51560

$2\sqrt{\frac{1}{\pi}} =$ 1.12837 91670 95512 57389 61589 03120

$\frac{1}{2}\sqrt{\frac{1}{\pi}} =$ 0.28209 47917 73878 14347 40397 25780

$\pi^2 =$ 9.86960 44010 89358 61883 44909 99876

$\frac{1}{\pi^2} =$ 0.10132 11836 42337 77144 38794 63209

$\frac{1}{2\pi^2} =$ 0.05066 05918 21168 88572 19397 31604

$\frac{1}{6\pi^2} =$ 0.01688 68639 40389 62857 39799 10534

$2\sqrt{\frac{1}{\pi}} =$ 1.12837 91670 95512 57389 61589 03120

$\frac{1}{6}\sqrt{\frac{1}{\pi}} =$ 0.09403 15972 57959 38115 80132 41926

$\frac{1}{8}\sqrt{\frac{1}{\pi}} =$ 0.07052 36979 43469 53586 85099 31445

$\frac{1}{6\pi} =$ 0.05305 16476 97298 44525 62945 87790

$\frac{360}{\pi} =$ 114.59155 90261 64641 75359 63096 28200

$\frac{2}{3}\pi =$ 2.09439 51023 93195 49230 84289 22186

$\frac{\pi}{24} =$ 0.13089 96938 99574 71826 92768 07636

$\frac{6}{\pi} =$ 1.90985 93171 02744 02922 66051 60470

$\sqrt[3]{\frac{6}{\pi}} =$ 1.24070 09819, &c.

$6\pi^2 =$ 59.21762 64065 36151 71300 69459 99256

$\sqrt[3]{6\pi^2} =$ 3.89777 707, &c.

$36\pi =$ 113.09733 55292 32556 58465 51617 98062

$\sqrt[3]{36\pi} =$ 4.83597 58620 4, &c.

$\frac{1}{3}\sqrt{\pi} =$ 0.22155 67313 63189 50341 22709 35418

$\frac{1}{2\pi} =$ 0.15915 49430 91895 33576 88837 63372

$\sqrt[3]{\frac{\pi}{6}} =$ 0.80599 59770 07, &c.

$\sqrt{231} =$ 15.19868 41535 70663 6

$\sqrt{282} =$ 16.79285 56237 466 + &c.

$\sqrt{10152} =$ 100.15713 374

$\sqrt{\frac{282}{.785398, \text{ &c.}}} =$ 18.94870 8, &c.

$\sqrt{277.274} =$ 16.65154 64747 4

Log. of $\pi =$ 0.49714 98726 94133 85435

TABLE VI.

Of various useful numbers, with their logarithms.

	Number.	Logarithm.
Modulus of the common logarithms	.43429448	$\bar{1}$.6377843
No. whose hyper. logarithm is unity	2.7182818	0.4342945
Force of gravity in the latitude of London $=g$.	32.19084	1.5077222
Length of second pendulum in inches . . .	39.1393	1.5926130
Feet in a statute mile	5280	3.7226339
Feet in a geographical mile	6075.6	3.7835892
Mean diameter of the earth in miles	7912	3.8982863
Mean circumference „ „	24856	4.3954312
Length of arc $1''=\sin 1''$	.000004848	$\bar{6}$.6855749
„ „ $2''=$ „ $2''$	.000009696	$\bar{6}$.9866049
„ „ $3''=$ „ $3''$	.000014544	$\bar{5}$.1626961
„ „ $1'=$ „ $1'$	.000290888	$\bar{4}$.4637261
„ „ $1°$	.017453293	$\bar{2}$.2418774
Sin of $1°$	.017452406	$\bar{2}$.2418553
360 degrees expressed in seconds	1296000	6.1126050
Radius reduced to seconds	206264.8	5.3144251
„ „ minutes	3437.74677	3.5362739
„ „ degrees	57.29578	1.7581226
Volume of a sphere whose diameter is unity .	.5235988	$\bar{1}$.7189986

TABLE VII.

A Table of the diameters, areas and circumferences of circles, and also the sides of equal squares; from which any one being given, the other three may be found by inspection.

Diam.	Area.	Circumference.	Side of eq. sq.
1.00	0.78539810	3.14159265	0.88622692
.25	1.22718463	3.92699081	1.10778365
.5	1.76714586	4.71238898	1.32934038
.75	2.40528187	5.49778714	1.55089711
2.	3.14159265	6.28318530	1.77245384
.25	3.97607820	7.06858347	1.99401058
.5	4.90873852	7.85398163	2.21556731
.75	5.93957361	8.63937979	2.43712404
3.	7.06858347	9.42477796	2.65868077
.25	8.29576810	10.21017612	2.88023750
.5	9.62112750	10.99557428	3.10179423
.75	11.04466167	11.78097245	3.32335096
4.	12.56637061	12.56637061	3.54490769
.25	14.18625432	13.35176877	3.76646442
.5	15.90431280	14.13716694	3.98802116
.75	17.72054606	14.92256510	4.20957789
5.	19.63495408	15.70796326	4.43113462
.25	21.64753687	16.49336143	4.65269135
.5	23 75829444	17.27875959	4.87424808
.75	25.96722677	18.06415775	5.09580482
6.	28.27433388	18.84955592	5.31736155
.25	30.67961575	19.63495408	5.53891828
.5	33.18307240	20.42035224	5.76047501
.75	35.78470382	21.20575041	5.98203174

Diam.	Area.	Circumference.	Side of eq. sq.
7.	38.48451001	21.99114857	6.20358847
.25	41.28249096	22.77654673	6.42514520
.5	44.17864669	23.56194490	6.64670193
.75	47.17297718	24.34734306	6.86825866
8.	50.26548245	25.13274122	7.08981539
.25	53.45616249	25.91813939	7.31137213
.5	56.74501730	26.70353755	7.53292886
.75	60.13204688	27.48893571	7.75448559
9.	63.61725123	28.27433388	7.97604232
.25	67.20063035	29.05973204	8.19759905
.5	70.88218424	29.84513020	8.41915578
.75	74.66191290	30.63052837	8.64071251
10.	78.53981633	31.41592653	8.86226925
.25	82.51589454	32.20132469	9.08382598
.5	86.59014751	32.98672286	9.30538270
.75	90.76257525	33.77212102	9.52693944
11.	95.03317777	34.55751918	9.74849617
.25	99.40195505	35.34291735	9.97005290
.5	103.86890711	36.12831551	10.19160964
.75	108.43403393	36.91371367	10.41316637
12.	113.09733553	37.69911184	10.63472310
.25	117.85881189	38.48451000	10.85627983
.5	122.71846303	39.26990816	11.07783656
.75	127.67628893	40.05530633	11.29939329
13.	132.73228961	40.84070449	11.52095002
.25	137.88646506	41.62610265	11.74250675
.5	143.13881527	42.41150082	11.96406348
.75	148.48934026	43.19689898	12.18562021
14.	153.93804002	43.98229714	12.40717695
.25	159.48491455	44.76769531	12.62873368
.5	165.12996385	45.55309347	12.85029041
.75	170.87318792	46.33849163	13.07184714
15.	176.71458676	47.12388980	13.29340388
.25	182.65416028	47.90928796	13.51496061
.5	188.69190875	48.69468613	13.73651734
.75	194.82783190	49.48008429	13.95807407
16.	201.06192982	50.26548245	14.17963080
.25	207.39420252	51.05088062	14.40118753
.5	213.82464998	51.83627878	14.62274426
.75	220.35327221	52.62167694	14.84430099
17.	226.98006922	53.40707511	15.06585772
.25	233.70504099	54.19247327	15.28741446
.5	240.52818753	54.97787143	15.50897119
.75	247.44950885	55.76326960	15.73052792
18.	264.46900494	56.54866776	15.95208465
.25	266.58667578	57.33406592	16.17364138
.5	268.80252140	58.11946409	16.39519811
.75	276.11654180	58.90486225	16.61675484
19.	283.52873699	59.69026041	16.83831157
.25	291.03910692	60.47565858	17.05986830
.5	298.64765163	61.26105674	17.28142503
.75	306.35437111	62.04645490	17.50298177
20.	314.15926535	62.83185307	17.72453850
.25	322.06233437	63.61725123	17.94609524
.5	330.06357816	64.40264939	18.16765197
.75	338.16299672	65.18804756	18.38920870
21.	346.36059005	65.97344572	18.61076543
.25	354.65635814	66.75884388	18.83232216
.5	363.05030101	67.54424205	19.05387889
.75	371.54241865	68.32964021	19.27543562
22.	380.13271108	69.11503837	19.49699235
.25	388.82117826	69.90043654	19.71854908
.5	397.60782021	70.68583470	19.94010581
.75	406.49263694	71.47123286	20.16166255
23.	415.47562843	72.25663103	20.38321928
.25	424.55679467	73.04202919	20.60477601

Diam.	Area.	Circumference.	Side of eq. sq.
.5	433.73613573	73.82742735	20.82633274
.75	443.01365154	74.61282552	21.04788945
24.	452.38934212	75.39822368	21.26944618
.25	461.86320745	76.18362184	21.49100291
.5	471.43524757	76.96902001	21.71255964
.75	481.10546239	77.75441817	21.93411637
25.	490.87385212	78.53981634	22.15567313
.25	500.74041655	79.32521450	22.37722986
.5	510.70515575	80.11061266	22.59878659
.75	520.76806971	80.89601083	22.82034332
26.	530.92915845	81.68140899	23.04190006
.25	541.18842196	82.46680715	23.26345679
.5	551.54586024	83.25220532	23.48501352
.75	562.00147328	84.03760348	23.70657025
27.	572.55526112	84.82300164	23.92812698
.25	583.20722369	85.60839981	24.14968371
.5	593.95736105	86.39379797	24.37124044
.75	604.80567318	87.17919613	24.59279717
28.	615.75216010	87.96459430	24.81435390
.25	626.79682177	88.74999246	25.03591063
.5	637.93965822	89.53539062	25.25746737
.75	649.18066943	90.32078879	25.47902410
29.	660.51985541	91.10618695	25.70058083
.25	671.95721616	91.89158511	25.92213756
.5	683.49275169	92.67698328	26.14369429
.75	695.12646198	93.46238144	26.36525102
30.	706.85834706	94.24777960	26.58680776
.25	718.68840688	95.03317777	26.80836449
.5	730.61664148	95.81857593	27.02992122
.75	742.64305085	96.60397409	27.25147794
31.	754.76763502	97.38937226	27.47303468
.25	766.99039394	98.17477042	27.69459141
.5	779.31132762	98.96016858	27.91614814
.75	791.73043607	99.74556675	28.13770488
32.	804.24771932	100.53096491	28.35926161
.25	816.86317729	101.31636307	28.58081834
.5	829.57681005	102.10176124	28.80237507
.75	842.38861759	102.88715940	29.02393180
33.	855.29859994	103.67255756	29.24548853
.25	868.30675696	104.45795573	29.46704526
.5	881.41308881	105.24335389	29.68860199
.75	894.61759542	106.02875205	29.91015872
34.	907.92027688	106.81415022	30.13171545
.25	921.32113305	107.59954838	30.35327219
.5	934.82016398	108.38494654	30.57482892
.75	948.41736968	109.17034471	30.79638565
35.	962.11275016	109.95574287	31.01794239
.25	975.90630540	110.74114103	31.23949912
.5	989.79803541	111.52653920	31.46105585
.75	1003.78794019	112.31193736	31.68261258
36.	1017.87601975	113.09733552	31.90416931
.25	1032.06227407	113.88273369	32.12572604
.5	1046.34670316	114.66813185	32.34728277
.75	1060.72930703	115.45353001	32.56883950
37.	1075.21008569	116.23892818	32.79039623
.25	1089.78903909	117.02432634	33.01195296
.5	1104.46616727	117.80972450	33.23350970
.75	1119.24147022	118.59572267	33.45506643
38.	1134.11494794	119.38052083	33.67662316
.25	1149.08660043	120.16591899	33.89817989
.5	1164.15642768	120.95131716	34.11973662
.75	1179.32442971	121.73671532	34.34129335
39.	1194.59060653	122.52211348	34.56285008
.25	1209.95495809	123.30751165	34.78440681
.5	1225.41748449	124.09290981	35.00596354
.75	1240.97818552	124.87830797	35.22752027

Diam.	Area.	Circumference.	Side of eq. sq.
40.	1256.63706144	125.66370614	35.44907701
.25	1272.39411208	126.44910430	35.67063374
.5	1288.24933751	127.23450246	35.89219048
.75	1304.20273770	128.01990063	36.11374721
41.	1320.25431266	128.80529879	36.33530394
.25	1336.40406240	129.59069695	36.55686067
.5	1352.65198690	130.37609512	36.77841740
.75	1368.99808617	131.16149328	36.99997413
42.	1385.44236022	131.94689144	37.22153086
.25	1401.98480903	132.73228961	37.44308759
.5	1418.62543261	133.51768777	37.66464432
.75	1435.36423096	134.30308593	37.88620105
43.	1452.20120412	135.08848410	38.10775779
.25	1469.13635202	135.87388226	38.32931452
.5	1486.16967468	136.65928042	38.55087125
.75	1503.30117212	137.44467859	38.77242798
44.	1520.53084433	138.23007675	38.99398471
.25	1537.85869131	139.01547491	39.21554144
.5	1556.28471306	139.80087308	39.43709817
.75	1572.80890957	140.58627124	39.65865490
45.	1590.43128088	141.37166940	39.88021164
.25	1608.15182692	142.15706757	40.10176837
.5	1625.97054775	142.94246573	40.32332510
.75	1643.88744335	143.72786390	40.54488183
46.	1661.90251374	144.51326206	40.76643856
.25	1680.01575889	145.29866022	40.98799530
.5	1698.22717880	146.08405839	41.20955203
.75	1716.53677348	146.86945655	41.43110876
47.	1734.94454294	147.65485471	41.65266549
.25	1753.45048716	148.44025288	41.87422222
.5	1772.05460615	149.22565104	42.09577891
.75	1790.75689992	150.01104920	42.31733568
48.	1809.55736847	150.79644797	42.53889241
.25	1828.45601175	151.58184553	42.76044914
.5	1847.45282982	152.36724369	42.98200587
.75	1866.54782267	153.15264186	43.20356261
49.	1885.74099031	153.93804002	43.42511934
.25	1905.83233270	154.72343818	43.64667607
.5	1924.42184986	155.50883635	43.86823280
.75	1943.90954179	156.29423451	44.08978953
50.	1963.49540848	157.07963268	44.31134627
.25	1983.17944995	157.96503084	44.53290300
.5	2002.96166619	158.65042900	44.75445973
.75	2022.84205720	159.43582717	44.97601646
51.	2042.82062300	160.22122533	45.19757319
.25	2062.89736352	161.00662349	45.41912992
.5	2083.07227884	161.79202166	45.64068665
.75	2103.34536893	162.57741982	45.86224338
52.	2123.71663382	163.36281798	46.08380012
.25	2144.18607346	164.14821615	46.30535685
.5	2164.75368786	164.93361431	46.52691358
.75	2185.41947703	165.71901247	46.74847031
53.	2206.18344098	166.50441064	46.97002704
.25	2227.04557969	167.28980880	47.19158377
.5	2248.00589318	168.07520696	47.41314050
.75	2269.06438143	168.86060513	47.63469723
54.	2290.22104447	169.64600329	47.85625386
.25	2311.47588225	170.43140145	48.07781069
.5	2332.82889481	171.21679962	48.29936743
.75	2354.28008215	172.00219778	48.52092416
55.	2375.82944427	172.78759594	48.74248089
.25	2397.47698115	173.57299411	48.96403763
.5	2419.22269280	174.35839227	49.18559436
.75	2441.06657922	175.14379043	49.40715109
56.	2463.00864041	175.92918860	49.62870782
.25	2485.04887637	176.71458676	49.85026455

Diam.	Area.	Circumference.	Side of eq. sq.
.5	2507.18728710	177.49998492	50.07182128
.75	2520.42387260	178.28538809	50.29337801
57.	2551.75863288	179.07078125	50.51493474
.25	2574.19156790	179.85617941	50.73649147
.5	2596.72257781	180.64157758	50.95804820
.75	2619.35196239	181.42697574	51.17960494
58.	2642.07942166	182.21237390	51.40116167
.25	2664.90505579	182.99777207	51.62271840
.5	2687.82886464	183.78317023	51.84427513
.75	2710.85084834	184.56856839	52.06583186
59.	2733.97100678	185.35396656	52.28738859
.25	2757.18933998	186.13936472	52.50894532
.5	2780.50584795	186.92476288	52.73050205
.75	2803.92058070	187.71016105	52.95205878
60.	2827.43338828	188.49555921	53.17364552
.25	2851.04442049	189.28095737	53.39517225
.5	2874.75362754	190.06635554	53.61672898
.75	2898.56100937	190.85175370	53.83828572
61.	2922.46656592	191.63715186	54.05984245
.25	2946.47029734	192.42255003	54.28139918
.5	2970.57220350	193.20794819	54.50295591
.75	2994.77228444	193.99334635	54.72451264
62.	3019.07054010	194.77874452	54.94606937
.25	3043.46697053	195.56414268	55.16762610
.5	3067.96157576	196.34954084	55.38918283
.75	3092.55435572	197.13493901	55.61073956
63.	3117.24531051	197.92033717	55.83229629
.25	3142.03444002	198.70573533	56.05385303
.5	3166.92174434	199.49113350	56.27540976
.75	3191.90722341	200.27653166	56.49696649
64.	3216.99087728	201.06192982	56.71852322
.25	3242.17270581	201.84732799	56.94007995
.5	3267.45270920	202.63272615	57.16163668
.75	3292.83088742	203.41812431	57.38319341
65.	3318.30724035	204.20352248	57.60475015
.25	3343.88176802	204.98892064	57.82630688
.5	3369.55447036	205.77431880	58.04786361
.75	3395.32534774	206.55971697	58.26942034
66.	3421.19439976	207.34511513	58.49097707
.25	3447.16162652	208.13051329	58.71253380
.5	3473.22702806	208.91591146	58.93409054
.75	3499.39060438	209.70130962	59.15564727
67.	3525.65235549	210.48670778	59.37720400
.25	3552.01228735	211.27210595	59.59876073
.5	3578.47038197	212.05750411	59.82031746
.75	3605.02665737	212.84290227	60.04187419
68.	3631.68110754	213.62830044	60.26343092
.25	3658.43373248	214.41369860	60.48498765
.5	3685.28453219	215.19909676	60.70654438
.75	3712.23350667	215.98449493	60.92810111
69.	3739.28065594	216.76989309	61.14965785
.25	3766.42597994	217.55529125	61.37121458
.5	3793.66947873	218.34068942	61.59277131
.75	3821.01115229	219.12608758	61.81432804
70.	3848.45100064	219.91148574	62.03588478
.25	3875.98902375	220.69688391	62.25744151
.5	3903.62522162	221.48228207	62.47899824
.75	3931.35959426	222.26768023	62.70055497
71.	3959.19214168	223.05307840	62.92211170
.25	3987.12286386	223.83847656	63.14366843
.5	4015.15176082	224.62387472	63.36522516
.75	4043.27883254	225.40927289	63.58678189
72.	4071.50407905	226.19467105	63.80833862
.25	4099.82750030	226.98006921	64.02989536
.5	4128.24909633	227.76546738	64.25145209
.75	4156.76886714	228.55086554	64.47300882

Diam.	Area.	Circumference.	Side of eq. sq.
73.	4185.38681274	229.33626370	64.69456555
.25	4214.10293309	230.12166187	64.91612228
.5	4242.91722821	230.90706003	65.13767901
.75	4271.82969810	231.69245819	65.35923574
74.	4300.84034275	232.47785636	65.58079247
.25	4329.94916219	233.26325452	65.80234920
.5	4359.15615638	234.04865268	66.02390593
.75	4388.46132535	234.83405085	66.24546267
75.	4417.86466910	235.61944901	66.46701940
.25	4447.36618760	236.40484717	66.68857613
.5	4476.96588088	237.19024534	66.91043287
.75	4506.66374893	237.97564350	67.13168960
76.	4536.45979178	238.76104166	67.35324633
.25	4566.35400937	239.54643983	67.57480306
.5	4596.34640174	240.33183799	67.79635979
.75	4626.43696897	241.11723615	68.01791652
77.	4656.62571078	241.90263432	68.23947325
.25	4686.91262745	242.68803248	68.46102998
.5	4717.29771890	243.47343064	68.68258671
.75	4747.78098511	244.25882881	68.90414344
78.	4778.36242610	245.04422697	69.12570018
.25	4809.04204185	245.82962513	69.34725691
.5	4839.81983238	246.61502330	69.56881364
.75	4870.79579767	247.40042146	69.79037037
79.	4901.66993776	248.18581962	70.01192710
.25	4932.74225260	248.97121779	70.23348383
.5	4963.91274221	249.75661595	70.45504056
.75	4995.18140659	250.34201411	70.67659729
80.	5026.54824574	251.32741228	70.89815403
.25	5058.01325966	252.11281044	71.11971076
.5	5089.57644835	252.89820860	71.34126749
.75	5121.23781181	253.68360677	71.56282422
81.	5152.99735004	254.46900493	71.78438096
.25	5184.85506304	255.25440309	72.00593769
.5	5216.81095081	256.03980126	72.22749442
.75	5248.86501335	256.82579942	72.44905115
82.	5281.01725068	257.61059758	72.67060788
.25	5313.26766276	258.39599575	72.89216461
.5	5345.61624962	259.18139391	73.11372134
.75	5378.06301124	259.96679207	73.33527807
83.	5410.60794764	260.75219024	73.55683480
.25	5443.25105880	261.53758840	73.77839153
.5	5475.99234474	262.32298656	73.99994827
.75	5508.83180544	263.10838473	74.22150500
84.	5541.76944092	263.89378269	74.44306173
.25	5574.80525116	264.67918105	74.66461846
.5	5607.93923618	265.46457922	74.88617519
.75	5641.17139596	266.24997738	75.10773192
85.	5674.50173054	267.03537554	75.32928866
.25	5707.93023987	267.82077371	75.55084539
.5	5741.45692397	268.60617187	75.77240212
.75	5775.08178284	269.39157003	75.99395885
86.	5808.80481648	270.17696820	76.21551558
.25	5842.62602489	270.96236636	76.43707232
.5	5876.54540807	271.74776452	76.65862904
.75	5910.56296602	272.53316269	76.88018578
87.	5944.67869876	273.31856085	77.10174251
.25	5978.89260623	274.10395901	77.32329924
.5	6013.20468849	274.88935718	77.54485597
.75	6047.61494552	275.67475534	77.76641270
88.	6082.12337734	276.46015350	77.98796943
.25	6116.72998392	277.24555167	78.20952616
.5	6151.43476526	278.03094983	78.43103289
.75	6186.23772138	278.81634799	78.65263962
89.	6221.13885226	279.60174616	78.87419635
.25	6256.13815792	280.38714432	79.09575309

Diam.	Area.	Circumference.	Side of eq. sq.
.5	6291.23563834	281.17254248	79.31730982
.75	6326.43129354	281.95794065	79.53886655
90.	6361.72512352	282.74333881	79.76042329
.25	6397.11712824	283.52873697	79.98198002
.5	6432.60730774	284.31413514	80.20353673
.75	6468.19566202	285.09953330	80.42509348
91.	6503.88219109	285.88493146	80.64669021
.25	6539.66689491	286.67032963	80.86820694
.5	6575.54977350	287.45572779	81.08976367
.75	6611.53082686	288.24112595	81.31132040
92.	6647.61005499	289.02652412	81.53287713
.25	6683.78745789	289.81192228	81.75443366
.5	6720.06303556	290.59732044	81.97599060
.75	6756.43678800	291.38271861	82.19754733
93.	6792.90871521	292.16811677	82.41910406
.25	6829.47881719	292.95351493	82.64066079
.5	6866.14709394	293.73891310	82.86221752
.75	6902.91354546	294.52431126	83.08377425
94.	6939.77817177	295.30970942	83.30533096
.25	6976.74097284	296.09510759	83.52688771
.5	7013.80194367	296.88050579	83.74844444
.75	7050.96109928	297.66590391	83.97000117
95.	7088.21842465	298.45130208	84.19155791
.25	7125.57992480	299.23670024	84.41311464
.5	7163.02759971	300.02209840	84.63467136
.75	7200.57944940	300.80749657	84.85622811
96.	7238.22947387	301.59289473	85.07778484
.25	7275.97767308	302.37829269	85.29934157
.5	7313.82404707	303.16369106	85.52089830
.75	7351.76859584	303.94908922	85.74245503
97.	7389.81131940	304.73448738	85.96401176
.25	7427.95221771	305.51988555	86.18556849
.5	7466.19129079	306.30528371	86.40712522
.75	7504.52853864	307.09068187	86.62868195
98.	7542.96396126	307.87608004	86.85023869
.25	7581.49755865	308.66147820	87.07179542
.5	7620.12933081	309.44687635	87.29335215
.75	7658.85927774	310.23227453	87.51490888
99.	7697.68739946	311.01767269	87.73646568
.25	7736.61369591	311.80307085	87.95802234
.5	7775.63816715	312.58846902	88.17957907
.75	7814.76081316	313.37386718	88.40113580
100.	7853.98163397	314.15926535	88.62269254
.25	7893.30062952	314.94466352	88.84424927
.5	7932.71779985	315.73006168	89.06580600
.75	7972.23314494	316.51545984	89.28736273

The preceding Tables were computed with great care by the author's esteemed friend, the late H. Goodwin, Esq., of Blackheath, a gentleman whose indefatigable perseverance and remarkable accuracy in reference to numerical computations cannot be too highly characterized. They are inserted here to supersede the necessity of consulting some erroneous tables of the areas, &c., of circles recently put into circulation.

Supposing the unit to be an inch, for example, the last Table exhibits the area, circumference, and side of an equal square, corresponding to diameters varying by a quarter of an inch, from 1 inch to 100.

But the same Table may also be made to serve for many intermediate diameters, as well as for diameters beyond its apparent reach, by simply recollecting that *the circumferences and sides of*

equal squares are as the diameters, while *the areas are as the squares of the diameters.*

Thus, divide any

circumference or side of equal square by { 2, 3, 4, 5, 6, 10, &c. } the quotient will give the { circumference or side of equal square to } { $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{10}$ } the original diameter.

Or multiply by 2, 3, 4, 5, &c., the product will give the circumference or side of equal square to 2, 3, 4, 5, &c., times the assumed diameter.

For areas, take $\frac{1}{4}$, $\frac{1}{9}$, $\frac{1}{16}$, $\frac{1}{25}$, &c., for $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, &c., the assumed diameter, and contrarily, 4 times, 9 times, 16 times, 25 times, 100 times, &c., an area, for the one which agrees to twice, 3 times, 4 times, 5 times, 10 times, &c., the diameter to which the assumed area corresponds.

Example. Find the area of a circle whose diameter is $8\frac{1}{8}$, or 8·125.

Area to diameter 16·25 is 207·39420252
Divide this by $2^2 = 4$

The quotient is the area required 51·84855063

Ex. 2. Find the area of a circle whose diameter is 8·1

Area to diameter 81, is 5152·99735004
Divide this by $10^2 = 100$

The quotient is the area required 51·5299735004

Ex. 3. The exterior and the interior diameters of a circular ring are 8·75 and 8·1. Required its area.

Area to diameter 8·75, from table 60·13204688
„ „ 8·1 „ „ 51·52997350

Their diff. is the area of the ring = 8·60207338

TABLE VIII.

Relations of the Arc, Abscissa, Ordinate, and Subnormal, in the Catenary; useful in the construction of Catenarian equilibrated arches for bridges or powder magazines.

Arc A P	Abscissa A G	Ordinate G P	Subnormal G K	Arc A P	Abscissa A G	Ordinate G P	Subnormal G K
1	1.00	0.02	25.00	13	12.47	3.18	23.98
2	2.00	0.08	24.97	14	13.36	3.65	23.86
3	2.99	0.18	24.94	15	14.23	4.16	23.72
4	3.99	0.32	24.91	16	15.07	4.68	23.55
5	4.97	0.50	24.86	17	15.90	5.23	23.38
6	5.95	0.71	24.79	18	16.72	5.81	23.22
7	6.92	0.96	24.71	19	17.52	6.40	23.05
8	7.87	1.25	24.60	20	18.31	7.01	22.89
9	8.82	1.57	24.50	21	19.08	7.65	22.72
10	9.75	1.93	24.37	22	19.84	8.31	22.55
11	10.67	2.31	24 25	23	20.59	8.98	22.38
12	11.58	2.73	24.12	24	21.32	9.66	22.21

TABLE VIII.—*Relations of the Catenary, continued.*

Arc A P	Abscissa A G	Ordinate G P	Subnormal G K	Arc A P	Abscissa A G	Ordinate G P	Subnormal G K
25	22.03	10.36	22.03	88	49.28	66.48	14.00
26	22.73	11.08	21.86	89	49.55	67.44	13.92
27	23.42	11.80	21.68	90	49.82	68.41	13.84
28	24.09	12.54	21.51	91	50.09	69.37	13.76
29	24.76	13.29	21.34	92	50.35	70.34	13.68
30	25.40	14.05	21.17	93	50.61	71.30	13.60
31	26.04	14.83	21.00	94	50.87	72.27	13.53
32	26.66	15.61	20.83	95	51.13	73.24	13.45
33	27.27	16.41	20.66	96	51.38	74.20	13.38
34	27.87	17.21	20.49	97	51.63	75.17	13.31
35	28.45	18.02	20.33	98	51.88	76.14	13.23
36	29.03	18.83	20.16	99	52.13	77.11	13.16
37	29.59	19.66	20.00	100	52.37	78.08	13.09
38	30.14	20.49	19.83	101	52.61	79.05	13.02
39	30.68	21.33	19.67	102	52.85	80.02	12.95
40	31.22	22.17	19.51	103	53.09	81.00	12.89
41	31.74	23.02	19.36	104	53.32	81.97	12.82
42	32.26	23.88	19.20	105	53.55	82.94	12.75
43	32.77	24.74	19.05	106	53.78	83.91	12.69
44	33.27	25.61	18.90	107	54.01	84.89	12.62
45	33.76	26.48	18.75	108	54.24	85.86	12.56
46	34.24	27.36	18.61	109	54.47	86.83	12.50
47	34.71	28.24	18.46	110	54.69	87.81	12.43
48	35.18	29.12	18.32	111	54.91	88.78	12.37
49	35.64	30.01	18.18	112	55.13	89.76	12.31
50	36.09	30.90	18.05	113	55.35	90.73	12.25
51	36.53	31.80	17.91	114	55.56	91.71	12.19
52	36.97	32.70	17.77	115	55.77	92.68	12.13
53	37.40	33.60	17.64	116	55.98	93.66	12.07
54	37.82	34.51	17.51	117	56.19	94.64	12.01
55	38.24	35.42	17.38	118	56.39	95.62	11.95
56	38.65	36.33	17.25	119	56.60	96.59	11.90
57	39.06	37.24	17.13	120	56.80	97.57	11.84
58	39.46	38.16	17.01	121	57.00	98.55	11.78
59	39.85	39.08	16.89	122	57.21	99.53	11.73
60	40.24	40.00	16.77	123	57.41	100.51	11.67
61	40.62	40.93	16.65	124	57.61	101.49	11.62
62	40.99	41.85	16.53	125	57.81	102.47	11.57
63	41.36	42.78	16.42	126	58.00	103.45	11.51
64	41.73	43.71	16.30	127	58.20	104.42	11.46
65	42.09	44.65	16.19	128	58.39	105.42	11.41
66	42.45	45.58	16.08	129	58.58	106.40	11.36
67	42.80	46.51	15.97	130	58.77	107.38	11.30
68	43.15	47.45	15.86	131	58 96	108.35	11.25
69	43.49	48.39	15.76	132	59.15	109.35	11.20
70	43.83	49.33	15.65	133	59.34	110.33	11.15
71	44.17	50.28	15.55	134	59.52	111.31	11.10
72	44.50	51.22	15.45	135	59.71	112.28	11.06
73	44.82	52.17	15.35	136	59.89	113.28	11.01
74	45.14	53.11	15.25	137	60.07	114.26	10.96
75	45.46	54.06	15.16	138	60.25	115.25	10.91
76	45.77	55.01	15.06	139	60.43	116.23	10.87
77	46.08	55.96	14.96	140	60.60	117.22	10.82
78	46.39	56.91	14.87	141	60.77	118.20	10.77
79	46.69	57.86	14.77	142	60.95	119.18	10.73
80	46.99	58.81	14.68	143	61.12	120.15	10.68
81	47.29	59.77	14.59	144	61.29	121.15	10.64
82	47.58	60.72	14.50	145	61.46	122.14	10.59
83	47.87	61.68	14.42	146	61.63	123.12	10.55
84	48.16	62.64	14.33	147	61.79	124.11	10.50
85	48.44	63.60	14.25	148	61.96	125.10	10.46
86	48.73	64.56	14.16	149	62.12	126.09	10.42
87	49.00	65.52	14.08	150	62.29	127.08	10.38

At page 185 there is given a table of relations of catenarian ırves, so arranged as to be useful in the erection of suspension idges, and in determining the weight and tension of the several ırts. The table now presented in this supplement agrees with at in principle, but presents a totally different aspect, in order that may be subservient to the construction of catenarian arches of uilibration, whether for bridges or for powder magazines.

It is very well known with regard to arches of equilibration, when e substance of the structure presses vertically downwards, by the rce of gravity, that for a parabolic arch assumed as the intrados, e principles of equiponderance require a similar and equal para- lic curve, situated throughout at the same vertical distance from e inner curve as the extrados; and in that case the curves may be sily constructed, and the joints of the voussoirs, being tangents the curve found, by the method explained in page 174. But is equally true, that upon the same general principles of equi- llence, a catenarian curve may be assumed for the arch, and the :rados and extrados be two similar and equidistant curves, pro- led that equidistance be measured upon the radius of curvature at ch point, and be but small compared with the span of the arch. ınstructions founded upon the knowledge of this fact have been ıg rejected, but have of late been reintroduced; on which account ıble VIII. is given, that the time of practical men may not be ıste in *needless* calculation.

Suppose that figure 245 represented a catenary, in which, as indi- ted in the Table, A P represented any portion of the curve A G, d G P the corresponding abscissa and ordinate, and G K the subnor- al, p representing the parameter, or constant horizontal tension at the rtex of the curve. Then, in addition to the equations exhibited in ges 181 and 182, it is known that $AP = \sqrt{(AG)^2 + 2pAG}$, d $$\frac{1}{p} = \frac{2AG}{(GP)^2} - \frac{2(AG)^3}{3(GP)^4} + \frac{26(AG)^5}{45(GP)^6} - \frac{662(AG)^7}{945(GP)^8} +, \text{ \&c.}$$

Now, the Table presents values of A G, G P, G K, corresponding several values of A P, the arc, on the supposition that P, the rameter, or tension at the vertex, is 25.

Suppose, as an illustration of the use of the Table, it were re- ired to construct a vault, whose semibase should be 8 (yards, for ample) and the height 10; we shall find its model by searching that rt of the table where the abscissa, A G, and the ordinate, G P, are the ratio of 10 to 8. This is, where the arc A P is 80; for there e abscissa and ordinate are 58·81 and 46·99, which are in the ratio 10·01 to 8, sufficiently near for practical purposes. Hence, then, the :h may be regarded as divided into 80 equal parts, and the table will esent the computed proportions for 80 voussoirs, on each side of the rtex, or rather for 79 voussoirs and half the key-stone. Or they ıy be reduced to 40, or to 20, or to 10, on each side the crown the structure. The actual values to the dimensions 10 and 8, will be ınd by multiplying each number in the table by $\frac{8}{46{\cdot}99}$, or its equi- lent, 0·1702; or, to have the dimensions in feet, let the multiplier $3 \times 0{\cdot}1702$ or 0·5106. Thus, we should have the values of A G, P, A P, for every voussoir from the vertex of the curve downwards, ıile the corresponding values of A G + G K would give the points from which the line K P must be drawn, to give in each case the rection of the joint. In the example assumed the value of the

parameter, p, would be $25 \times 0{\cdot}1702$ or $4{\cdot}255$; and this would be the measure of the horizontal thrust.

The curve thus sketched will be posited in the middle of the arch, half way between the intrados and the extrados. Trace above and below this, at the distance $\frac{1}{2}\ t$ (half the thickness at the crown), measured upon the respective positions of the joints, two curves parallel to the mean catenary, and you will obtain the proposed arch of equilibration.

In cases where the proposed ratio of the height and semi-span of the arch, cannot be found with sufficient accuracy in the tables, it may be approximated to by the usual methods employed in reference to proportional parts. As, suppose the semi-span and the altitude were to be equal; this condition lies between 60 and 61 in the values of the arch; and, by the method just alluded to, we shall find that, when the arc is equal to 60·44, the height and half base are each 40·40, the horizontal thrust being 25.

If an equilibrated circular arch were to be erected, of the same span and height, and the same thickness at the crown, the horizontal thrust would be 31·6. Whence, by the way, it appears that an equilibrated catenarian arch in these proportions, would produce a less horizontal thrust than an equilibrated circular arch: contrary to the opinion of Bossut, *Equilibres des Voûtes.*

TABLE IX.

Of the Lengths and Vibrations of Pendulums.

[See Note at foot of page 223.]

Length inches.	Time of vibration in seconds.	No. Vibr. per sec.	Length inches.	Time of vibration in seconds.	No. Vibr. per sec.
1.0	0.1598	6.256	3.5	0.2990	3.344
1.1	0.1676	5.965	3.6	0.3032	3.297
1.2	0.1751	7.711	3.7	0.3074	3.252
1.3	0.1822	5.487	3.8	0.3115	3.209
1.4	0.1891	5.287	3.9	0.3157	3.167
1.5	0.1957	5.108	4.0	0.3196	3.128
1.6	0.2021	4.945	4.1	0.3236	3.089
1.7	0.2084	4.798	4.2	0.3275	3.052
1.8	0.2144	4.663	4.3	0.3314	3.016
1.9	0.2203	4.538	4.4	0.3352	2.982
2.0	0.2260	4.423	4.5	0.3390	2.949
2.1	0.2316	4.317	4.6	0.3428	2.916
2.2	0.2370	4.217			
2.3	0.2424	4.125	4.7	0.3465	2.885
2.4	0.2476	4.038	4.8	0.3502	2.855
2.5	0.2527	3.956	4.9	0.3538	2.826
2.6	0.2577	3.879	5.0	0.3574	2.797
2.7	0.2626	3.807	5.1	0.3609	2.770
2.8	0.2674	3.738	5.2	0.3644	2.743
2.9	0.2721	3.673	5.3	0.3679	2.717
3.0	0.2768	3.612	5.4	0.3713	2.692
3.1	0.2814	3.553	5.5	0.3748	2.667
3.2	0.2859	3.497	5.6	0.3782	2.643
3.3	0.2903	3.443	5.7	0.3816	2.620
3.4	0.2947	3.392	5.8	0.3849	2.597

Length inches.	Time of vibration in seconds.	No. Vibr. per sec.	Length inches.	Time of vibration in seconds.	No. Vibr. per sec.
5.9	0.3882	2.575	11.3	0.5373	1.861
6.0	0.3915	2.554	11.4	0.5396	1.853
6.1	0.3947	2.533	11.5	0.5420	1.845
6.2	0.3980	2.512	11.6	0.5444	1.837
6.3	0.4012	2.492	11.7	0.5467	1.829
6.4	0.4043	2.472	11.8	0.5490	1.821
6.5	0.4075	2.453	11.9	0.5514	1.813
6.6	0.4106	2.435	12.0	0.5537	1.806
6.7	0.4137	2.416	12.1	0.5560	1.798
6.8	0.4168	2.399	12.2	0.5583	1.791
6.9	0.4198	2.381	12.3	0.5605	1.783
7.0	0.4229	2.364	12.4	0.5628	1.776
7.1	0.4259	2.347	12.5	0.5651	1.769
7.2	0.4289	2.331	12.6	0.5673	1.762
7.3	0.4318	2.315	12.7	0.5696	1.755
7.4	0.4348	2.299	12.8	0.5718	1.748
7.5	0.4377	2.284	12.9	0.5741	1.741
7.6	0.4406	2.269	13.0	0.5763	1.735
7.7	0.4435	2.254	13.1	0.5785	1.728
7.8	0.4464	2.240	13.2	0.5807	1.721
7.9	0.4492	2.225	13.3	0.5829	1.715
8.0	0.4521	2.211	13.4	0.5851	1.709
8.1	0.4549	2.198	13.5	0.5873	1.703
8.2	0.4577	2.184	13.6	0.5894	1.696
8.3	0.4605	2.171	13.7	0.5916	1.690
8.4	0.4632	2.158	13.8	0.5938	1.684
8.5	0.4660	2.145	13.9	0.5959	1.678
8.6	0.4687	2.133	14.0	0.5980	1.672
8.7	0.4714	2.121	14.1	0.6001	1.666
8.8	0.4741	2.108	14.2	0.6023	1.660
8.9	0.4768	2.097	14.3	0.6044	1.654
9.0	0.4795	2.085	14.4	0.6065	1.648
9.1	0.4821	2.073	14.5	0.6086	1.642
9.2	0.4848	2.062	14.6	0.6107	1.637
9.3	0.4874	2.051	14.7	0.6128	1.631
9.4	0.4900	2.040	14.8	0.6149	1.626
9.5	0.4926	2.029	14.9	0.6170	1.620
9.6	0.4952	2.019	15.0	0.6191	1.615
9.7	0.4978	2.008	15.1	0.6211	1.609
9.8	0.5003	1.998	15.2	0.6231	1.604
9.9	0.5029	1.988	15.3	0.6252	1.599
10.0	0.5054	1.978	15.4	0.6272	1.594
10.1	0.5079	1.968	15.5	0.6293	1.589
10.2	0.5105	1.958	15.6	0.6313	1.584
10.3	0.5130	1.949	15.7	0.6333	1.579
10.4	0.5155	1.939	15.8	0.6353	1.574
10.5	0.5179	1.930	15.9	0.6373	1.569
10.6	0.5204	1.921	16.0	0.6393	1.564
10.7	0.5228	1.912	16.1	0.6413	1.559
10.8	0.5253	1.903	16.2	0.6433	1.554
10.9	0.5277	1.894	16.3	0.6453	1.549
11.0	0.5301	1.886	16.4	0.6473	1.544
11.1	0.5325	1.877	16.5	0.6493	1.539
11.2	0.5349	1.869	16.6	0.6532	1.535

Length inches.	Time of vibration in seconds.	No. Vibr. per sec.	Length inches.	Time of vibration in seconds.	No. Vibr. per sec.
16.7	0.6551	1.531	22.1	0.7514	1.330
16.8	0.6571	1.526	22.2	0.7531	1.327
16.9	0.6590	1.521	22.3	0.7548	1.324
17.0	0.6609	1.517	22.4	0.7565	1.321
17.1	0.6629	1.512	22.5	0.7582	1.318
17.2	0.6648	1.508	22.6	0.7598	1.315
17.3	0.6648	1.504	22.7	0.7615	1.313
17.4	0.6667	1.499	22.8	0.7632	1.310
17.5	0.6686	1.495	22.9	0.7649	1.307
17.6	0.6705	1.491	23.0	0.7665	1.304
17.7	0.6724	1.487	23.1	0.7682	1.301
17.8	0.6743	1.482	23.2	0.7699	1.298
17.9	0.6762	1.478	23.3	0.7715	1.296
18.0	0.6781	1.474	23.4	0.7732	1.293
18.1	0.6800	1.470	23.5	0.7748	1.290
18.2	0.6819	1.466	23.6	0.7765	1.287
18.3	0.6837	1.462	23.7	0.7781	1.285
18.4	0.6856	1.458	23.8	0.7798	1.282
18.5	0.6875	1.454	23.9	0.7814	1.279
18.6	0.6893	1.450	24.0	0.7830	1.277
18.7	0.6912	1.446	24.1	0.7847	1.274
18.8	0.6930	1.442	24.2	0.7863	1.271
18.9	0.6949	1.439	24.3	0.7879	1.269
19.0	0.6967	1.435	24.4	0.7895	1.266
19.1	0.6985	1.431	24.5	0.7911	1.263
19.2	0.7003	1.427	24.6	0.7927	1.261
19.3	0.7022	1.424	24.7	0.7944	1.259
19.4	0.7040	1.420	24.8	0.7960	1.256
19.5	0.7058	1.416	24.9	0.7976	1.253
19.6	0.7076	1.413	25.0	0.7992	1.251
19.7	0.7094	1.409	25.1	0.8008	1.248
19.8	0.7112	1.405	25.2	0.8024	1.246
19.9	0.7130	1.402	25.3	0.8040	1.243
20.0	0.7148	1.398	25.4	0.8056	1.241
20.1	0.7166	1.395	25.5	0.8071	1.238
20.2	0.7184	1.391	25.6	0.8087	1.236
			25.7	0.8103	1.234
20.3	0.7201	1.388	25.8	0.8119	1.231
20.4	0.7219	1.384	25.9	0.8134	1.229
20.5	0.7237	1.381	26.0	0.8150	1.226
20.6	0.7254	1.378	26.1	0.8166	1.224
20.7	0.7272	1.375	26.2	0.8181	1.222
20.8	0.7289	1.371	26.3	0.8197	1.219
20.9	0.7307	1.368	26.4	0.8212	1.217
21.0	0.7324	1.365	26.5	0.8228	1.215
21.1	0.7342	1.361	26.6	0.8244	1.213
21.2	0.7359	1.358	26.7	0.8259	1.211
21.3	0.7377	1.355	26.8	0.8275	1.208
21.4	0.7394	1.352	26.9	0.8290	1.206
21.5	0.7411	1.349	27.0	0.8305	1.204
21.6	0.7428	1.346	27.1	0.8321	1.201
21.7	0.7446	1.343	27.2	0.8336	1.199
21.8	0.7463	1.339	27.3	0.8351	1.197
21.9	0.7480	1.336	27.4	0.8367	1.195
22.0	0.7497	1.333	27.5	0.8382	1.193

Length inches.	Time of vibration in seconds.	No. Vibr. per sec.	Length inches.	Time of vibration in seconds.	No. Vibr. per sec.
27.6	0.8397	1.191	33.6	0.9265	1.079
27.7	0.8412	1.189	33.7	0.9279	1.077
27.8	0.8427	1.186	33.8	2.9293	1.076
27.9	0.8443	1.184	33.9	0.9306	1.074
28.0	0.8458	1.182	34.0	0.9320	1.072
28.1	0.8473	1.180	34.1	0.9334	1.071
28.2	0.8488	1.178	34.2	0.9347	1.069
28.3	0.8503	1.176	34.3	0.9361	1.068
28.4	0.8518	1.173	34.4	0.9375	1.066
28.5	0.8533	1.171	34.5	0.9389	1.065
28.6	0.8548	1.169	34.6	0.9402	1.063
28.7	0.8563	1.167	34.7	0.9415	1.062
28.8	0.8578	1.165	34.8	0.9429	1.060
28.9	0.8593	1.163	34.9	0.9443	1.059
29.0	0.8607	1.161	35.0	0.9456	1.057
29.1	0.8622	1.159	35.1	0.9470	1.055
29.2	0.8637	1.157	35.2	0.9483	1.054
29.3	0.8652	1.155	35.3	0.9497	1.052
29.4	0.8667	1.154	35.4	0.9510	1.051
29.5	0.8682	1.152	35.5	0.9523	1.050
29.6	0.8696	1.150	35.6	0.9537	1.048
29.7	0.8711	1.148	35.7	0.9550	1.047
29.8	0.8726	1.146	35.8	0.9563	1.045
29.9	0.8741	1.144	35.9	0.9577	1.044
30.0	0.8755	1.142	36.0	0.9590	1.042
30.1	0.8769	1.140	36.1	0.9603	1.041
30.2	0.8784	1.138	36.2	0.9617	1.039
30.3	0.8798	1.136	36.3	0.9630	1.038
30.4	0.8813	1.135	36.4	0.9643	1.036
30.5	0.8827	1.133	36.5	0.9657	1.035
30.6	0.8842	1.131	36.6	0.9670	1.034
30.7	0.8856	1.129	36.7	0.9683	1.032
30.8	0.8870	1.127	36.8	0.9696	1.031
30.9	0.8885	1.125	36.9	0.9709	1.029
31.0	0.8899	1.123	37.0	0.9722	1.028
31.1	0.8914	1.121	37.1	0.9736	1.027
31.2	0.8928	1.120	37.2	0.9749	1.025
31.3	0.8942	1.118	37.3	0.9762	1.023
31.4	0.8956	1.116	37.4	0.9775	1.024
31.5	0.8971	1.114	37.5	0.9788	1.021
31.6	0.8985	1.112	37.6	0.9801	1.020
31.7	0.8999	1.111	37.7	0.9814	1.018
31.8	0.9013	1.109	37.8	0.9827	1.017
31.9	0.9027	1.107	37.9	0.9840	1.016
32.0	0.9042	1.105	38.0	0.9853	1.014
32.1	0.9056	1.104	38.1	0.9866	1.013
32.2	0.9070	1.102	38.2	0.9879	1.012
32.3	0.9084	1.100	31.3	0.9892	1.010
32.4	0.9098	1.099	38.4	0.9905	1.009
32.5	0.9112	1.097	38.5	0.9918	1.008
32.6	0.9126	1.095	38.6	0.9931	1.006
32.7	0.9140	1.094	38.7	0.9943	1.005
32.8	0.9154	1.092	38.8	0.9956	1.004
32.9	0.9168	1.090	38.9	0.9969	1.003
33.0	0.9182	1.089	39.0	0.9982	1.001
33.1	0.9196	1.087	39.1	0.9995	1.000
33.2	0.9210	1.085	39.2	1.0001	0.9993
33.3	0.9224	1.084			
33.4	0.9237	1.082			
33.5	0.9251	1.080			

TABLE X.

Specific Gravities.

METALS.	Specific Grav.	Weight cub. foot in avoird. lbs.	METALS.	Specific Grav.	Weight cub. foot in avoird. lbs.
Arsenic	5.763	360.14	Pure cast silver	10.474	654.62
Cast antimony	6.702	368.87	Same hammered	10.510	656.87
Cast zinc	7.190	449.37	Cast lead	11.352	709.50
Cast iron	7.207	450.44	Mercury	13.568	848.00
Cast tin	7.291	455.69	Trinket gold	15.709	981.81
Bar iron	7.788	487.75	Gold coin	17.647	1102.94
Cast nickel	7.807	487.94	Pure cast gold	19.258	1203.62
Cast cobalt	7.811	488.19	Same hammered	19.361	1210.06
Hard steel	7.816	488.50	Pure platinum	19.500	1218.75
Soft steel	7.833	499.56	Same hammered	20.336	1271.00
Cast brass	8.395	524.69	Platinum wire	21.041	1315.06
Cast copper	8.788	549.25	Platinum laminated, or beat into leaves.	22.069	1379.31
Cast bismuth	9.822	613.88			

All metals become more dense or heavy by hammering.

STONES, EARTHS, &c.

Ambergris	.926		Alabaster	2.730	
Amber	1.078		Marble	2.742	171.38
Phosphorus	1.714		Porphyry	2.765	172.81
Brick	2.000	125.00	Emerald	2.775	
Sulphur	2.033	127.06	Chrysolite, jewellers'	2.782	
Opal	2.114		Chalk	2.784	174.00
Gypsum, opaque	2.168	135.50	Jasper	2.816	
Stone, paving	2.416	151.00	Basaltes (Giant's Causeway)	2.864	179.00
Mill-stone	2.484	155.20			
Stone, common	2.520	157.50	Hone, white razor	2.876	179.75
Flint and spar	2.594	162.12	Limestone	3.179	198.68
Crystal	2.653		Flint glass	3.329	
Granite, red Egyptian	2.654	165.87	Diamond	3.521	
Glass, green	2.642		Beryl	3.549	
Glass, white	2.892		Sapphire	3.994	
Glass, bottle	2.733		Topaz	4.011	
Pebble	2.664	166.50	Garnet	4.189	
Slate	2.672	167.00	Ruby	4.283	
Pearl	2.684				

RESINS, GUMS, &c.	Specific Grav.	LIQUIDS.	Specific Grav.
Wax	.897	Oil of turpentine	.87
Tallow	.945	Muriatic ether	.87
Bees' wax	.965	Olive oil	.91
Camphor	.989	Burgundy wine	.99
Honey	1.456	Wine of Bourdeaux	.99
Bone of an ox	1.659	Port	.99
Ivory	1.822	Distilled water	1.00
		Sea water	1.02
		Milk	1.03
LIQUIDS.		Beer	1.03
		Madeira wine	1.03
Air, at the earth's surface, about	.001$\frac{2}{7}$	Nitric acid	1.27
Sulphuric ether	.716	Water from the Dead Sea	1.24
Alcohol absolute	.797	Nitrous acid	1.550
Liquid bitumen	.848	Sulphuric acid	1.896

WOODS.	Specific Grav.	Weight cub. foot in avoird. lbs.	WOODS.	Specific Grav.	Weight cub. foot in avoird. lbs.
.....	.240	15.00	Teak	.745	46.56
ır	.383	23.94	Maple and Riga fir ...	.750	46.87
ı	.544	34.00	Ash and Dantzic oak.	.760	47.50
and N. E. fir ...	.556	34.75	Yew, Dutch	.788	49.25
gany, Honduras.	.560	35.00	Apple tree	.793	49.56
.....	.579	36.18	Alder	.800	50.00
w	.585	36.56	Yew, Spanish	.807	50.44
.....	.596	37.25	Mahogany, Spanish...	.852	53.25
Pine	.660	41.25	Oak, Canadian	.872	54.50
tree	.661	41.31	Box, French	.912	57.00
ut	.671	41.94	Logwood	.913	57.06
Forest fir	.694	43.37	Oak, English	.970	60.62
tree	.695	43.44	Ditto, 60 years old ...	1.170	73.12
ı	.696	43.50	Ebony	1.331	83.18
ge wood	.705	44.06	Lignum vitæ	1.333	83.31
y tree	.715	44.68			

ıce a cubic foot of water at the temperature of 40° Fahrenheit, hs 1000 ounces avoirdupois, or 62½ lbs., the numbers in the ding Table under the head Specific Gravity, exhibit very nearly respective weights in avoirdupois ounces of a cubic foot of the al substances. We have also given in another column the ht in avoirdupois pounds of a *cubic foot* of each. These, of ıe, are *medium* specific gravities and weights; for there are tions, sometimes, indeed, considerable ones, between the specific ties of different specimens of the same kind of substance.

ıese additional columns will evidently facilitate the labour of ıg *the magnitude of a body from its weight*, or *the weight of a from its magnitude*. In this respect too the following partis will often be of utility.

(1) 430.25 cubic inches of cast iron weigh 1 cwt.
397.60 ..... bar iron
368.88 ..... cast brass
352.41 ..... cast copper
272.8 ..... cast lead

(2) 14.835 cubic feet of paving stone weigh a ton
14.222 ..... common stone
13.505 ..... granite
13.070 ..... marble
12.874 ..... chalk
11.273 ..... limestone
64.460 ..... elm
64.000 ..... Honduras mahogany
51.650 ..... Mar Forest fir
51.494 ..... beech
47.762 ..... Riga fir
47.158 ..... ash, and Dantzic oak
42.066 ..... Spanish mahogany
36.205 ..... English oak

TABLE XI.

The weight of Materials frequently employed in Construction.

MATERIAL.	1 cubic foot.	1 cubic yard.	1 super. ft., 1 in. thick.	A rod 1 in. sq., 1 ft. long.	A rod 1 in. dia., 1 ft. long.
	lbs.	lbs.	lbs.	lbs.	lbs.
Ash	47.5	...	3.96	0.330	.260
Beech	45.3	...	3.78	.315	.247
Brass	523.0	...	43.60	3.630	2.850
Brickwork, in mortar	117.0	3159	...	...	...
Brickwork, in cement	125.0	3375	...	...	...
Cast iron	450.0	...	38.40	3.200	2.510
Chalk	144.7	3897	...	...	...
Clay	125.0	3375	...	...	...
Copper	549.0	...	45.70	3.810	2.990
Earth, common	95.0 to 125.0	2565 to 3375	...	...	...
Elm	34.0	...	2.83	.236	.185
Fir, red or yellow	34.8	...	2.90	.242	.190
Fir, white	29.3	...	2.45	.204	.160
Granite, Aberdeen	164.0	4428	...	...	...
Gravel	120.0	3240	...	...	...
Iron, wrought	475.0	...	40.00	3.300	2.610
Lead	709.5	...	59.30	4.940	3.880
Mahogany	35.0	...	2.92	.243	.191
Marble	169.0	4563	14.00	1.170	.919
Oak, English	52.0	...	4.32	.360	.283
Pine, American	26.8	...	2.23	.186	.146
Slate	172.0	...	14.30	1.190	.935
Steel	490.0	...	40.80	3.400	2.670

XII.—PRINCIPLES OF CHRONOMETERS.

Clockwork regulated by a simple balance, as well as clockwork regulated by a pendulum vibrating in the arc of a circle, is of itself inadequate to the accurate mensuration of time. For two reasons; first, because the vibrations in greater or smaller arcs are not performed in equal times: and secondly, because the length of the pendulum is varied by heat and cold.

Clockwork, regulated by a pendulum vibrating in the arc of a cycloid, is also inadequate to the accurate mensuration of time.

The isochronism of the vibrations of a cycloidal pendulum in greater and smaller arcs is true only on the hypothesis that the pendulum moves in a non-resisting medium, and that the whole mass of the pendulum is concentrated in a point, both of which positions are false. For these reasons the application of the cycloid in practice has been entirely relinquished.

Modern time-keepers owe almost the whole of their superiority over those formerly made to two things; 1st, the application of a thermometer; 2ndly, the particular construction of the escapement.

Metals expand by heat and contract by cold. This is proved experimentally by the pyrometer. Metallic bars of the same kind

are found to expand in proportion to their length. Metals of different kinds expand in different proportions: thus the expansion of iron and steel are as 3, copper 4½, brass 5, tin 6, lead 7. Hence pendulum rods, expanding and contracting by the successive changes of temperature, affect the going of the clocks to which they are applied.

Various have been the contrivances to correct the errors of pendulums from their contraction and expansion by heat and cold; the principal of these are described under the subject pendulums, pages 228 and 229.

The balance of a watch is analogous to the pendulum in its properties and use.

The simple balance is a circular annulus, equally heavy in all its parts, and concentrical with the pivots of the axis on which it is mounted. This balance is moved by a spiral spring, called the balance-spring, the invention of the ingenious Mr. Hook.

The pendulum requires a less maintaining power than the balance. Hence the natural isochronism of the pendulum is less disturbed by the relatively small inequalities of the maintaining power.

The spring's elastic force, which impels the circumference of the balance, is directly as the tension of the spring; that is, the weights necessary to counterpoise a spiral spring's elastic force, when the balance is wound to different distances from the quiescent point, are in the direct ratio of the arcs through which it is wound.

The vibrations of a balance, whether through great or small arcs, are performed in the same time. For the accelerating force is directly as the distance from the point of quiescence; hence, therefore, the motion of the balance is analogous to that of the pendulum, vibrating in cycloidal arcs.

The time of the vibration of a balance is the same as if a quantity of matter, whose inertia is equal to that by which the mass contained in the balance opposes the communication of motion to the circumference, described a cycloid whose length is equal to the arc of vibration described by the circumference, the accelerating force being equal to that of the balance. Because in both cases the spaces described would be equal, as also the accelerating forces in corresponding points, and therefore the times of description.

If 1 denote the accelerating force of gravity, L the length of a pendulum vibrating seconds in a cycloid, a the semi-arc of vibration of the balance, T the time of vibration, and F the accelerating force of the balance, then will

$$T = \sqrt{\frac{a}{L \times F}}.$$

Let $\frac{1}{2}g$ be the space which a body falling freely from a state of rest describes in 1″, and $p = 3{\cdot}141593$ the circumference of a circle whose diameter is unity, then will

$$T = \sqrt{\frac{p^2\, a}{g\, F}}.$$

In this expression for the time of vibration, the letter a denotes the length of the semi-arc of vibration; if this arc should be expressed by a number of degrees, $c°$, and r be the radius of the ba-

lance, then a will $= \frac{p\,r\,c}{180}$; and, this quantity being substituted for a, the time of a vibration will be

$$T = \sqrt{\frac{p^3 c}{g\,F \times 180}};$$

if the given arc be 90°, we shall have

$$T = \sqrt{\frac{p\,c\,r}{2\,g\,F}}.$$

If the spring's elastic force, when wound through the given angle or arc $a = 90°$ from the quiescent position, be $= P$; the weight of the balance, and the parts which vibrate with it $= W$, the distance of the center of gyration from the axis of motion $= \varrho$, then will

$$T = \sqrt{\frac{W\,p^3\,\varrho^2}{2\,p\,r\,g}}.$$

These are expressions for the time of a vibration, whatever may be the figure of the balance, the other conditions remaining the same as above stated. If the balance be an annulus, or a cylindrical plate, $\varrho = \frac{r}{\sqrt{2}}$, and the time of vibration will be

$$T = \sqrt{\frac{W\,p^3\,r}{4\,P\,g}}.$$

The times of vibration of different balances are in a ratio compounded of the direct subduplicate ratios of their weights and semi-diameters, and the inverse subduplicate ratio of the tensions of the springs or of the weights which counterpoise them, when wound through a given angle.

The times of vibrations of different balances are in a ratio compounded of the direct simple ratio of the radii, and direct subduplicate ratio of their weights, and the inverse subduplicate ratio of their absolute forces of the springs at a given tension.

Hence the absolute force of the balance spring, the diameter and weight of the balance being the same, is inversely as the square of the time of one vibration.

The absolute force or strength of the balance spring, the time of one vibration, and the weight of the balance being the same, is as the square of the diameter and the balance.

The weight of the balance, the strength of the spring and time of vibration being the same, is inversely as the square of the diameter.

Hence a large balance vibrating in the same time, with the same spring, will be much lighter than a small one.

If the rim of the balance be always of the same breadth and thickness, so that the weight shall be as the radius, the strength of the spring must be as the cube of the diameter of the balance, that the time of vibration may continue the same.

If a balance be made with two balls joined by a rod, and the weights and distances of these balls from their common center of motion be unequal, but such that each separately would vibrate in the same time; the center of gravity of these balls will not coincide with their center of motion, nor will they poise each other.

The momentum of the balance is increased better by increasing its diameter than its weight.

A stronger balance spring is preferable to a weaker. Because the force of this spring upon the balance remaining the same, whilst the disturbing force varies, the errors arising from the variation will be less, as the fixed force is greater.

The longer a detached balance continues its motion the better. Because, 1st, The friction in this case is less, and therefore the natural isochronism of the vibration is less disturbed. 2ndly. When applied to the watch, it requires a less maintaining power, and therefore the variations in the intensity of the maintaining power will be less. 3rdly. The maintaining power being less, the friction of the wheel-work will be less, and therefore the motion more regular. 4thly. The pressure on the escapement will be less, and therefore the oscillations of the balance less disturbed.

The greater the number of vibrations performed by a balance in a given time is, the less susceptible is it of external agitations.

Slow vibrations are preferable to quick vibrations: but there is a limit; for, if the vibrations be too slow, the watch will be liable to stop.

If we regarded only the effect of external agitations, balances that vibrate quick should be preferred to such as vibrate slow; but they are attended with two inconveniences, greater than that which we would avoid. 1st. In two balances of the same weight and diameter, the friction on the pivots increases with the number of vibrations. 2ndly. It appears by experience that the motion of the same detached balance continues longer, when its vibrations are slow, than when they are quick.

A balance should describe as large arcs as possible, as suppose 240°, 260°, 300°, or an entire circle.

First, because the momentum of the balance is thus increased; and therefore the inequalities in the force of the maintaining power bear a less proportion to it, and of consequence will have less influence. 2ndly. The balance is less susceptible of external agitations. 3rdly. A given variation in the extent of the vibrations produces a less variation in the going of the machine. But care must be taken, that in these great vibrations, the spring shall neither touch any obstacle, nor its spires touch each other in contracting.

The times of vibration in larger arcs are sometimes shorter, sometimes longer, than in less arcs.

A uniform spiral spring may be rendered perfectly isochronal, by adjusting its length and number of spires.

This is the opinion of Mr. Berthoud. His reasoning seems to be this: if the spring forming a spiral of a certain species be so disposed, that when wound through different angles, the accelerating elastic forces of the spires, from the center towards the circumference, increase faster than they ought to do in order to render the vibrations isochronal, it may be otherwise so disposed, namely, by making the spires approach more nearly to equality with each other in succession, that the law shall vary in such a manner as absolute isochronism requires. But as the fundamental property of springs, namely, that *as the tension is, so is the force*, is determined by experiment, so must this property likewise be ascertained in the same manner. Accordingly, Berthoud tells us, that having attached to a balance a spiral of very large folds, making but three turns, and whose diameter was 15 lines, the angles through which it was wound being successively 5°, 10°, 15°, 20°, 25°, 30°, 35°, 40°, 45°, 60°, 120°, the

counterpoising weights in grains were 10½, 21, 32, 42, 54, 65, 76, 88, 99, 134, 278. The same spring forming very small spires, making five turns in eight lines diameter, the angles through which it was wound being the same as before, the counterpoising weights were 11, 22, 33, 45, 56, 67, 78, 89, 100, 133, 250 grains. These experiments, he tells us, were made with great care; and they show that the same spiral, its length continuing unchanged, when folded in large and small spires, has a sufficient difference in its progression to vary its isochronism: when folded in large spires, according to the first experiment, the vibrations in larger arcs are accelerated; and by the second experiment, when folded in narrow spires, they are rendered slower.

A spiral spring may be rendered isochronal by a proper adjustment of its strength and thickness in different parts.

A spiral spring which is not isochronal may be rendered such by the addition of two auxiliary springs, whose points of quiescence are properly adjusted.

This was the ingenious invention of Mr. Mudge; the theory of which construction is delivered in the Phil. Trans. for the year 1794, by Mr. Atwood.

The influence of the maintaining power on the balance, in restoring the motion which it loses by friction, or otherwise, may be either constant or interrupted.

This depends on the escapement; when the action of the maintaining power is constant, the escapement is called either the *recoil* or the *dead-beat;* when it is interrupted, the escapement is said to be *detached.*

By *escapement* is understood the means by which the action of the wheels is applied to maintain the vibration of the balance; and it consists of the balance-wheel and pallets.

Pallets are small plates or levers attached to the axis or verge of the balance, which receive the impulse of the balance-wheel produced by the maintaining power, and thus continually renew the motion which the balance loses by friction, or other resistance.

In a *recoil escapement*, when one tooth of the balance-wheel drops off the first pallet, the other acting tooth falls on the inclined plane of the other pallet, which meeting it obliquely, causes the balance-wheel to recoil, from which circumstance this escapement derives its name.

In the *dead-beat escapement*, when one tooth of the balance-wheel drops off the inclined plane of the first pallet, the other acting tooth immediately falls upon the convex surface of the other pallet, which surface being concentrical with the axis of the balance, the wheel continues at rest until, by the motion of the pallet or cylinder, the inclined plane of the tooth comes to act upon the face of this latter pallet or edge of the cylinder, which then, by its pressure on that edge, throws the cylinder round, and thus gives motion to the balance; then instantly entering the cavity of the cylinder, it falls upon the concave surface, and for the same reason as before continues at rest, until the balance-spring drives the cylinder round in a contrary direction to what it did before, so as that the inclined plane of the tooth may act on the second edge of the cylinder; which pressure throws the cylinder round in the contrary direction, and the tooth gets out of the cavity, and at that instant the subsequent tooth falls upon the convex surface, and so on. From the quiescence of the

balance-wheel during the interval of time that elapses between the falling of the acting tooth on the surface and its pressure on the edge of the cylinder, this escapement is called the *dead-beat*.

In the *detached escapement* the motion of the maintaining power is suspended during almost the whole time of vibration; just at the end of the return of the balance it unlocks the wheel-work, and a tooth of the balance-wheel, immediately acting on the pallet, restores the motion which the balance had lost; and having given its impulse, the wheel-work is instantly locked again, and the balance performs its vibration freely and disengaged from all other parts of the machine.

In the escapement of recoil, the vibrations are quicker than if the balance or pendulum vibrated freely. For the recoil shortens the ascending part of the vibration by contracting the extent of the arc; and the reaction of the wheel accelerates the descending part of the vibration.

In the dead-beat escapement, the vibrations are slower than when they are performed in a detached state. For the pressure of the tooth on the surface of the cylinder retards that part of the vibration which is performed while the cylinder, by the motion of the balance spring, revolves so far as to bring the tooth to the edge of the cylinder: and if the maintaining power be increased, the pressure of the tooth on the cylinder may become so great as entirely to stop the motion. When the tooth has communicated its impulse to the edge of the cylinder, it moves almost freely; and as the tooth does not yet press with its entire force on the next surface, the cylinder will indeed describe a larger arc, and therefore on that account the time may be shortened; but when it has consumed all the impulse of the wheel, it returns by the sole force of elasticity; now the pressure of the tooth causes a friction which diminishes the tendency to return to the point of rest, so that the balance performs its vibrations slower.

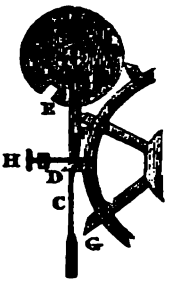
In the escapement of recoil, if the maintaining power be increased, the vibrations will be performed in larger arcs, but in less time. Because the greater pressure of the crown wheel on the pallet will cause the balance to vibrate through larger arcs; and the time, on this account, will be less increased, than it will be diminished by the acceleration of the balance by that pressure, and the diminution of the time of recoil.

In the escapement of the cylinder or dead-beat, an increase of the maintaining power renders the vibrations larger, and at the same time slower. Because the greater pressure of the tooth on the edge of the cylinder throws it round through a greater arc; and its increased pressure on both surfaces of the cylinder retards its motion.

The upper part of the first marginal diagram exhibits the anchor recoil; the lower, Graham's dead-beat escapement.

The second diagram represents Mr. Arnold's watch escapement. The pin A, projecting from the verge or axis of the balance, moving towards B, carries before it the spring B, and with it the stronger spring C, so as to set at liberty the tooth D, which rests on a pallet projecting from the spring. The angle E of the principal pallet has then just passed the tooth F, and is impelled by it until the tooth G arrives at the detent. In the return of the balance, the pin A passes easily by the detent, by

forcing back the spring B. The screw H serves to adjust the position of the detent, which presses against it.



The escapement can render those vibrations only isochronal, whose inequality proceeds from the maintaining power, and not such as are produced by external agitations.

The effect of external agitations on the balance may be counteracted by the double escapement.

In this escapement, two equal balances are so connected, that they vibrate through equal angles, but in contrary directions; by which means, the one must always be accelerated as much as the other is retarded by any external agitation. But, as Mr. Cummins observes, when balances are connected by means of teeth, there arises a resistance which, however small, when applied in this most delicate part, will tend to diminish the momentum of the balances.

That escapement is best in which the duration of the action of the balance-wheel on the pallets is least with respect to the time of vibration.

Hence the detached escapement is the best, which appears to have been the invention of an ingenious artist, Mr. Thomas Mudge, who made a watch on this construction for the late King of Spain, Ferdinand VI., in the year 1755.

The time of the vibration of the balance is increased by heat, and diminished by cold.

First, because the length of the spiral spring is increased by heat, and therefore its force diminished, and the contrary by cold. 2ndly. The diameter of the balance is increased by heat, and therefore also the time of vibration; and the contrary by cold.

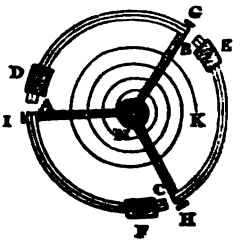
That balance is the most perfect which, without the compensation of a thermometer, is most subject to the influence of heat and cold.

Because the obstructions from oil and friction act as a compensation to the expansion or contraction of the spring and balance; therefore that balance which is most affected, is freest from the influence of oil and friction.

The errors in the going of a watch, arising from the change of temperature, may be corrected by varying the length of the balance spring.

Nevertheless, as it is extremely difficult to form an isochronal spiral, any variation in its length is dangerous, because we shall thus probably lose that point which determines its isochronism.

The errors in the going of a watch, occasioned by the variation of temperature, may be corrected by varying the diameter of the balance.



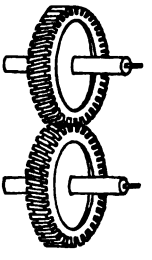
This may be effected by dividing the rim of the balance into two or more separate parts, G D, I F, H E, each of which is composed of two plates of metal of different expansibility, riveted together, the least expansible being nearest the center N, and carrying at one end D, F, E, a weight; whilst the other is connected either with the rim of the balance, or one of its radii. Now if the temperature increase, the exterior plate expanding more than the interior, the compound will become more concave towards the center; and conse-

itly the end which carries the weight will approach the center ie balance, and on that account the vibrations will be rendered ker. At the root of each thermometer, there is a screw G, I, H, hich the diameter of the balance may be increased or diminished, to alter the time kept by the chronometer, without interfering the adjustment for heat and cold; and if the magnitude and ion of the weights be properly regulated, they will correct the r arising from the variation of the diameter of the balance caused he variation of temperature. (*M. Young's Analysis.*) The er who wishes to acquire *practical* knowledge on this subject, advantageously consult *Hatton's Introduction to the Mechanical of Clock and Watch Work.*

XIII.—SELECT MECHANICAL EXPEDIENTS.

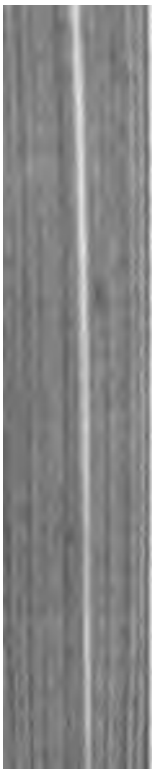
lthough a full account of the principal contrivances for trans-ing motion, and changing its rate, its direction, or its character, ld carry us much beyond the assigned limits; yet it seems ad-ble to give a few of these, which are, therefore, here presented.

. *Spiral Gear.* In the ordinary cases, the teeth of wheels are cut ss their circumferences in a direction parallel to the axis. But, in spiral gear, now used a little in this country, and more in the American states, (especially in cotton-s,) the teeth are cut obliquely, so that if they e continued they would pass round the axis like threads of a screw. By reason of this disposi-, the teeth come in contact only in the line of centers, and thus operate, in great measure, iout friction. It must, however, be remarked, the action of these wheels is compounded of two es, one of which acts in the direction of the plane he wheel, the other in the direction of its axis.



his spiral gearing is sometimes applied to clock-work, and has this uliarity, that it admits of a smaller pinion than any other gearing.

. *Change of rotatory velocity.* It is sometimes necessary that a hine should be propelled with a velocity which is not equable, continually changes in a given ratio. Thus, in cotton-mills, it is essary that the speed of certain parts of the machinery should tinually decrease from the beginning to the end of an operation. accomplish this, two conical drums of equal size are placed with r axes parallel to each other, and with their larger diameters in osite directions. They are connected by a belt, which is so re-ated by proper mechanism, that it is gradually moved from one remity of the conic frustums or drums to the other; and thus ng upon circles of different diameter, causes a continual change relocity in the driven cone with relation to that which drives it. Thus, if the drum on the axis *a b*, Fig. 245*, drives the wheel on axis A B, and the belt commences its operation at the ends *a* A, driven conic frustum will first revolve slower than that which es it, and will continue to move slower until the belt has reached middle of both, when the rotatory motions of both will be equal: r that, the cone which is driven will turn quickest, and will so tinue, turning quicker and quicker both with respect to the other, , in fact, until the belt reaches the ends *b*, B.



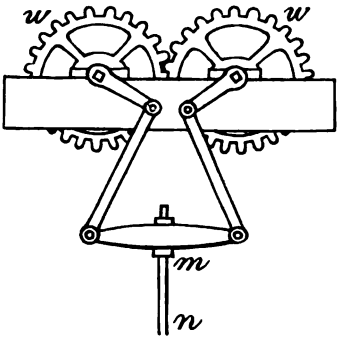
driven axis will revolve with a velocity which is
due to the relative size of the wheel which is
locked, and that which drives it. Suppose, for example
meters of the wheels are as 1, 2, 3, 4, and 5, upon the
and the corresponding wheels, upon the driven axis, are a
5, 4, 3, 2, 1. Then, when the wheel 1 drives the wh
tatory velocity of the latter will be one-fifth of that o
When 2 drives II, the rotatory velocity communicated
that of the wheel 2. When 3 drives III, the velo
equal. When 4 drives IV, the rotatory velocity of the
double that of the former. And, when 5 drives V, t
the latter will be five times that of the former. Diff
tions in the diameters of the wheels will, of course,
proportions in the velocities. We owe this beautiful c
the late Mr. Bramah.

It is sometimes requisite that a wheel or axis
with different velocities in different parts of one and
volution. This may be accomplished by an eccentric
acting upon and driving a long pinion. Thus, if the
in fig. 246 rotates uniformly upon a center of mot
is not the center of the wheel, and the teeth of this
play into the leaves of the long pinion *p q*, since the p
crown wheel pass in contact with the pinion with differ
as their distances from the center of motion c vary, t
will turn with an unequable or varying velocity, depend
eccentricity of the center of motion c.

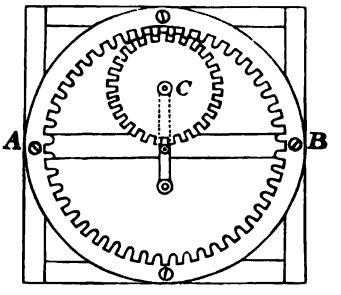
3. *Cams and Wipers.* These are contrivances b
which beams placed vertically, or inclined aslant upw
made to advance over a small space in the direction of
and then recede in the opposite direction; and so or
Eccentric circles, hearts, ellipses, portions of circles, a
epicycloids, serve to communicate these kinds of motion

In Fig. 250, a triple cam is applied to a tilt or trip hammer, turning upon a center: there are three epicycloidal cams, or *wipers* (as in this case they are often called,) causing three strokes of the hammer for one revolution of the wheel, to whose circumference these wipers are attached at equal distances.

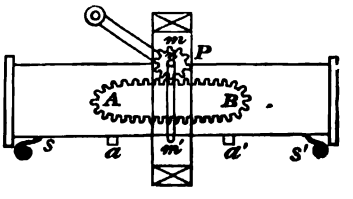
4. *Parallel Motions* is a term given to the contrivances by which, especially in steam engines, circular motion, whether continued or alternate, is converted into alternate rectilinear motion, and *vice versâ*. A moveable parallelogram is often, and very successfully, employed for this purpose; as shown in plates VII. and VIII. From among the numerous other contrivances for this purpose, we shall select only one, which is very simple and elegant, and may be used in saw mills, and other reciprocating machines, as well as in steam engines. This is the invention of Dr. Cartwright. The reciprocating motion of the piston rod or other rod *m n*, in the same rectilinear course, is insured by connecting it with two equal cranks arranged in opposition to each other, and having their axes geared together by two equal teethed wheels, *w w*, which play regularly into each other.



5. *Epicycloidal Wheel.* This is another very beautiful method of converting circular into alternate motion, or the contrary. A B is a fixed wheel, having teeth disposed uniformly on its inner rim. C is a toothed wheel of half the diameter of the fixed wheel, its center C revolving about the center of the said fixed wheel. While this revolution of the wheel C is going on, any point whatever on its circumference will describe a straight line; or will pass and repass through a diameter of the moving circle once during each revolution. This is an elegant application of the well-known mathematical property, that, if a circle rolls on the inside of another of twice its diameter, the epicycloid described is a right line. In practice, the piston rod, or other reciprocating part, may be attached to any point on the circumference of the wheel C.

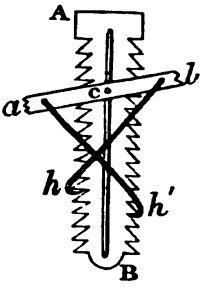


6. *Double Rack and Pinion.* This is a contrivance for an alternating motion with a gradual change. A B is a double rack, with circular ends, fixed to a beam that is capable of moving in the direction of its length. The rack is driven by a pinion P, which is susceptible of moving up and down in a groove *m m'*, cut in the cross piece. When the pinion



has moved the rack and beam until the end B is reached, the projecting *a* meets the spring *s*, and the rack is pressed against the pinion. Then the pinion working in the circular end of the rack, will be forced down the groove *m m'* until it works in the lower side of the rack, and moves the beam back in the opposite direction; and thus the motion is continued. The motion of the pinion in the groove will be diminished, if, instead of a double rack, there be used a single row of pins which are parallel to the axis of the pinion: this plan is sometimes adopted in the machines called *mangles*.

7. *The Universal Lever.* This is a French invention, and is often, from the name of the inventor, called *Lever de la Garousse.* It consists of a bar, *a b*, moving upon a center C, and having a moveable catch, or hook, *h h'* attached to each side, and acting upon the oblique teeth of a double rack; so that, as *b* and *a* alternately rise and fall in the reciprocal motion, the hooks *h* and *h'* successively lay hold of the teeth of the beam A B, and draw it up in the direction B A.



8. *The Tachometer.* This is a very ingenious contrivance, which we owe to Mr. Donkin, for the purpose of measuring small variations in velocity. The contrivance is, in fact, hydrodynamical, but we mention it here, and the simplicity of its principle will render it easy of comprehension.

If a cup with any fluid, as mercury, be placed upon a spindle, so that the brim of the cup shall revolve horizontally round its center, then the mercury in the cup will assume a concave form, that is, the mercury will rise on the sides of the cup, and be depressed in the middle; and the more rapid the rotation of the cup, the more will the surface of the mercury be depressed in the middle and rise at the sides; the figures being those of hollow paraboloids. Now, if the mouth of this cup be closed, and a tube inserted into it, terminated in the cup by a ball-shaped end, and half filled with some coloured liquid, as coloured spirits of wine; then it is evident that the more the surface of the mercury is depressed the more the fluid in the tube will fall, and *vice versâ*. Consequently, the velocity of rotation of the cup, and of the spindle to which it is attached, will be indicated by the height of the liquid in the tube; and, indeed, absolutely measured by it, when the apparatus has been subjected to the adequate preparatory experiments.

For a more minute description, see Trans. Soc. of Arts, vol. xxvi., or the article *Tachometer* in the PANTOLOGIA.

Two plates, exhibiting a great variety of contrivances for converting rotatory, reciprocating, and other motions, one into the other, and thus facilitating the construction of machinery, are given in my Treatise of Mechanics.

XIV. — OBSERVATIONS ON THE EFFECT OF OLD LONDON BRIDGE ON THE TIDES, ETC.

In the first volume of Dr. Hutton's Tracts, 8vo., there are inserted some curious papers drawn up by Mr. Robertson, of Christ's

Hospital, and others, on London Bridge, and on the probable consequences, in reference to the tides, of erecting a new bridge across the Thames, viz., Blackfriars. Such documents are not only interesting as matters of scientific history, but become valuable in process of time; as the comparison of facts with theoretic predictions is subservient to the correction of the theory itself. With a similar object in view, I here introduce abridged accounts of some valuable facts with regard to the motion and level of the tides in the Thames at London, collected in 1820 and 1821, when the project of a new bridge of five arches, instead of the old bridge, originally of 20 arches, and a very contracted waterway, came first under consideration.

LONDON BRIDGE.

Result of the levels of tides observed from 23rd September to 25th October, 1820, between the entrance of the London Docks and Westminster Bridge. Also the transverse sections of the river Thames at London Bridge, Southwark Bridge, Blackfriars Bridge, Waterloo Bridge, and Westminster Bridge, collected from four drawings of the above surveys, made by Mr. Francis Giles, under the direction of Mr. James Montague, pursuant to an order of the Select Committee of Bridge House Lands, of the 8th September, 1820.

London Docks to London Bridge.

The high water of spring tides at the entrance of the London Docks, averaged a level of 1·5 inch higher and 10 minutes earlier time, than at the lower side of London Bridge.

The low water of spring tides at the same place averaged a level of 3 inches lower, and 9 minutes earlier time, than at London Bridge.

The high water of neap tides at the London Docks averaged a level of one inch, and 8 minutes earlier time, than at London Bridge.

The low water of neap tides at the London Docks averaged a level of 2 inches lower, and 14 minutes earlier time, than at London Bridge.

London Bridge.

High water of the highest spring tides occurs at three or four o'clock.—High water of the lowest neap tides occurs at eight or nine o'clock.

Spring tides flow four or five hours, and ebb seven to eight and a half hours.—Neap tides flow five, to five and a half hours, and ebb six and a half to eight hours.

The high water of spring tides produced an average fall through London Bridge of 8 inches, but the greatest fall at high water was 1 foot 1 inch.—October 24th.

The low water of spring tides produced a fall of 4 feet 4 inches, but the greatest fall at low water was 5 feet 7 inches.—September 27th.

The high water of neap tides produced a fall of 5 inches.

The low water of neap tides produced a fall of 2 feet 1 inch, but the least fall at low water was 1 foot 1 inch.—October 16th.

The flood of spring tides of October 21st and 23rd, produced slack water through the bridge in about 40 minutes after low water below bridge, from which time a head gradually increased (below bridge) to

1 foot 10 inches at half flood, and then regularly decreased to about 8 inches at high water.—The first flow of these tides, nevertheless, began above bridge about 20 minutes after low water time below bridge, although the water was then about 2 feet 6 inches higher above than below bridge; the time of low water below bridge averaged 10 minutes earlier than above bridge.

The ebb of these tides produced slack water at the bridge about 30 minutes after high water, and then gradually sunk to their greatest fall at low water.—The time of high water of October 21st and 23rd, was the same below as above bridge; but the average time of high water spring tides is 9 minutes earlier below than above bridge.

The flood of neap tide, October 30th, produced slack water through the bridge, in about two hours after low water time below bridge, (when there was some land flood in the river,) from which time a head gradually increased (below bridge) to 1 foot 3 inches at two-thirds flood, and then regularly decreased to 4 inches at high water.—The first flow of this tide, nevertheless, began above bridge about 1 hour after low water time below bridge, although the water was then 1 foot higher above than below bridge; but the average time of low water below bridge is 32 minutes earlier than above bridge.

The ebb of this tide produced slack water at the bridge about 15 minutes after high water above bridge, and then gradually sunk to its greatest fall at low water.—The time of high water of October 30th was 15 minutes earlier below than above bridge, and the average time of high water neap tides is 15 minutes earlier below than above bridge.

London Bridge to Westminster Bridge.

The high water line from the upper side of London Bridge to Westminster Bridge is generally level, unless influenced by winds and land floods.

The time of high water is about 10 minutes earlier at London than Westminster bridge.

	in.	pts.			min.	
The mean low water line has a fall of	4	0	from Westminster to Waterloo Bridge,	time	7	later at Westminster than at Waterloo Bridge.
Ditto . .	4	3	from Waterloo to Blackfriars Bridge .	. .	6	Ditto.
Ditto . .	3	2	from Blackfriars to Southwark Bridge	. .	5	Ditto.
Ditto . .	0	5	from Southwark to London Bridge .	. .	4	Ditto.
Total foot	1	0	0		22	

Areas of the Transverse Sections of the River Thames at London Bridge.

		Feet.	Abate for Water-works. Feet.	Feet.
At an extraordinary high water or level of 2 feet above the average spring tide high water mark, at the Hermitage entrance to the London Docks, as settled by the Corporation of the Trinity House, August, 1800.		8,130	1,000	7,130
At the Trinity High Water mark, or	Datum	7,360	...	...
	Under the Datum. ft. in.			
At an average Spring Tide High Water	below Bridge, or Level of 0 6·5	7,122	850	6,272
At Ditto Spring Tide High Water	above Bridge, or Level of 1 2·0	6,837	810	6,027
At Ditto Neap Tide High Water	above Bridge, or Level of 4 3·0	5,293	600	4,693

	Under the Datum. ft.	in.	Feet.	Abate for Water-works. Feet.	Feet.
At an average Spring and Neap Tide Low Water above Bridge, or Level of	14	5·0	1,488	...	1,488
At Ditto Neap Tide Low Water below Bridge, or Level of	16	5·0	1,030	...	1,030
At Ditto Spring Tide Low Water below Bridge, or Level of	18	9·0	540	...	540

At Southwark, Blackfriars, Waterloo, and Westminster Bridges.

	Southwark Bridge.	Blackfriars Bridge.	Waterloo Bridge.	Westminster Bridge.
At the above level of extraordinary high water	15,260	15,460	19,822	16,750
At the Trinity high water mark, or datum	13,940	14,117	17,707	15,198
At an average Spring Tide high water, or level of 1 foot 2 inches under the ditto	13,170	12,975	16,447	14,015
At ditto Neap tide high water, or level of 4 feet 3 inches under the ditto	11,135	10,590	13,116	11,380
At ditto Spring and Neap tide low water	5,012	3,724	3,382	3,720

London, 12th March, 1821.—(Published in a letter addressed to G. H. Sumner, Esq., M.P., by a scientific Architect.)

Gradation of the Ebbing and Flowing of the Tide at London Bridge, taken above and below, on the 29th of July, 1821; being the day of the new moon.

ABOVE BRIDGE.

Low Water at Pepper Alley, 50 minutes past nine o'clock in the morning.

FLOOD TIDE.	Ft.	In.
Depth of water when flood commenced	6	0
1st hour rose	2	11
2nd hour „	3	0
3rd hour „	2	10
4th hour „	2	8
45 minutes „	1	0
4 hours and 45 minutes ...	18	5

High Water at Pepper Alley, 35 minutes past two o'clock in the afternoon.

EBB TIDE.	Ft.	In.
1st hour fell	2	1
2nd hour „	2	7
3rd hour „	2	0
4th hour „	1	9
5th hour „	1	5
6th hour „	1	2
7th hour „	1	0
55 minutes „	0	11
Depth at low water	5	6
7 hours and 55 minutes ...	18	5

BELOW BRIDGE.

Low Water at Coxes' Quay, 30 minutes past nine o'clock in the morning.

FLOOD TIDE.	Ft.	In.
Depth of water when flood commenced	1	3
1st hour rose	5	9
2nd hour „	5	4
3rd hour „	2	9
4th hour „	2	5
48 minutes „	1	4
4 hours and 48 minutes ...	18	10

High Water at Coxes' Quay, 18 minutes past two o'clock in the afternoon.

EBB TIDE.	Ft.	In.
1st hour fell	2	1
2nd hour „	4	4
3rd hour „	3	1
4th hour „	2	7
5th hour „	2	3
6th hour „	1	9
7th hour „	1	6
59 minutes „	0	11
Depth left	0	4
7 hours and 59 minutes ...	18	10

*** The object of this statement was to show that the old bridge tended to retain the water above bridge and assist the navigation up the river.

Difference between the Levels of High and Low water Spring Tides, between Rotherhithe and Battersea, in the year 1820.

	ft.	in.
Rotherhithe, Old Horse Ferry	21	10
London, Old Bridge	18	2
Blackfriars	14	9
Westminster	12	6
Vauxhall	12	2
Battersea	11	6

From Battersea Bridge to London Bridge, 5 miles: from London Bridge to Old Horse Ferry, 1¼ miles. From London Bridge to the Nore, 44 miles.

These observations, taken previous to the removal of Old London Bridge, are rendered more interesting and important by those which have been subsequently made by Mr. Thomas Page, Civil Engineer, some of which, contained in two letters addressed by him to Lord Lincoln upon the subject of the River Thames, have been published in the First Report of the Commissioners appointed by her Majesty to inquire into the means of improving the Metropolis. The results deduced from these observations are of the greater importance, since they appear to have been quite unnoticed by previous writers upon the Thames, although involving principles of the greatest practical moment in any measures for the improvement of that river, and the want of attention to which, is in many parts lamentably attested by its present condition.

XV.—PROFESSOR FARISH ON ISOMETRICAL PERSPECTIVE.

In the course of lectures which I deliver in the University of Cambridge, I exhibit models of almost all the more important machines which are in use in the manufactures of Britain.

The number of these is so large, that had each of them been permanent and separate, on a scale requisite to make them work, and to explain them to my audience, I should, independently of other objections, have found it difficult to have procured a warehouse large enough to contain them. I procured, therefore, an apparatus consisting of what may be called a system of the first principles of machinery; that is, the separate parts, of which machines consist. These are made chiefly of metal, so strong, that they may be sufficient to perform even heavy work: and so adapted to each other, that they may be put together at pleasure, in every form, which the particular occasion requires.

Those parts are various: such as loose brass wheels, the teeth of which all fit into one another: axes, of various lengths, on any part of which the wheel required may be fixed: bars, clamps, and frames; and whatever else might be necessary to build up the particular machines which are wanted for one lecture. These models may be taken down, and the parts built up again in a different form for the lecture of the following day. As these machines, thus constructed for a temporary purpose, have no permanent existence in themselves, it became necessary to make an accurate representation of them on paper, by which my assistants might know how to put them together without the necessity of my continual superintendence. This might

ave been done by giving three orthographic plans of each; one on he horizontal plane, and two on vertical planes at right angles to ach other. But such a method, though in some degree in use mong artists, would be liable to great objections. It would be unntelligible to an inexperienced eye; and even to an artist, it shows ut very imperfectly that which is most essential, the connection of he different parts of the engine with one another; though it has the dvantage of exhibiting the lines parallel to the planes, on which the rthographic projections are taken on a perfect scale.

This will be easily understood by supposing a cube to be the object epresented. The ground plan would be a square representing both he upper and lower surfaces. And the two elevations would also e squares on two vertical planes, parallel to the other sides of the ube. The artist would have exhibited to him three squares; and e would have to discover how to put them together in the form of cube, from the circumstance of there being two elevations and a round plan. This method, therefore, giving so little assistance on so ssential a point, I thought unsatisfactory.

The taking a picture on the principles of common perspective, was he next expedient that suggested itself. And this might be adapted o the exhibition of a model by taking a kind of bird's-eye view of he object, and having the plane of the picture, not as is most comon in a drawing, perpendicular to the horizon, but to a line, drawn rom the eye, to some principal part of the object. For example: in aking the picture of a cube, the eye might be placed in a distant oint on the line which is formed by producing the diagonal of the ube. But to this common perspective there are great objections. he lines, which in the cube itself are all equal, in the representation re unequal. So that it exhibits nothing like a scale. And to comute the proportions of the original from the representation, would e exceedingly difficult, and, for any useful purpose, impracticable: here is equal difficulty too, in computing the angles which represent he right angles of the cube. Neither does the representation appear orrect, unless the eye of the person who looks at it be placed exctly in the point of sight. It is true, that as we are continually in he habit of looking at such perspective drawings, we get the habit f correcting, or rather overlooking the apparent errors which arise om the eye being out of the point of sight, and are therefore not ruck with the appearance of incorrectness, which, if we were unxcustomed to it, we should feel at once.

The kind of perspective which is the subject of this paper, though ible, in a slight degree, to the last mentioned inconvenience, till the e becomes used to it, I found much better adapted to the exhibition machinery; I therefore determined to adopt it, and set myself to vestigate its principles, and to consider how it might most easily be ought into practice.

It is preferable to the common perspective on many accounts, for ch purposes. It is much easier and simpler in its principles. It also, by the help of a common drawing table, and two rulers*, in-

* It is unnecessary to describe the drawing table any further than by obrving that it ought to be so contrived, as to keep the paper steady on which e drawing is to be made.

There should be a ruler in the form of the letter T to slide on one side of e drawing table. The ruler should be kept, by small prominences on the der side, from being in immediate contact with the paper, to prevent its otting the fresh drawn lines as it slides over them. And a second ruler, by

comparably more easy, and, consequently, more accurate in its application; insomuch, that there is no difficulty in giving an almost perfectly correct representation of any object adapted to this perspective, to which the artist has access, if he has a very simple knowledge of its principles, and a little practice.

It further represents the straight lines which lie in the three principal directions, all on the same scale. The right angles contained by such lines are always represented either by angles of 60 degrees, or the supplement of 60 degrees. And this, though it might look like an objection, will appear to be none on the first sight of a drawing on these principles, by any person who has ever looked at a picture. For he cannot for a moment have a doubt, that the angle represented is a right angle, on inspection.

And we may observe further, that an angle of 60 degrees is the easiest to draw of any angle in nature. It may be instantly found by any person who has a pair of compasses, and understands the first proposition of Euclid. The representation, also, of circles and wheels, and of the manner in which they act on one another, is very simple and intelligible. The principles of this perspective, which, from the peculiar circumstance of its exhibiting the lines in the three principal dimensions on the same scale, I denominate "*Isometrical*," will be understood from the following detail:

Suppose a cube to be the object to be represented. The eye placed in the diagonal of the cube produced. The paper on which the drawing is to be made, to be perpendicular to that diagonal, between the eye and the object, at a due proportional distance from each, according to the scale required. Let the distance of the eye, and consequently that of the paper, be indefinitely increased, so that the size of the object may be inconsiderable in respect of it.

It is manifest, that all the lines drawn from any points of the object to the eye, may be considered as perpendicular to the picture, which

means of a groove near one end on its under side, should be made to slide on the first. The groove should be wider than the breadth of the first ruler, and so fitted, that the second may at pleasure be put into either of the two positions represented in Plate XIII. Fig. 1, so as to contain with the former ruler, in either position, an angle of 60 degrees. The groove should be of such a size, that when its shoulders *a* and *d* are in contact with, and rest against the edges of the first ruler, the edge of the second ruler should coincide with *d e*, the side of an equilateral triangle described on *d g*, a portion of the edge of the first ruler; and when the shoulders *b* and *c* rest against the edges of the first ruler, the edge of the second should lie along *g e*, the other side of the equilateral triangle. The second ruler should have a little foot at *k* for the same purpose as the prominences on the first ruler, and both of them should have their edges divided into inches, and tenths, or eighths of inches.

It would be convenient if the second ruler had also another groove *r s*, so formed that when the shoulders *r* and *s* are in contact with the edges of the first ruler, the second should be at right angles to it. For representing circles in their proper positions, the writer made use of the inner edge of rims cut out from cards, into isometrical ellipses as represented in the figure; of these he had a series of different sizes, corresponding to his wheels. Such a series might be cut by help of the concentric ellipses in Fig. 5; but he thinks that it would be an easier way to make use of that set of concentric ellipses as they stand, by putting them in the proper place under the picture, if the paper on which the drawing is made be thin enough for the lines to be traced through, as by the help of them the several concentric circles will go to the representation of one which might be drawn at once. It is difficult to execute them separately with sufficient accuracy to make them correspond. For this purpose a separate plate of Fig. 5 should be had, and one edge of the paper on the drawing table should be loose, to admit of the concentric ellipses being slid under it to the proper place, as described at page 101.

becomes, therefore, a species of orthographic projection. It is manifest, the projection will have for its outline an equiangular and equilateral hexagon, with two vertical sides, and an angle at the top and bottom. The other three lines will be radii drawn from the center to the lowest angle, and to the two alternate angles; and all these lines and sides will be equal to each other both in the object and representation: and if any other lines parallel to any of the three radii should exist in the object, and be represented in the picture, their representations will bear to one another, and to the rest of the sides of the cube, the same proportion which the lines represented bear to one another in the object.

If any one of them, therefore, be so taken as to bear any required proportion to *its* object, *e. g.* 1 to 8, as in my representations of my models, the others also will bear the same proportion to *their* objects; that is, the lines parallel to the three radii will be reduced to a scale.

I omit the demonstration of this and some other points, partly for the sake of brevity, and partly because a geometrician will find no difficulty in demonstrating them himself, from the nature of orthographic projection; and a person, who is not a geometrician, would have no interest in reading a demonstration.

For the same reason, it is unnecessary to show that the three angles at the center are equal to one another, and each equal to 120 degrees, twice the angle of an equilateral triangle; and the angle contained between any radius and side is 60 degrees, the supplement of the above, and equal to the angle of an equilateral triangle. All this follows immediately from Euclid, B. IV. Prop. 15, on the inscription of a hexagon in a circle.

In models and machines, most of the lines are actually in the three directions parallel to the sides of a cube, properly placed on the object. And the eye of the artist should be supposed to be placed at an indefinite distance, as before explained, in a diagonal of the cube produced.

Definitions.

The last-mentioned line may be called the *line of sight.*

Let a certain point be assumed in the object, as for example C, Fig. 2, Plate XIII., and be represented in the picture, to be called the *regulating point.* Through that point on the picture may be drawn a vertical line, C E, Fig. 2, and two others, C B, C G, containing with it, and with one another, angles of 120°, to be called the *isometrical lines*, to be distinguished from one another by the names of the *vertical*, the *dexter*, and the *sinister* lines. And the two latter may be called by a common name—the *horizontal isometrical* lines. Any other lines, parallel to them, may be called respectively by the same names. The plane passing through the dexter and vertical lines may be called *the dexter isometrical plane;* that passing through the vertical and sinister lines, the *sinister plane;* and that through the dexter and sinister lines, the *horizontal plane.*

By the use of the simple apparatus described above in the note, the representation of these lines in the objects may be drawn on the picture, and measured to a scale, with the utmost facility, the point at the extremity being first found, or assumed. The position of any point in the picture may be easily found, by measuring its three distances, namely, first its perpendicular distance from the *regulating*

horizontal plane (that is, the horizontal plane passing through the regulating point); secondly, the perpendicular distance of that point where the perpendicular meets the horizontal plane, from the regulating dexter line; and, thirdly, of the point where that perpendicular meets the dexter line from the regulating point; and then taking those distances reduced to the scale, first, along the dexter line; secondly, along the sinister line; and, thirdly, along the vertical line in the picture. These three may be called the *dexter distance* of the point, its *sinister distance*, and its *altitude*. And it is manifest they need not be taken in this order, but in any other that may be more convenient to the artist, there being six ways in which this operation may be varied.

If any point in the same isometrical plane with the point required to be found, is already represented in the picture, that point may be assumed as a new regulating point, and the point required found by taking two distances; and if the new assumed regulating point is in the same isometrical line with the point, it is found by taking only one distance. And this last simple operation will be found in practice all that is necessary for the determination of most of the points required. Thus any parallelopiped, or any frame work, or other object with rafters, or lines lying in the isometrical directions, may be most easily and accurately exhibited on any scale required. But if it be necessary to represent lines in other directions, they will not be on the same scale, but may be exhibited, if straight lines, by finding the extremities as above, and drawing the line from one to the other; or sometimes more readily in practice by help of an ellipse, as hereafter described.

If a curved line be required, several points may be found sufficient to guide the artist to that degree of exactness which is required.

The method of exhibiting the representations of any machines, or objects, the lines of which lie, as they generally do, in the isometrical directions, that is, parallel to the three directions of the lines of the cube, is as has been already shown; and likewise the mode of representing any other straight lines, by finding their extremities; or curved lines, by finding a number of points.

But in representing machines and models, there are not only isometrical lines, but also many wheels working into each other, to be represented. These, for the most part, lie in the isometrical planes; and it is fortunate that the picture of a circle in any one of these planes is always an ellipse of the same form, whether the plane be horizontal, dexter, or sinister; yet they are easily distinguished from each other by the position in which they are placed on their axle, which is an isometrical line, always coinciding with the minor axis of the ellipse.

This will be obvious from considering the picture of a cube with a circle inscribed in each of its planes, Fig. 3, and considering these circles as wheels on an axle. The two other lines (or spokes of the wheel) in the ellipse, which are drawn respectively through the opposite points of contact of the circle with the circumscribing figure, are isometrical lines also; for the points of contact bisect the sides of the circumscribing parallelogram, and therefore the lines are parallel to the other sides. They give likewise the true diameter of the wheels, reduced to the scale required. It further appears, from the nature of orthographic projection, that the major axis of the ellipse is to the minor axis as the longer to the shorter diagonal of the circumscribing parallelogram, that is (since the shorter diagonal divides it into two

equilateral triangles,) as the square root of three to one; as appears from Euclid, B. I., Prop. 47: and since the sum of the squares of the conjugate diameters in an ellipse is always the same, if we put $\sqrt{1}$ for the minor axis, $\sqrt{3}$ for the major, and i for the isometrical diameter, we shall have $2i^2 = 1 + 3, = 4$, and $i = \sqrt{2}$.

Therefore the minor axis, the isometrical diameter, and the major axis, may be represented respectively by $\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}$, or nearly by 1, 1·4142, 1·7321; or more simply, though not so nearly, by 28, 40, 49.

These lines may be geometrically exhibited by the following construction:—

Let A B, Fig. 4, be equal to B D, and the angle at B, a right angle. In B A produced, take B α = to A D, draw α D, and produce both it and α B. Then will B D, B α, and α D, be respectively to one another, as $\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}$, by Euclid I. 47. Therefore if $\alpha\beta$ be taken equal to the isometrical diameter of the ellipse required, $\beta\delta$ drawn perpendicular to it will be the minor axis, and $\alpha\delta$ the major axis. The ellipse itself, therefore, may be drawn by an elliptic compass, as that instrument may be properly set if the major and minor axes are known. If it is to represent a wheel on an axle, care must be taken to make the minor axis lie along that axle. In the absence of the instrument it may be drawn from the concentric ellipses, Fig. 5, which may be placed under the paper, in the position above described, and seen through it, if the paper be not too thick; and in this method the smaller concentric circles of the wheel may be described at the same time, as they may be seen through the paper; or if they should not be exactly of the right size, it would be easy to describe them by hand between the two nearest concentric ellipses; and thus also the height of the cogs of a wheel in the different parts of it may be exhibited longer and narrower towards the extremities of the minor axis. Their width may be determined from the divisions of the ellipse. In most cases this may be done with sufficient accuracy from the circumference of the ellipse being divided into eight equal divisions of the circle, by the two axes, and two isometrical diameters, each of which parts may be subdivided by the skill of the artist; and not only the face of the wheel in front may be thus exhibited, but the parts of the back circles also, which are in sight, may be exhibited by pushing back the system of concentric ellipses on the minor axis or axle through a distance representing the breadth of the wheel, and then tracing both the exterior and the interior circles of the wheel, and of the bush on which it is fixed, as far as they are visible. Care should be taken to represent the top of the teeth, or cogs, by isometrical lines, parallel to the axle, in a face-wheel, or tending to a proper point in the axle in a bevil-wheel. And nearly in the same way may the floats of a water-wheel be correctly represented. If a series of concentric ellipses, such as are given in Fig. 5, be not at hand, it will still be easy for an artist to draw the ellipses with sufficient accuracy for most purposes, by drawing through the proper point in the axle the major and minor axes, and the two isometrical diameters, thus making eight points in the circumference to guide him.

If in any case it should become necessary to represent a circle, which does not lie in an isometrical plane, we may observe that the major axis will be the same in whatever plane it lies: and it will be the picture of that diameter, which is the intersection of the circle with the plane parallel to the picture, passing through its center.

And the major axis will bear to the minor axis the proportion of radius to the sine of the inclination of the line of sight to the plane of the circle. We may observe further, that the diameters of the ellipse, which are to the major axis as $\sqrt{2}$ to $\sqrt{3}$, when such exist, are isometrical lines *.

And the representation of every other line, parallel and equal to any diameter of the circle, may be exhibited by drawing it equal and parallel to the corresponding diameter in the ellipse. If it should be desired to divide the circumference of an ellipse into degrees, or any number of parts representing given divisions of the circle, it may be done by the following method:—

Let an ellipse be drawn, Fig. 6, and on its major axis, A G, a circle described, with its circumference divided into degrees or parts in any desired proportion, at B, C, D, E, F, &c., from which points draw perpendiculars to the major axis. They will cut the periphery of the ellipse in corresponding points. It would be difficult, however, in this way, to mark, with sufficient accuracy, the degrees which lie near the extremities of the major axis. But the defect may be supplied by transferring those degrees in a similar way, from a graduated circle, described on the minor axis. In this manner, an isometrical ellipse may be formed into an isometrical circular instrument, or an isometrical compass, which may show bearings or measure angles on the picture, in the same manner as a real compass or circular instrument would do in nature.

It may be often useful to have a scale to measure distances, not only in the isometrical directions, but in others also. And this may be done by a series of similar concentric ellipses, as in Fig. 7, dividing the isometrical diameters into equal portions. The other diameters will be so divided as to serve for a scale for all lines parallel to them respectively.

Thus, in the isometrical squares, exhibited in Fig. 2, distances measured on the longer diagonal, or its parallels, would be measured by the divisions on the major axis, those depending on the shorter diagonal by the divisions on the minor axis.

To describe a cylinder lying in an isometrical direction, the circles at its extremities should be represented by the proper isometrical ellipses, and two lines touching both should be drawn: and in a similar way, a cone, or frustum of a cone, may be described. A globe is represented by a circle, whose radius is the semi-major axis of the ellipse representing a great circle.

It would not be difficult to devise rules for the representation of many other forms which might occur in objects to be represented. But the above cases are sufficient to include almost every thing which

* We may remark, that if a cone be described, having its vertex at C which lies in the line of sight, Fig. 2, and passing through the three radii C B, C E, C G, all the straight lines in the superficies of that cone passing through C, and all other lines parallel to any of them, are isometrical, as well as those parallel to the three principal isometrical lines, C B, C E, C G; and no other lines but these can be on the same scale. But though these multiply the number of isometrical lines infinitely, it is of little practical use, because it is only those which are parallel to the three principal lines, that can be easily distinguished at sight, to be isometrical.

We may further remark, that if a line be drawn through the point C parallel to any given line whatever, and that line be made to revolve round the line of sight, at the same angular distance from it, so as to describe the surface of a cone, all other lines parallel to it, in any of its positions, will be isometrical as they respect one another.

occurs in the representation of models, of machines, of philosophical instruments, and indeed of almost any regular production of art.

Buildings may be exhibited by this perspective as correctly, in point of measurement, as by plans and elevations, under the advantage of having the full effect of a picture.

A bridge, or any circular or gothic arch, consisting of portions of circles lying in isometrical planes, may be represented by portions of isometrical ellipses, which will easily be adapted and drawn upon the principles already explained, by which wheels are exhibited on their axles. The centers of those circles must be found with which the centers of the ellipses must be made to coincide, their minor axes lying along the lines drawn from those centers perpendicular to the planes of the circles. The shaft of a pillar consists of a frustum of a cone and a cylinder united; or perhaps of a cylinder alone, or a congeries of cylinders; and we have already shown the method of exhibiting these, as well as their bases. And on the same principles, the position and size of the volutes and ornaments of the capital may be found, and such guiding points as will make it easy to trace their forms. Thus the different courts and edifices of a cathedral, a college, or a palace, may be correctly depicted; and even the rooms and internal structure, though less in the form of a picture, may be exhibited in such a way as to enable an architect, or his employer, to contemplate their situation, their ornaments, furniture, or any other circumstance belonging to their appearance, and to mark down exactly what he would have done, in such a way as could hardly be misunderstood by an attentive agent, though at a distance.

But in thus exhibiting buildings as transparent, and their interior laid open, there is a danger of being confused by a multiplicity of lines, which is a difficulty in a building containing many rooms, that would need some address to get over. It is better adapted to exhibit the inside of a single room; of a cathedral, for instance, the aisles and transepts of which would not cause any great perplexity.

In the same manner, a plan of a city might be given, which would not only represent its streets and squares, as well (by the help of the scale above described, Fig. 7,) as a common plan, but also a picture of its churches and public buildings, and even its private houses, if such were the design contemplated by the artist, as they would almost all become visible when looked down upon from the commanding height which this perspective supposes. And such a single exhibition, if well executed, might give a better idea of a distant capital than a volume of description.

In the instances which have been given, most of the lines are isometrical. But the art is applicable to many cases, where there are few or none such. It may be necessary, in many of them, to draw isometrical lines, or isometrical ellipses, by way of a guide, to determine the position of certain lines and points, to enable the artist to describe with accuracy what he has in view. And there is scarce any form so anomalous as to preclude the artist from taking advantage of these methods of ascertaining such lines or points in it as will give him much assistance in representing it with precision. If the intention be merely to make a picture, the guiding lines may be obliterated as soon as they have served the purpose designed, or they may be retained in some cases, and their lengths or diameters noted down in figures, if it be wished, to give ready information. And often, if the artist wishes to provide materials to enable him at his leisure to give accurate descriptions or exact drawings, the

rudest exhibition of such lines may completely serve his purpose, provided he notes down on the spot such measurements with accuracy, however unexact the lines may be on which they are recorded. In many cases, it may be expedient to take liberties with this perspective, or with the picture, which will make it suit the purpose designed. And this will produce no confusion, provided those liberties are explained: for instance, it may often be expedient to make the scale in the vertical direction larger, sometimes very considerably so, than in the horizontal. It may in some cases be necessary to represent on paper what is hid in nature. What has been said on the internal structure of buildings is an instance of this, as well as what we shall observe on the exhibition of subterraneous objects. We shall proceed to give some examples of these observations.

To give such a representation of an Etruscan vase, as would enable an artist to model it exactly, would be exceedingly easy. Let a vertical line be drawn to represent the axis of the vase, Fig. 8, and let points be taken in that axis, corresponding to the centers of the principal circles of the vase; through which the horizontal isometrical lines may be drawn representing the radii of those circles, by the help of which the isometrical ellipses representing them are easily drawn. These will become a complete guide to the artist. He may assist himself by looking at the object along the line of sight, and then, if he has any skill in drawing, he will find no difficulty in tracing the outline from one of these to the other, with sufficient correctness. If he is unskilled in the art, of course he must be at the trouble of finding a larger number of ellipses to guide him. And in a similar manner, any solid formed by the revolution of a plane figure round one of its sides may be represented.

The laying down the timbers of a ship, or making a picture of one, shall be another example.

Let a vertical isometrical plane be conceived to pass through its keel, and to be intersected by the perpendicular planes passing through the ribs, and by planes parallel to the decks. The isometrical lines, which are the intersections of these, may be measured in the ship, and represented with their proper measures noted down in the picture, which will afford the means of representing the ribs, and laying them down in their proper places.

If this should be designed for the purpose of constructing a ship from a given model, it might be sufficient to represent the ribs only on one side; those on the other side being the exact counterparts. If the purpose should be to make use of these lines for a drawing, they need be marked but very faintly, and the artist will have little difficulty when guided by them to fill up the representation by hand.

A regular fortification, which we will suppose to have eight bastions, will afford another example.

A person not conversant in such a subject, is in general puzzled with plans and sections, and has very little idea of what is meant to be conveyed.

But he would easily understand it if he should see every thing exhibited in a correct picture, especially where he has the view of his object varied, as in a fortification, such as has been proposed. Let an isometrical ellipse be drawn expressing the internal circumference of the place; and another concentric one, which marks the salient angles of the fortification on the principles already explained. Draw other guiding lines to every necessary point; the lines of the fortifi-

cation may be easily transferred from a common plan to the isometrical by the help of the scale of concentric ellipses, described above, Fig. 7, which will serve also to lay down the length of the bastions and curtains, &c., in whatever direction they lie. Find the elevations of every part on the isometrical scale; and thus the body of the place, the ditches, counterscarp, covered way, glacis, ravelins, and all the outworks, will be represented to the eye as they appear in reality, and in every varied position, with the advantage of having all the admeasurements laid down with geometrical precision.

If the artist should think the vertical lines in such an exhibition too small to give a correct idea of all the minute elevations, there would be no harm in his increasing the scale in that dimension in any desired proportions.

The face of a hilly or mountainous country, like Switzerland, or the district of the lakes in the north part of England, will afford another example.

Isometrical horizontal lines may be drawn representing lines in the level from which the height of the mountains is to be reckoned, so that vertical lines drawn from the summits of the mountains may meet them, on which the heights may be marked (as well as recorded in figures, if required). And the mountains themselves may be drawn in their topographical situation. Their bearings may be marked by the help of the isometrical compass described in page 102. It would be easier to transfer them from a common map to the isometrical plan; and thus the face of the country might be represented just as it would appear from the commanding height which the isometrical perspective supposes.

Yet, as the slopes of hills and mountains are seldom so steep as the line of sight, it might sometimes suit the purpose to represent the height of elevations as twice or three times the reality, in order that mountains might project an outline on the plane behind; otherwise, the summit might be projected on the mountain itself, which would, in a degree, destroy the effect of a picture.

This art might be advantageously employed also for tracing what is below the surface of the earth, as well as what is above it. It may be applied to geological purposes, and give not only the order of the strata, but their variations and their geographical situations. And for this purpose it might be useful to increase the vertical scale, in a great proportion, above the horizontal. It would be easy to mark the dip or rise of the strata, as well as of the earth above them: to represent their various disruptions, to show the situation and extent of fissures and metallic veins, to mark the boundaries where the upper strata have been swallowed up, or cease to appear, or where the under strata push up towards the day. It would be easy to mark the variations in the thickness of the strata in different places, and to record the result of experiments made at any point, by boring or sinking shafts, which might be done by drawing a vertical line downward, so as to represent the thickness of the laminæ, which might be marked by different colours. By such a method, the geologist might obtain a map of the country, which might exhibit, at one view, the general results of all the experiments and inquiries that had been made relative to that science; and the owner of an estate might record in a small compass, all that is known respecting its minerals, and be able, from a comprehensive view of them all, to judge of the probability of success in sinking a shaft, or driving a level. He might also make good use of this perspective in tracing his shafts and drifts in

all their windings, elevations, and depressions, and comparing them with the surface above, marking also the veins and strata in which they run. For, if the artist knows what is beneath the surface, he has no difficulty in representing it as transparent. He must be careful, however, not to perplex himself by lines too much multiplied, and take advantage of his being able to tint the lines with different colours, for the purposes of distinction; and he must use a considerable address in throwing out such lines as would be of little use, and retaining such as will produce the effect of a picture, which should be well preserved in order to make the exhibition easily intelligible.

If he should wish to make a drawing of minerals or crystals, this perspective would be well suited to the purpose.

The point, however, on which the writer of this paper can speak with the greatest confidence is on the representation of machines and philosophical instruments; having been himself so much in the habit of practically applying to them the principles that have been detailed: and this he has exemplified in the plates.

The correct exhibition of objects would be much facilitated by the use of this perspective, even in the hands of a person who is but little acquainted with the art of drawing; and the information given by such drawings is much more definite and precise than that obtained by the usual methods, and better fitted to direct a workman in execution *.

* The author has transcribed this interesting paper from the first volume of the Transactions of the Cambridge Philosophical Society. The method is peculiarly deserving of the attention, and in many cases the adoption, of mechanics and engineers.

Some useful exemplifications of isoperimetrical perspective are given in the 18th and 19th volumes of the Mechanic's Magazine; and while this sheet was going through the press, I learnt that Mr. Jopling is now printing a small treatise on the subject, for the use of mechanics and artificers.

INDEX.

—Those numbers which have *App.* prefixed to them, refer to the Appendix.

A.

ıents, equilibrium of, 196.
ated velocity, 211.
n, algebraical, 54.
— arithmetical, 5.
— of complex fractions, 24.
— of decimal fractions, 22.
— of surds, 68.
— of vulgar fractions, 17.
mechanical, 294.
a mechanical agent, 297.
cific gravity of, *App.* 80.
ocity of, into a vacuum, 280.
ight and equilibrium of, 278.
er, specific gravity of, *App.* 80.
, specific gravity of, *App.* 80.
weight of, *App.* 81.
, 52.
ical characters, 4, 52.
—— division, 58.
—— multiplication, 56.
—— subtraction and addition, 54.
aic notation and definitions, 52.
gris, specific gravity of, *App.* 80.
, specific gravity of, *App.* 80.
ons, on the strength of horses, 360.
of friction of earth, 201.
of repose, 206.
, definitions relating to, 98.
propositions relating to, 99.
raced, definition of, 164.
r measure, 124.
———— French, 126.
strength as a mechanical agent, 348.
cite coal, 344.
my, weight of, *App.* 80.
res, discharge from, 269.
ree wood, weight of, *App.* 81.
, stability of, 205.
table for the construction of equi-
ted catenarian, *App.* 73.

Archimedes' screw, 290.
Arcs, table of multiple, 139.
Arithmetic, 1.
Arithmetical progression, 85.
———— symbols, 4.
Arithmetic, logarithmic, 40.
Arnold's, Mr., escapement, *App.* 87.
Arsenic, weight of, *App.* 80.
Ash, cohesion of, 377.
—— elasticity of, 383, 384.
—— transverse strength of, 387.
—— weight of, *App.* 81, 82.
Asymptotes, definition of, 164, 175.
Atmosphere, density of, 279.
———— height of a homogeneous, 280.
———— machines for raising water by the pressure of, 281.
———— pressure of, 278.
———— temperature of, 280.
Attraction, capillary, 258.
Atwood's, Mr., escapement, *App.* 86.
Avoirdupois weight, 123.

B.

Barker's mill, 296.
Barlow's, Professor, experiments on the cohesion of timber, 377.
Barometer, measuring heights by means of, 158, 279.
Basalts, specific gravity of, *App.* 80.
Beams, formula for the deflexion of, 389.
——— of uniform strength, 386.
——— strength with distributed load, 388.
——— transverse strength of, 385.
Bed of rivers, action of the stream upon, 265.
Beech, cohesion of, 376, 377.
——— elasticity of, 383, 384.
——— transverse strength of, 388.

Beech, weight of, *App.* 81, 82.
Beer, specific gravity of, *App.* 80.
Bees wax, specific gravity of, *App.* 80.
Bernoulli's experiments on spirit springs, *App.* 85.
Berry, specific gravity of, *App.* 80.
Bevan's experiments on cohesion, 378.
——— experiments on human strength, 356.
——— experiments on the power of horses, 363, 365.
——— experiments on torsion, 392.
——— experiments on the modulus of elasticity, 383.
Bidone, M., on the motion of water in canals, 269.
Binomials, Sir I. Newton's rule for involving, 61.
Bismuth, weight of, *App.* 80.
Bitumen, specific gravity of, *App.* 80.
Bodies, Platonic, 111.
Bone, specific gravity of, *App.* 80.
Borda on moving air, 293.
Bouguer's formula for animal strength, 348.
Box, cohesion of, 377.
Boxwood, weight of, *App.* 81.
Bramah's experiments on torsion, 392.
——— hydrostatic press, 252.
Brass, cohesion of, 376.
—— elasticity of, 384.
—— transverse strength of, 387.
—— weight of, *App.* 80, 81, 82.
Breast waterwheels, 296.
Bricks, cohesion of, 378.
——— crushing strength of, 380.
——— specific gravity of, *App.* 80.
Brickwork, weight of, *App.* 82.
Bridge, Old London, 267.
——— effect of, on the tides, *App.* 92.
——— Rochester, 267.
——— Westminster, 267.
Bridges, effects of the piers of, 266.
——— equilibrium of suspension, 207.
——— rise of water occasioned by the piers of, 267.
Buat's experiments, 262, 263, 264, 265, 266, 271, 348.
Buchanan on human strength, 357.

C.

Caloric of vaporization, 307.
Camphor, specific gravity of, *App.* 80.
Cams, *App.* 90.
Canals, effect of a horse on, 364.
——— motion of water in, 262, 269.
——— resistance to traction on, 346, 347.
Cane, elasticity of, 384.
Capacity, measures of, 121.
Capillary attraction, 258.
——— tubes, 259.
Cast iron, cohesive strength of, 375, 376.
——— crushing strength of, 380.
——— elasticity of, 384.
——— transverse strength of, 385.
——— weight of, *App.* 80, 81, 82.
Catenarian curves, table of the relations of, 185; *App.* 73.
Catenary, 180.
Catgut, cohesion of, 378.
Cedar, specific gravity of, *App.* 81.
Center, motion about a fixed, 220.
——— of gravity, 192.
——— of gyration, 229.
——— of oscillation, 220.
——— of percussion, 220.
——— of pressure, 250.
Central forces, 232.
Chalk, weight of, *App.* 80, 81, 82.
Characters, arithmetical and algebraical, 4, 52.
Cherry tree, specific gravity of, *App.* 81.
Christiana deal, cohesion of, 377.
Chronometers, principles of, *App.* 82.
Chrysolite, specific gravity of, *App.* 80.
Circle, equation of the, 176.
——— definitions relating to, 104.
——— divisions of the, 124.
——— mensuration of, 127.
——— propositions relating to, 105.
Circles, tables of the areas, &c. of, *App.* 66.
Circular arcs, mensuration of, 128.
——— sectors, mensuration of, 128.
——— segments, mensuration of, 128.
Circulating decimals, 19.
Circumscribed figures, 104.
Cissoid, 177.
Clarke's, Mr. T., pump, 289.
Clay, weight of, *App.* 82.
Clockwork, *App.* 82.
Coals, anthracite, 344.
—— consumed by steam engines, 343.
—— Derbyshire, 344.
—— Newcastle, 344.
—— tables of the quantity consumed by steam engines, 344.
—— Welsh, 344.
Cobalt, weight of, *App.* 80.
Coefficients of friction, table of the, 206.
Cohesion, modulus of, 378.
——— of timber, 376, 377.
Cohesive strength of cast iron, 374.
——— of wrought iron, 376.
Collision of bodies in motion, 236.
Columns, strength of, 379.
Compensation pendulums, 228.
Complex fractions, 23.

Computation of formulæ, 95.
Conchoid, 176.
Condensing steam engine, 312.
——— with two cylinders, working expansively, 330.
——— with two cylinders, 329.
——— working without expansion, 327.
Conduit pipes, motion of water in, 262.
Cone, mensuration of, 130.
Conical pendulum, 233.
Conic sections, 162.
Conjugate hyperbola, 162.
Conoids, mensuration of, 131.
Copper, cohesion of, 376.
——— crushing strength of, 380.
——— weight of, *App.* 80, 81, 82.
Cork, elasticity of, 384.
—— specific gravity of, *App.* 81.
Cornish engines, duty of, 345, 346.
——— pumping engine, 334.
Cosecants, table of logarithmic, *App.* 19.
Cosine, expressions for, 138.
Cosines, table of logarithmic, *App.* 19.
Cotangents, table of logarithmic, *App.* 19.
Coulomb's experiments on human strength, 355.
——— experiments on windmills, 301.
Craigleith, freestone, crushing strength of, 380.
Crushing force of iron, 380.
Crystal, specific gravity of, *App.* 80.
Cubitt's, Mr., experiments, 375, 390, 391.
Curvature, radius of, 164.
Curve, algebraical, 175.
Curves, catenarian, table of the relations of, 185.
——— properties of, 175.
——— transcendental, 175.
Cycloid, 178.
Cylinder, mensuration of, 130.

D.

Dead-beat escapement, *App.* 86, 87.
Deal, cohesion of, 377.
—— crushing strength of, 380.
—— elasticity of, 383, 384.
Decimal fractions, 18.
——— system, 3.
Decimals, circulating, 19.
Deflexion of beams, 389.
Degree of latitude, length of, 120.
——— longitude, length of, 120.
Density of the atmosphere, 279.
Derbyshire coal, 344.
Desagulier's estimate of horse power, 361.
——— estimate of human strength, 354.
Detached escapement, *App.* 86, 88.
Diagonal scales, 113.
Diamond, specific gravity of, *App.* 80.
Differences of logarithms, table of, *App.* 1.
Dines, Mr., formulæ for the deflexion of beams, 389.
—— formula for the transverse strength of beams, 386.
Directrix, definition of, 164.
Discharge from apertures, 269.
——— of pipes, 268.
Distances, application of trigonometry to the measurement of, 152.
——— mechanical methods of measuring, 158.
Division, algebraical, 58.
——— arithmetical, 10.
——— by logarithms, 45.
——— of complex fractions, 26.
——— of decimal fractions, 22.
——— of surds, 69.
——— of vulgar fractions, 17.
Divisions of the circle, 124.
Donkin's, Mr., tachometer, *App.* 92.
Drummond's pendulum, 229.
Duloug and Arago's formulæ relating to steam, 304.
Dunlop's experiments on torsion, 392.
Duodecimals, 27.
Duodecimal system, 3.
Dynamics, 211.
——— definition of, 188.

E.

Earth, angle of friction of, 201.
——— common, weight of, *App.* 82.
——— diameter of the, *App.* 66.
——— pressure of, against walls, 198.
——— specific gravity of, *App.* 80.
Ebony, weight of, *App.* 81.
Effect, mechanical, 212.
Effluence of liquids, 260.
Elastic fluids, weight and equilibrium of, 278.
Elasticity, modulus of, 381.
——— of bodies, 381.
——— subjected to a transverse strain, 388.
Elder tree, weight of, *App.* 81.
Ellipse, definitions of the, 162.
——— equations of the, 165.
——— mensuration of the, 129.
——— problems relating to the, 167.

Ellipse, properties of the, 165.
Elm, cohesion of, 377.
—— crushing strength of, 380.
—— elasticity of, 383, 384.
—— transverse strength of, 388.
—— weight of, *App.* 81, 82.
Elongation of bodies exposed to a tensile strain, 381.
Embankments, 253.
Emerald, specific gravity of, *App.* 80.
Emerson's estimate of human strength, 355.
Engine, fire, 288.
——— steam, 302.
——————— theory of the, 316.
Engines, atmospheric, 341.
——— classification of, 322.
——— condensing, working without expansion, 327.
——— Cornish, duty of, 345, 346.
——————— pumping, 334.
——— description of the various kinds, 322.
——— expansive, general expressions for, 321.
——— formulæ for calculating their power, 322.
——— locomotive, without expansion, 323.
——— locomotive, working expansively, 328.
——— stationary condensing, working expansively, 329.
——— stationary non-condensing, without expansion, 322.
——— stationary non-condensing, working expansively, 327.
——— unexpansive, general expressions for, 321.
——— Woolfe's, 330.
English and French weights and measures, 125.
Epicycloid, 179.
Epicycloidal wheel, *App.* 91.
Equation of time, 124.
Equations in general, 81.
——————— of the ellipse, 165.
——————— quadratic, 77.
——————— simple, 70.
——————— solution of, by trial-and-error, 82.
Equilibrium of air and elastic fluids, 278.
——————— of non-elastic fluids, 249.
——————— of piers, or abutments, 196.
——————— of polygons, 202.
——————— of structures, 196.
——————— of suspension bridges, 207.
——————— statical, 189.
Escapements, *App.* 86.
Euler's formulæ for animal strength, 348.
Evolute, 176.
Evolution, 29.
——————— by logarithms, 47.
——————— in algebra, 63.
——————— of surds, 69.
Expansive engines, general expressions for, 321.
——————————— maximum effect of, 321.
——————— steam engines, 315.
Expedients, select mechanical, *App.* 89.
Exponents, fractional and negative, 88.
Extermination, 73.
Eytelwein's inquiries, 268.

F.

Factors, table of, for reducing old measures to new, 122.
——— table of useful, *App.* 64.
Fairbairn's, Mr., experiments, 385.
Farey's self-acting flood-gate, 255.
Farish, Professor, on isometrical perspective, *App.* 96.
Figures, circumscribed and inscribed, 104.
Fir, cohesion of, 377.
— elasticity of, 383, 384.
— transverse strength of, 388.
— weight of, *App.* 81, 82.
Fire engine, 288.
Flint glass, specific gravity of, *App.* 80.
—— specific gravity of, *App.* 80.
Floating bodies, 254.
Flood-gate, Farey's self-acting, 255.
Fluidity, 248.
Fluids, elastic, weight and equilibrium of, 278.
Fluids, pressure and equilibrium of non-elastic, 249.
Focus, definition of, 164.
Force, absolute and permanent, 349.
—— of gravity, 213.
——————————— in different latitudes, 215.
——— of the wind, 292.
Forces, central, 232.
Forcing pump, 287.
Formulæ, computation of, 95.
——————— for calculating the power of steam engines, 322.
——————— for the temperature and pressure of steam, 304.
——————— trigonometrical, 134.
Fractional and negative exponents, 88.
Fractions, complex, 28.
——————————— addition of, 24.
——————————— division of, 26.
——————————— multiplication of, 25.
——————————— reduction of, 23.
——————————— subtraction of, 25.

Fractions, decimal, 18.
——————— addition and subtraction of, 22.
——————— multiplication and division of, 22.
——————— reduction of, 19.
——— vulgar, 13.
——————— addition and subtraction of, 17.
——————— multiplication and division of, 17.
——————— reduction of, 14.
——————— theorems for abbreviating, 15.
French and English weights and measures, 125.
Friction, angle of, for earth, 201.
——— coefficients of, 206.
——— of water in horizontal pipes, 270.
Functions of π, table of, *App.* 64.

G.

Garnet, specific gravity of, *App.* 80.
Gear, spiral, *App.* 89.
Geographical mile, 120.
——————— length of a, *App.* 66.
Geometrical definitions, 98.
Geometry, plane and solid, 98.
——— practical, 112.
Giles, Mr., tidal observations on the Thames, *App.* 93.
Glass, specific gravity of, *App.* 80.
—— tube, elasticity of, 384.
Gold, elasticity of, 383.
—— weight of, *App.* 80.
Governor, conical, 233.
Grahame, Mr. W., on the resistance to traction on canals, 365.
Graham's escapement, *App.* 87.
——— pendulum, 228.
Granite, crushing strength of, 380.
——— weight of, *App.* 80, 81, 82.
Grass, cohesion of, 378.
Gravel, weight of, *App.* 82.
Gravities, table of specific, *App.* 80.
Gravity, center of, 192.
——— force of, 213.
——————— at London, *App.* 66.
——————— in different latitudes, 215.
——— motion of bodies under the action of, 214.
——— specific, 249.
——————— rules for finding the, 257.
Gums, specific gravity of, *App.* 80.
Gypsum, specific gravity of, *App.* 80.
Gyration, center of, 229.

H.

Hachette's experiments on human strength, 357.
Hamel's, Du, experiments on the cohesion of ropes, 377.
Hassenfratz's estimate of human strength, 359.
Heat, latent, of steam, 307.
—— mechanical agents depending upon, 302.
Height of an homogeneous atmosphere, 280.
Heights, application of trigonometry to the measurement of, 152.
——— measuring by the barometer, 158, 279.
——— mechanical methods of measuring, 158.
Hempen fibres, cohesion of, 377, 378.
Hire, De la, on the strength of horses, 359.
Hodgkinson's, Professor, experiments, 374, 379, 385, 389.
——————— form of section, 385.
——————— formulæ for transverse strength of beams, 385.
Homogeneous atmosphere, height of, 280.
Hone, specific gravity of, *App.* 80.
Honey, specific gravity of, *App.* 80.
Horse-power of a steam engine, 343.
Horses, strength of, 359.
Horizontal waterwheels, 294.
Hungarian machine, 291.
Hydraulic mean depth, 263.
——— ram, 291.
Hydrodynamics, 260.
——————— definition of, 188.
Hydrostatical paradox, 251.
——————— press, 252.
Hydrostatics, 248.
——————— definition of, 188.
Hyperbola, 162.
——————— mensuration of, 130.
——————— problems relating to the, 170.
——————— properties of the, 168.
Hyperbolas, conjugate, 162.
Hyperbolic logarithms, 93, 169.
Hyperboloids, mensuration of, 131.

I. J.

Jasper, weight of, *App.* 80.
Ice, cohesion of, 378.

Ice, elasticity of, 384.
Inclined plane, 244.
——— planes, motion on, 217.
Inscribed and circumscribed figures, 104.
Involute, 176.
Involution, 28.
———— by logarithms, 47.
———— in algebra, 60.
———— of surds, 69.
Iron, cohesion of, 375, 376.
—— crushing strength of, 380.
—— elasticity of, 383, 384.
—— transverse strength of, 385, 387.
—— weight of, *App.* 80, 81, 82.
Isometrical perspective, *App.* 96.
Ivory, specific gravity of, *App.* 80.

L.

Lambton's, Gen., length of scale, 120.
Lance wood, elasticity of, 384.
Langsdorf's theorem, 268.
Larch, cohesion of, 377.
—— elasticity of, 383, 384.
——— specific gravity of, *App.* 81.
—— transverse strength of, 388.
Latent heat of steam, 307.
Latitude, length of a degree of, 120.
Latitudes, force of gravity in different, 215.
Latus rectum, 164.
Lead, cohesion of, 376.
—— crushing strength of, 380.
—— elasticity of, 384.
—— transverse strength of, 387.
—— weight of, *App.* 80, 81, 82.
Lean's reports of the Cornish engines, 345, 346.
Leather, cohesion of, 378.
Length, French measures of, 125.
———— measures of, 120.
——— particular measures of, 120.
Leslie, Prof., on cohesion, 376.
——— on human strength, 357.
——— on the modulus of elasticity, 383.
Levers, 240.
Lever, universal, *App.* 92.
Lifting pumps, 287.
Lignum vitæ, elasticity of, 384.
———— specific gravity of, *App.* 81.
Limestone, crushing strength of, 380.
———— elasticity of, 384.
———— weight of, *App.* 80, 81.
Lines, definitions relating to, 98.
——— propositions relating to, 99.
——— rectangles of, 99.
Liquids, motion and effluence of, 260.
——— specific gravity of, *App.* 80.
Locomotive engines without expansion, 323.
———— engines, working expansively, 328
Logarithmic arithmetic, 40.
———— differences, table of, *App.* 1.
———— sines, tangents, &c., table of, *App.* 19.
———— tables use of, 42.
Logarithms, 90.
———— division by, 45.
———— evolution and involution by, 47.
———— hyperbolic, 93, 169.
———— modulus of the common, *App.* 66.
———— multiplication by, 45.
———— of numbers, table of, *App.* 1.
———— of useful numbers, table of, *App.* 66.
———— proportion by, 46.
———— table for converting common into hyperbolic, and *vice versâ*, 94.
Logwood, specific gravity of, *App.* 81.
London Bridge, effect of, on the tide, *App.* 92.
———— old, 267; *App.* 92.
Longitude, length of a degree of, 120.

M.

Machine, Hungarian, 291.
———— Zurich, 290.
Machines for raising water by the pressure of the atmosphere, 281.
Macneill on the wear of roads, 368.
Madeira wine, specific gravity of, *App.* 80.
Mahogany, 377.
———— elasticity of, 383, 384.
———— weight of, *App.* 81, 82.
Mann's, Abbé, method of measuring streams, 275.
Maple, weight of, *App.* 81.
Marble, crushing strength of, 380.
—— weight of, *App.* 80, 81, 82.
Marine surveyor, 277.
Mariotte's law, 279.
Materials, strength of, 370.
———— to resist tension, 392.
Matting, cohesion of, 378.
Maximum effect of expansive engines, 321.
Measure, angular, 124.
Measurement, standards of, 119.
Measurers, stream, 273.
———— Pitot's, 274.
Measures, 119.

s, comparison of English and
h, 125.
— French, angular, 126.
——— solid, 126.
——— square, 125.
— of length, 120.
——— French, 125.
——— particular, 120.
— of solidity and capacity, 121.
— of surface, 121.
— of time, 124.
— of weight, 123.
— table of factors for reducing
new, 122.
ng heights and distances, appli-
of trigonometry to, 152.
— heights and distances by me-
al methods, 158.
— heights by the barometer,
179.
cal agents, 294.
— agent, animal strength as a,

— definitions, 187.
— effect, 212.
— expedients, *App.* 89.
— method of measuring heights
istances, 158.
— powers, 239.
——— table of the, 247.
cs, 187.
ır, cohesion of, 377.
— elasticity of, 383.
chulze's experiments on the
350.
tion, 119.
— of solids, 130.
— of superficies, 127.
, weight of, *App.* 80.
pecific gravity of, *App.* 80.
me, 227.
ter, 152, 158.
ngth of a geographical, *App.* 66.
gth of a statute, *App.* 66.
stical, 120.
ecific gravity of, *App.* 80.
:ker's, Parent's, or Segner's, 296.
me, specific gravity of, *App.*

nd, 298.
d plus signs, 53.
of cohesion, 378.
of elasticity, 381.
of the common logarithms, *App.*

definition of, 188.
m, definition of, 212.
n the strength of French stone-
s, 359.
bout a fixed center, 220.
Motion, laws of, uniform and variable, 212.
——— of bodies under the action of gravity, 214.
——— of liquids, 260.
——— of water in conduit pipes and canals, 262.
——— of water in rivers, 263.
——— on inclined planes, 217.
——— over fixed pulleys, 216.
——— uniformly accelerated, 214.
Motions, parallel, *App.* 91.
Mudge's, Mr., escapement, *App.* 86, 88.
Muriatic ether, specific gravity of, *App.* 80.
Multiple arcs, table of, 139.
Multiplication, algebraical, 56.
——— arithmetical, 7.
——— by logarithms, 45.
——— of complex fractions, 25.
——— of decimal fractions, 22.
——— of surds, 68.
——— of vulgar fractions, 17.
——— table, 8.

N.

Nautical mile, 120.
Navier's estimate of human strength, 359.
——— formula for the specific volume of steam, 309.
Negative and fractional exponents, 88.
Newcastle coal, 344.
Newton's, Sir I., rule for involving a binomial, 61.
Nickel, weight of, *App.* 80.
Nitric acid, specific gravity of, *App.* 80.
Nitrous acid, specific gravity of, *App.* 80.
Normal, definition of a, 164.
Notation, algebraical, 52.
——— Roman, 2.
——— usual system of, 2.
Numbers, prime, 49.
——— properties of, 49.
——— table of the logarithms of, *App.* 1.
——— table of useful, with their logarithms, *App.* 66.

O.

Oak, cohesion of, 376, 377.
—— crushing strength of, 380.
—— elasticity of, 383, 384.
—— transverse strength of, 387, 388.
—— weight of, *App.* 81, 82.
Oil of turpentine, specific gravity of, *App.* 80.

Olive oil, specific gravity of, *App.* 80.
Opal, specific gravity of, *App.* 80.
Orange wood, specific gravity of, *App.* 81.
Oscillation, center of, 220.
Overshot waterwheels, 295.

P.

Page's, Mr., tidal observations on the Thames, *App.* 96.
Pambour on the steam engine, 317.
Pambour's formulæ relating to steam, 304.
Paper, cohesion of, 378.
Parabola, definition of, 162.
——— mensuration of, 129.
——— properties of the, 171.
Paraboloids, mensuration of, 131.
Paradox, hydrostatical, 251.
Parameter, definition of, 163.
Parallel motions, *App.* 91.
Parallelopipedon, mensuration of, 130.
Parent's mill, 296.
Pear, cohesion of, 377.
—— tree wood, specific gravity of, *App.* 81.
Pearl, specific gravity of, *App.* 80.
Pebble, specific gravity of, *App.* 80.
Pendulum, conical, 233.
——— length of seconds, *App.* 66.
Pendulums, compensation, 228.
——— simple and compound, 221.
——— table of the lengths and vibrations of, *App.* 76.
Percussion, 236.
——— center of, 220.
Perspective, isometrical, *App.* 96.
Phosphorus, specific gravity of, *App.* 80.
Piers, equilibrium of, 196.
—— of bridges, effects of, 266.
——— table of the rise of water occasioned by, 267.
Pine, elasticity of, 383, 384.
—— transverse strength of, 387, 388.
—— weight of, *App.* 82.
Pinion and rack, double, *App.* 91.
Pipes, discharge of, 268.
—— motion of water in, 262
Pitch pine, specific gravity of, *App.* 81.
Pitot's stream measurer, 274.
Plane geometry, 98.
—— inclined, 244.
—— triangles, solution of the cases of, 148.
Planes, definitions relating to, 109.
——— motion on inclined, 217.
——— propositions relating to, 108.
Platinum, elasticity of, 383.
Platinum, weight of, *App.* 80.
Platonic bodies, 111.
——— mensuration of, 132.
Playfair on Barker's mill, 297.
Ploughing, strength of horses exerted in, 363.
Plus and minus signs, 53.
Pneumatics, 278.
——— definition of, 188.
Polygons, definitions relating to, 102.
——— equilibrium of, 202.
——— mensuration of, 127.
——— table of, 118.
——— propositions relating to, 103.
Poon, specific gravity of, *App.* 81.
—— transverse strength of, 387.
Poplar wood, specific gravity of, *App.* 81.
Porphyry, specific gravity of, *App.* 80.
Power, horse, of a steam engine, 343.
—— of steam engines, formulæ for calculating, 322.
Powers and roots, 28.
——— of trigonometrical quantities, 146.
—— mechanical, 239.
——— table of the mechanical, 247.
Practical geometry, 112.
Press, hydrostatic, 252.
Pressure, center of, 250.
——— of earth against walls, 198.
——— of non-elastic fluids, 249.
——— of steam, formulæ for, 304.
——— table of, 311.
——— of the atmosphere, 278.
——— machines for raising water by, 281.
Pressures, parallel, 190.
——— resultant of, 190.
Prime numbers, 49.
Principles of chronometers, *App.* 82.
Prism, mensuration of, 130.
Problems relating to the ellipse, 167.
——— hyperbola, 170.
——— parabola, 173.
Progression, 84.
——— arithmetical, 85.
——— geometrical, 86.
Properties of numbers, 49.
Proportion, 32.
——— by logarithms, 46.
Propositions in trigonometry, 146.
Pulleys, 243.
——— motion over fixed, 216.
Pump, forcing, 287.
—— lifting, 287.
—— quicksilver, 289.
—— spiral, 290.
—— suction, 281.
Pyramid, mensuration of, 130.

Q.

rals, definitions relating to,
—— mensuration of, 127.
—— propositions relating to,
equations, 77.
r, 179.
r pump, 289.

R.

pinion, double, *App.* 91.
curvature, 165.
ctor, definition of, 164.
effect of a horse on, 364.
resistance to traction on, 347.
raulic, 291.
s bar, length of, 120.
termination of, 37.
., length of scale, 120.
apement, *App.* 86, 87.
s of lines, 99.
of complex fractions, 23.
of decimal fractions, 19.
of surds, 65.
of vulgar fractions, 14.
table of human strength, 357.
adulum, 229.
Mr. G., experiments on cohe-
'6.
experiments on torsion, 392.
able of the angle of, 206.
ecific gravity of, *App.* 80.
led plane triangles, 151.
tion of the stream on the bed
.
tion of water in, 263.
le of the velocities of, 265.
fect of a horse on, 364.
stance to traction on, 347, 368.
r of, 368.
, Dr., experiments, 262, 266,
bridge, 267.
otation, 2.
l powers, 28.
—— of trigonometrical
ies, 146.
l, elasticity of, 384.
principles of, 229.
table of the velocity of the
293.
ciety's standard, 120.
ecific gravity of, *App.* 80.
'hree, 34.
Rule of Three, by logarithms, 46.
Rumsey's improvement on Barker's mill, 296.
Ryegate stone, elasticity of, 384.

S.

Sails of a windmill, angle for the, 298.
Sapphire, specific gravity of, *App.* 80.
Satin wood, elasticity of, 384.
Saumarez's, Mr., marine surveyor, 277.
Sauveur's estimate of the strength of horses, 361.
Scales, diagonal, 113.
Schemnitz vessels, 291.
Schulze's experiments upon the strength of men, 350.
Screw, 245.
—— Archimedes', 290.
—— endless, 246.
Sea water, specific gravity of, *App.* 80.
Secant's table of the logarithms of, *App.* 19.
Segner's mill, 296.
Sexagesimal system, 4.
Shuckburgh's, Sir G., length of scale, 120.
Signs, minus and plus, 53.
Silver, weight of, *App.* 80.
Simple equations, 70.
Sines, expressions for, 138.
—— table of logarithmic, *App.* 19.
Slate, cohesion of, 378.
—— elasticity of, 384.
—— weight of, *App.* 80, 82.
Smeaton on the friction of pipes, 270.
—— on water-wheels, 295.
—— on windmills, 298.
Smeaton's estimate of human strength, 355.
Solid geometry, 98.
Solidity, measures of, 121.
Solid measure, French, 126.
Solids, definitions relating to, 109.
—— mensuration of, 130.
—— propositions relating to, 110.
Sound, velocity of, 161, 281.
Southern's formulæ relating to steam, 304.
Specific gravities, table of, *App.* 80.
—— gravity, 249.
—— —— rules for finding, 257.
—— volume of steam, 308.
—— —— table of, 311.
Spheres, mensuration of, 131.
Spheroids, mensuration of, 131.
Spiral gear, *App.* 89.
—— pump, 290.

Springs, Berthoud's experiments upon, *App.* 85.
Square measure, French, 125.
Stability of arches, 205.
Statical equilibrium, 189.
Statics, 189.
——— definition of, 188.
Statute mile, length of a, *App.* 66.
Standards of measurement, 119.
Standard yard, length of, 120.
Steam, definitions of terms relating to, 303.
——— engines, 302.
——— classification of, 322.
——— coals consumed by, 343.
——— condensing, 312.
——— expansive, 315.
——— formulæ for calculating their power, 322.
——— general description of, 312.
——— horse power of, 343.
——— theory of, 316.
——— formulæ for the pressure and temperature of, 304.
——— latent heat of, 307.
——— properties of, 303.
——— specific volume of, 308.
——— table of the pressure, temperature and specific volume of, 311.
Steel, elasticity of, 383, 384.
——— weight of, *App.* 80, 82.
Stone, common, weight of, *App.* 81.
——— paving, weight of, *App.* 80, 81.
Stream, action of, on the beds of rivers, 265.
——— measurers, 273.
——— Pitot's, 274.
Strength, animal, 348.
——— of horses, 359.
——— of materials, 370.
——— to resist torsion, 392.
Structures, equilibrium of, 196.
Subtraction, algebraical, 54.
——— arithmetical, 6.
——— of complex fractions, 25.
——— of decimal fractions, 22.
——— of surds, 68.
——— of vulgar fractions, 17.
Suction pump, 281.
Sulphuric acid, specific gravity of, *App.* 80.
——— ether, specific gravity of, *App.* 80.
Sulphur, specific gravity of, *App.* 80.
Superficies, mensuration of, 127.
Surds, 65.
——— addition of, 68.
——— division of, 69.
Surds, evolution of, 69.
——— involution of, 69.
——— multiplication of, 68.
——— reduction of, 65.
——— subtraction of, 68.
Surface, measurs of, 121.
Surveyor, marine, 277.
Suspension bridges, equilibrium of, 207.
Sycamore, cohesion of, 376.
——— elasticity of, 383.
Syphon, 261.

T.

Table IV., use of, 140.
——— of factors for reducing old measures to new, 122.
——— of logarithmic differences, *App.* 1.
——— of logarithmic sines, tangents, &c., *App.* 19.
——— of multiple arcs, 139.
——— of polygons, 118.
——— of specific gravities, *App.* 80.
——— of the angle of friction of earth, 201.
——— of the areas, &c., of circles, *App.* 66.
——— of the coefficients of friction, and of the angle of repose, 206.
——— of the effect of a horse on canals, railways, and roads, 364.
——— of the friction of water in horizontal pipes, 270.
——— of the lengths and vibrations of pendulums, *App.* 76.
——— of the logarithms of numbers, *App.* 1.
——— of the mechanical powers, 247.
——— of the pressure, temperature, and specific volume of steam, 311.
——— of the relations of catenarian curves, 185; *App.* 73.
——— of the resistance to traction on canals, railways, and roads, 347.
——— of the rise of water occasioned by the piers of bridges, 267.
——— of the velocities of rivers, 265.
——— of the thickness of walls, 201.
——— of useful factors, *App.* 64.
——— numbers, with their logarithms, *App.* 66.
Tables for converting common logarithms into hyperbolic, and *vice versâ*, 94.
——— logarithmic, use of, 42.
——— of the quantity of coals consumed by steam engines, 344.
——— trigonometrical, 139.
Tachometer, *App.* 92.
Tallow, specific gravity of, *App.* 80.

nts, table of logarithmic, *App.* 19.
cohesion of, 376, 377.
elasticity of, 384.
specific gravity of, *App.* 81.
transverse strength of, 387.
erature of steam, formulæ for, 304.
——— of steam, table of, 311.
——— of the atmosphere, 280.
es, sectional areas of, *App.* 94.
— tidal observations on, *App.* 98.
ems for abbreviating vulgar fractions, 15.
y of the steam engine, 315.
, the Rule of, 34.
observations on the Thames, *App.*

, effect of old London bridge on
, *App.* 92.
er, cohesion of, 376, 377.
equation of, 124.
measures, 124.
ohesion of, 376.
rushing strength of, 380.
lasticity of, 384.
ransverse strength of, 387.
veight of, *App.* 80.
e joint, 242.
, specific gravity of, *App.* 80.
m, strength of materials to resist,
.
verse strength of beams, 385.
——————— of materials, 385.
old on horse power, 362, 364.
—— on the modulus of elasticity,
.
—— on the steam engine, 317.
—— on the strength of materials,
).
gles, definitions relating to, 100.
—— mensuration of, 127.
—— plane, solution of the cases of,
.
—— propositions relating to, 100.
—— right angled plane, 151.
nometrical definitions and formulæ,
.
——————— quantities, roots, and
rers of, 146.
——————— tables, 139.
nometry, 134.
——————— application of, to measurheights and distances, 152.
——————— general propositions, 146.
et gold, specific gravity of, *App.*

weight, 123.
, capillary, 259.
ike roads, resistance to traction on,
.

U.

Under-shot water-wheels, 294.
Unexpansive engines, general expressions for, 321.
Uniform motion, laws of, 212.
Universal lever, *App.* 92.

V.

Vacuum, velocity of air into, 280.
Valve on moving air, 293.
Vaporization, caloric of, 307.
Variable motion, laws of, 212.
Velocities of rivers, table of, 265.
——————— of streams, contrivances to measure, 273.
Velocity of sound, 161, 281.
Vibrations of pendulums, table of the, *App.* 76.
Vis inertia, definition of, 188.
Vis viva, definition of, 188.
Volume, specific, of steam, 308.
——————————————— table of, 311.
Vulgar fractions, 13.

W.

Walls, pressure of earth against, 198.
——— table of the thickness of, 201.
Walnut, specific gravity of, *App.* 81.
Water as a mechanical agent, 294.
——— friction of, in pipes, 270.
——— machines for raising, 281.
——— motion of, in pipes and canals, 262.
——— motion of, in rivers, 263.
———————, over weirs, 271.
——— rise of, occasioned by the piers of bridges, 267.
——— snail, 290.
——— specific gravity, *App.* 81.
——— wheels, breast, 296.
——————— horizontal, 294.
——————— overshot, 295.
——————— undershot, 294.
Wax, specific gravity of, *App.* 80.
Wedge, 245.
Weight, avoirdupois, 123.
——— comparison of English and French, 125.
——— French, 126.
——— measures of, 123.
——— of the air, 278.
Weights, 119.
Weirs, motion of water over, 271.
Welsh coal, 344.

Westminster bridge, 267.
Whalebone, cohesion of, 378.
———— elasticity of, 384.
Wheel and axle, 242.
—— epicycloidial, *App.* 91.
Whitehurst, Mr., 291.
Whiteland's improvement on Barker's mill, 297.
Wicksteed's experiments on fuel, 344.
Willow, elasticity of, 384.
——— specific gravity of, *App.* 81.
Wind, force of the, 292.
—— Rouse's table of the velocity of the, 293.
—— mills, 298.
———— Coulomb's experiments on, 301.
———— Smeaton's maxims concerning, 298.
Wine, specific gravity of, *App.* 80.
Wipers, *App.* 90.
Woolfe's steam engine, 330.
Wrought iron, cohesive stre
———— elasticity of,
———— transverse
387.
———— weight of, *A*

Y.

Yard, length of standard, 1
Yew, elasticity of, 384.
—— specific gravity of, *A*
Yorkshire paving, elasticity

Z.

Zinc, transverse strength of,
—— weight of, *App.* 80.
Zurich machine, 290.

ERRATA.

Page	Line	
133	24	*For* D being = AC = AE, &c., *read* D being = AB,
135	20	*For* AD, *read* A′D.
137	2	*In three places, for* HD, *read* A′D.
155	27	*For* (Fig. 108), *read* (Fig. 108*a*).
159	11	*For* (Fig. 103), *read* (Fig. 106).
167	last	*For* D, *read* E.
171	6	*For* KH and KI, *read* GH and GI.
191	32	*For* EG, *read* FG.
199	20	*For* IH = LH, *read* $P_4 . \overline{AZ} = W . \overline{AX}$.
208	27 & 29	*For s, read* s^2.
213	45	*After* 1 to ·9994, *insert* for the first mile.
237	35	*For* $\frac{B+b}{BV}$, *read* $\frac{BV}{B+b}$.
238	8	*For* $\frac{2BV+(B-b)v}{B+b}$, *read* $\frac{2BV+(b-B)v}{B+b}$.
238	13	*For* $\frac{2BV-(B-b)v}{B+b}$, *read* $\frac{2BV+(B-b)v}{B+b}$.
238	17	*For* $\frac{2Bv}{B+b}$, *read* $\frac{2BV}{B+b}$.
316	29	*For* sin (90 *n*), *read* $\surd n$.
316	34	*For* ·383, *read* ·500; *and for* 4·6, *read* 6·0.
316	36	*For* ·924, *read* ·866; *and for* 11·1, *read* 10·4.
335	20	*For* E, *read* F.

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Fig. 8

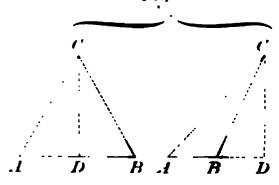


Fig. 9.

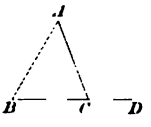


Fig. 10

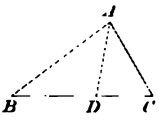
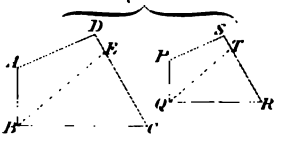


Fig. 19

Fig. 20.



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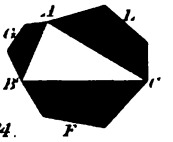
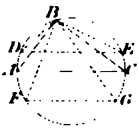


Fig. 32

Fig. 33.



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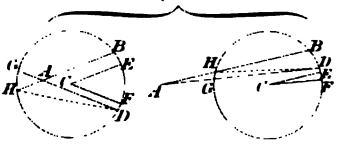

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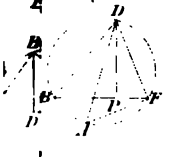

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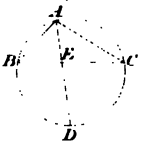

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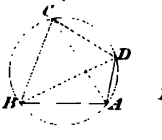

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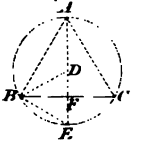

Fig. 58

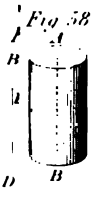

Fig. 59.

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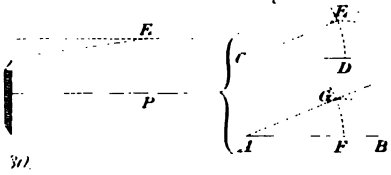
Fig. 62.

Fig. 71.



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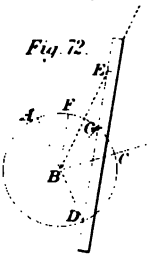

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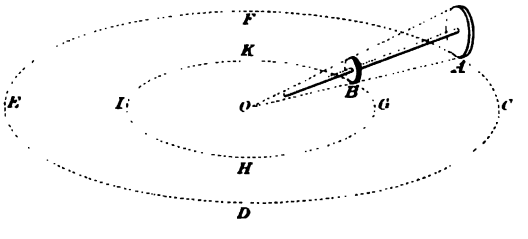

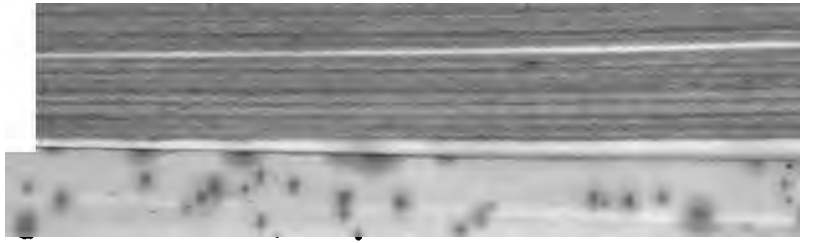

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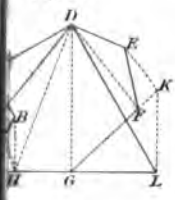

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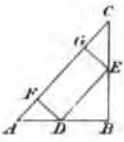

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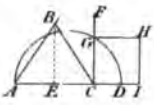

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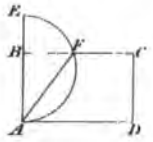
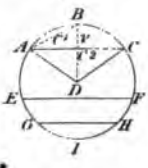


Fig.97.

Fig.98.



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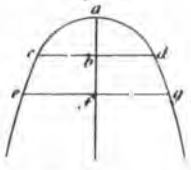

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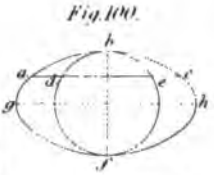

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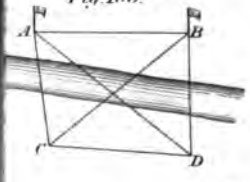

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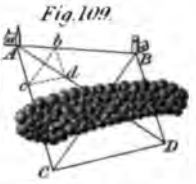

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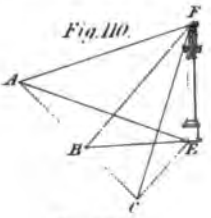

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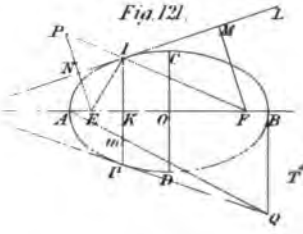

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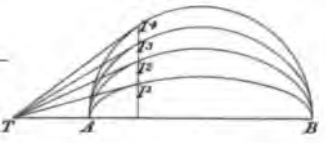

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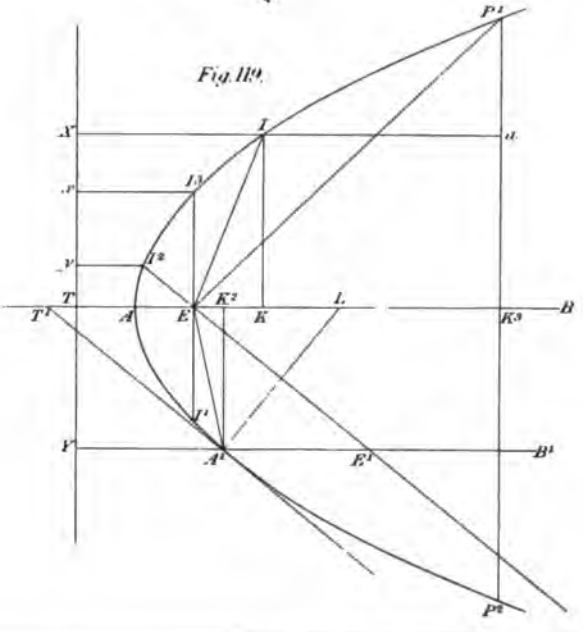

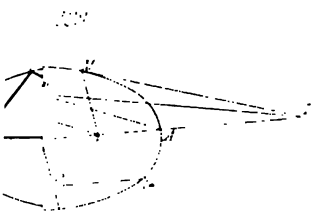

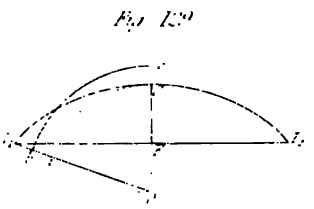

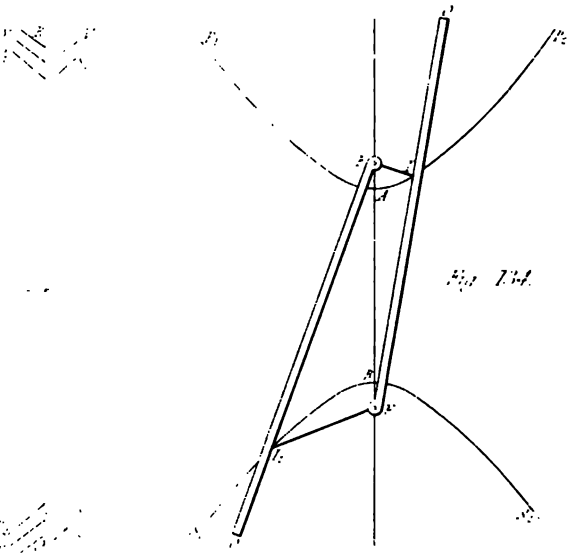

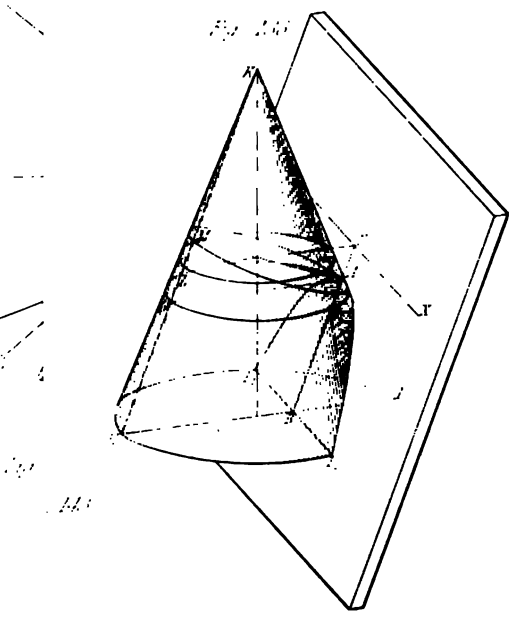

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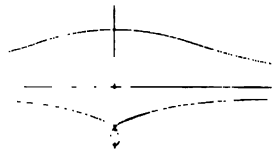

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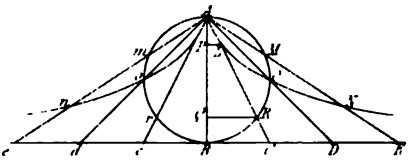

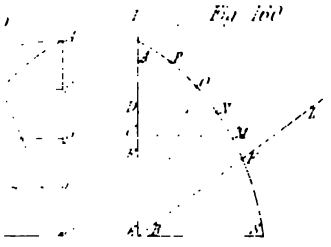

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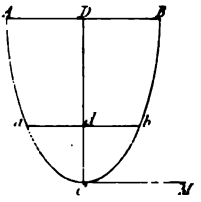

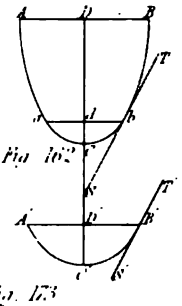

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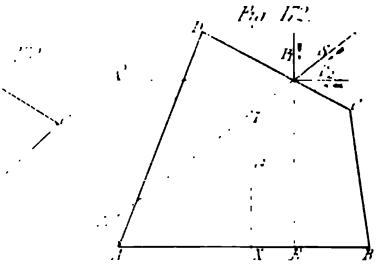

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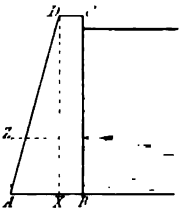

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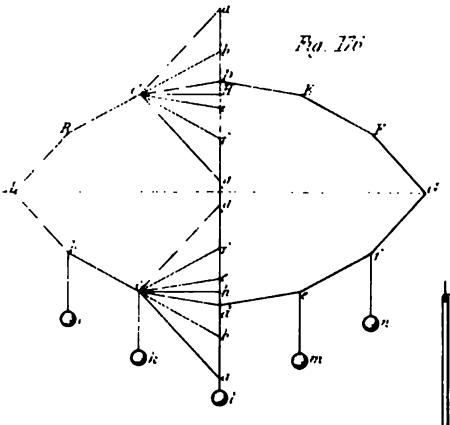


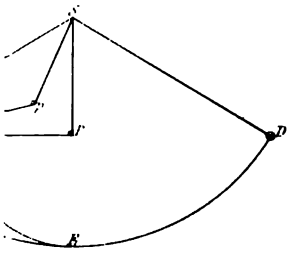
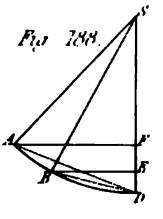


Fig. 188.



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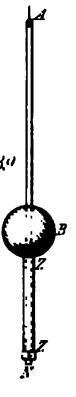

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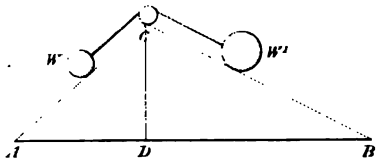

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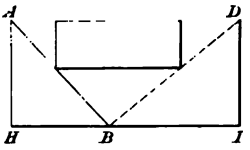

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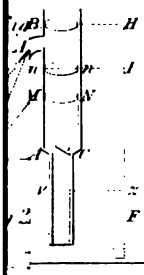

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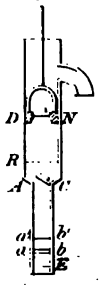

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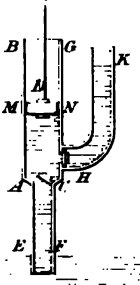

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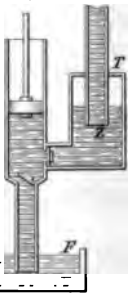

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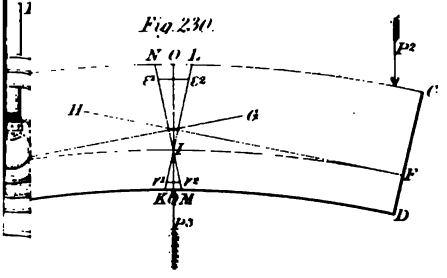


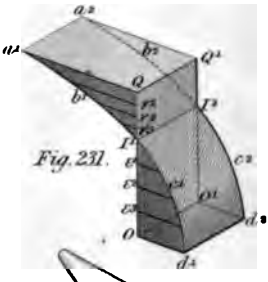


Fig. 231.

Fig. 232.

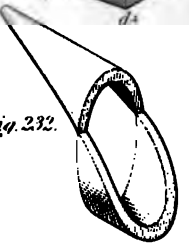
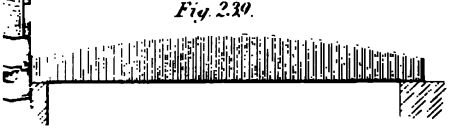


Fig. 238.

Fig. 239.



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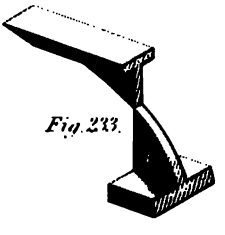

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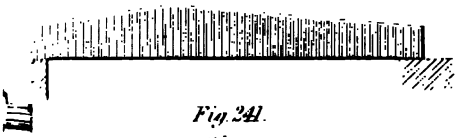
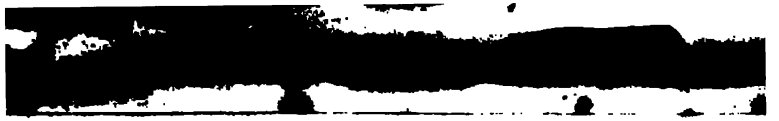
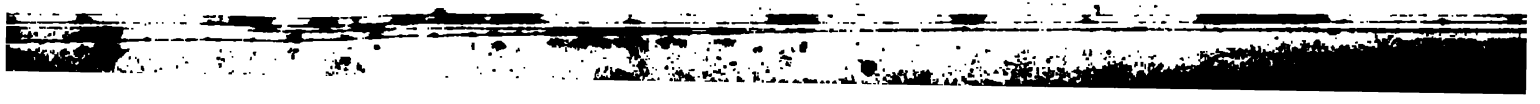


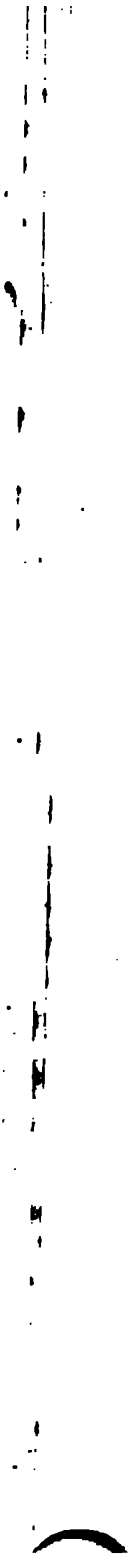
Fig. 241.

Fig. 234.

Fig. 235







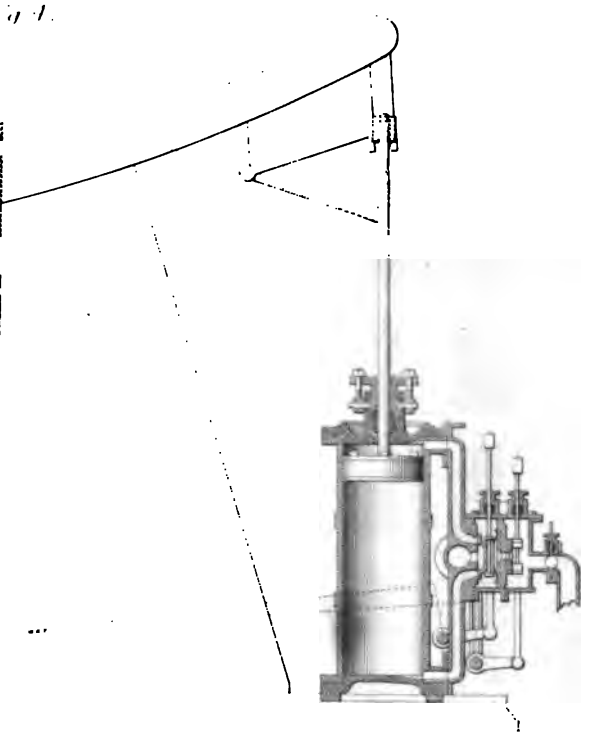
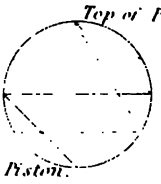
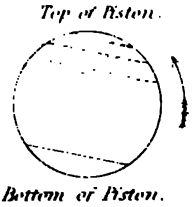
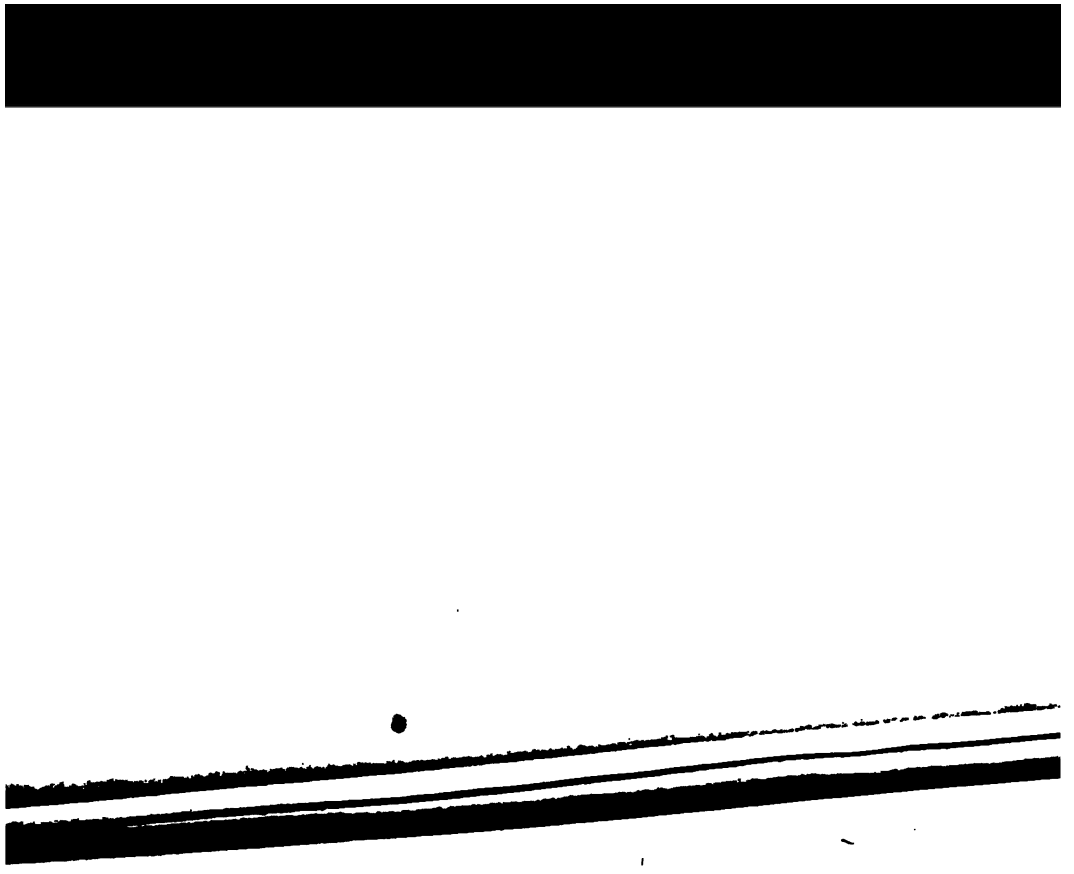
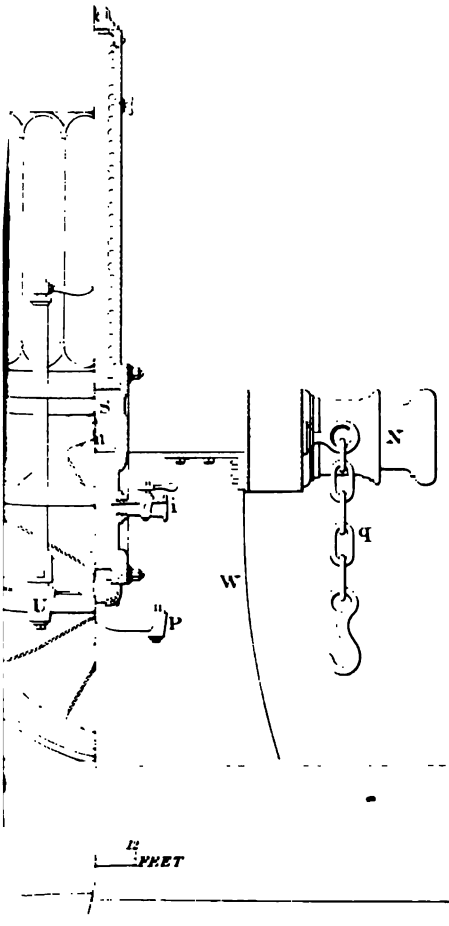


Fig. 6.


Top of Piston.
Bottom of Piston.


Top of Piston.
Bottom of Piston.





W. A. Beaver, sc.

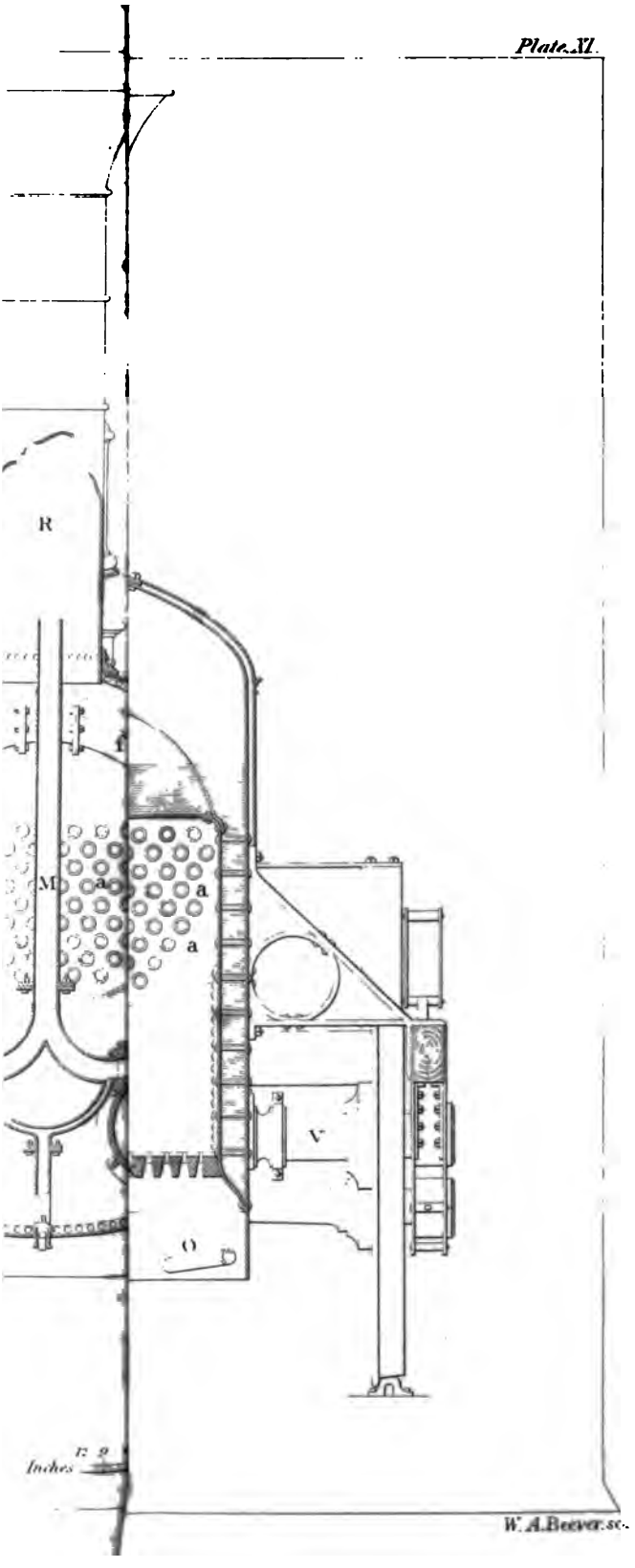


Plate XI.
R
M
a
a
a
V
O
Inches
W. A. Beever sc.